

## THE THEORY OF A METRICAL FIXED POINT THEOREM: THEORETICAL AND APPLICATIVE RELEVANCES

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**Abstract.** In this paper we present some examples of a theory of a metrical fixed point theorem. These theories are in connection with:

- Picard operators and weakly Picard operators
- Bessaga and Janos operators
- data dependence of the fixed points
- sequences of operators and fixed points
- operatorial inequalities
- comparison principle
- fibre generalized contractions
- well-posedness of the fixed point problem
- limit shadowing property
- applications.

**Key Words and Phrases:** generalized metric spaces, generalized contractions, infinite matrices, Picard operators, weakly Picard operators, Bessaga operators, Janos operators, data dependence, operatorial inequalities, comparison principle, fibre generalized contractions, topological isomorphisms, well-posedness of fixed point problem, limit shadowing property.

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### 1. INTRODUCTION

Instead of any title explanations we invite the readers to look over the following questions:

- what does it mean the theory of a theorem?

- what does it mean the theory of a fixed point theorem?
- what does it mean the theory of a metrical fixed point theorem?

under the umbrella of the following opinions:

- I.A. Rus ([58], pp. 16-18): "We have mentioned in §2 that a lot of metrical fixed point theorems are known. For example, only in the paper of Rhoades [1] (see [52]), 125 such theorems are given. For a better understanding of this accumulation of results we propose in this paper to study the position of the conditions from the "existence and uniqueness" in metrical fixed point theorems in relation with the following problems.

**Problem 1.4.1.** Let  $X$  be a nonempty set and  $f : X \rightarrow X$  be a mapping. Which are the conditions on  $f$  such that every periodic point of  $f$  is a fixed point, i.e.

$$F_f = F_{f^n}, \quad \forall n \in \mathbb{N}^* ?$$

**Problem 1.4.2.** Let  $f : (X, d) \rightarrow (X, d)$  be a mapping that satisfies the conditions of some existence and uniqueness fixed point theorem, such that the fixed point can be obtained by successive approximations, i.e.,  $F_f = \{x^*\}$  and  $x^* = \lim_{n \rightarrow \infty} f^n(x_0)$ ,  $x_0 \in X$ . Let  $g : X \rightarrow X$  be a mapping which approximates the mapping  $f$ . More precisely, we assume that

$$d(f(x), g(x)) \leq \eta, \quad \text{with given } \eta \in \mathbb{R}_+, \quad \forall x \in X.$$

Let  $y_n = g^n(x_0)$ . The problem is to give an estimate of  $d(x^*, y_n)$ .

**Problem 1.4.3.** Let  $f, f_n : (X, d) \rightarrow (X, d)$  be such that  $F_{f_n} \neq \emptyset, \forall n \in \mathbb{N}^*$  and  $F_f = \{x^*\}$ . If the sequence  $(f_n)_{n \in \mathbb{N}}$  converges uniformly to  $f$  (denoted by  $f_n \xrightarrow{\text{unif.}} f$  and  $x_n \in F_{f_n} \forall n \in \mathbb{N}^*$ , which are the conditions on  $f$  such that  $(x_n)_{n \in \mathbb{N}}$  converges to  $x^*$  ?

**Problem 1.4.4.** Let  $(X, d)$  be a metric space,  $(Y, \tau)$  be a topological space and  $f : X \times Y \rightarrow X$  be a mapping. Assume that the mapping  $f(\cdot, y)$  has a unique fixed point,  $x_y^*$ , for every  $y \in Y$ . We define the mapping

$$P : Y \rightarrow X, \quad y \mapsto x_y^*.$$

Which are the conditions on  $f$  such that the mapping  $P$  is continuous ?

**Problem 1.4.5.** Let  $(X, \|\cdot\|)$  be a Banach space and  $f : X \rightarrow X$  be a mapping. Which are the conditions on  $f$  such that  $1_X - f$  is surjective ?

- K. Deimling ([19], pp. 187): "While Banach's theorem has found numerous interesting applications, sometimes after tricky reformulation of the given

problems, the same cannot be claimed for most of its variants that we have seen. Most parts of this branch of fixed point theory are therefore definitely pure and often poor mathematics, and much depends on personal taste and aesthetics aspects.”

- P.P. Zabrejko ([74], pp. 826): ”... it is natural to try to present such variant of the fixed point principle for contractive operators in  $K$ -metric and  $K$ -normed linear spaces that allows to comprehend all usual and new nontrivial applications and, of course, to catch directions in which the further development of the theory turns out to be useful for investigations of the concrete linear and nonlinear problems...”

- W.A. Kirk ([35], pp. 1): ”A complete survey of all that has been written about contraction mappings would appear to be nearly impossible, and perhaps not really useful. In particular the wealth of applications of Banach’s contraction mapping principle is astonishingly diverse. We only attempt to touch on some of the high points of this profound and seminal development in metric fixed point theory.”

- I.A. Rus ([60], pp. 10): ”Our study centres around these generalized contractions in the terms of Picard operators, weakly Picard operators, Bessaga operators and Janos operators.”

- I.A. Rus ([62], pp. 191): ”The theory of weakly Picard operators is useful to study some equations for which the method of successive approximations works. In terms of weakly Picard operators, some classical results take a very simple form.”

The aim of this paper is to present some examples of a theory of a metrical fixed point theorem. Throughout this paper we follow the terminologies and notations in [62]. As a model for our study we take the following well known results:

**Theorem 1.1.** *Let  $(X, d)$  be a complete metric space and  $A : X \rightarrow X$  an  $\alpha$ -contraction. Then:*

(i)  $F_A = \{x_A^*\}$  and  $A^n(x) \xrightarrow{d} x_A^*$  as  $n \rightarrow \infty$ ,  $\forall x \in X$ , i.e.,  $A$  is a Picard operator (briefly PO);

(ii)  $F_A = F_{A^n} = \{x_A^*\}$ ,  $\forall n \in \mathbb{N}^*$ , i.e.,  $A$  is a Bessaga operator;

(iii)  $d(A^n(x), x_A^*) \leq \frac{\alpha^n}{1 - \alpha} d(x, A(x))$ ,  $\forall x \in X$  and  $n \in \mathbb{N}^*$ ;

(iv)  $d(x, x_A^*) \leq \frac{1}{1-\alpha} d(x, A(x)), \forall x \in X$ , i.e.,  $A$  is  $\frac{1}{1-\alpha}$ -PO;

(v)  $\sum_{n \in \mathbb{N}} d(A^n(x), A^{n+1}(x)) \leq \frac{1}{1-\alpha} d(x, A(x)), \forall x \in X$ , i.e.,  $A$  is a good PO;

(vi)  $\sum_{n \in \mathbb{N}} d(A^n(x), x_A^*) \leq \frac{1}{1-\alpha} d(x, x_A^*), \forall x \in X$ , i.e.,  $A$  is a special PO;

(vii)  $F_A = \{x_A^*\}$  and if  $x_n \in X, n \in \mathbb{N}$  are such that

$$d(x_n, A(x_n)) \rightarrow 0 \text{ as } n \rightarrow \infty,$$

then

$$x_n \rightarrow x_A^* \text{ as } n \rightarrow \infty,$$

i.e., the fixed point problem for the operator  $A$  is well posed;

(viii)  $x_n \in X, n \in \mathbb{N}$  and

$$(d(x_{n+1}, A(x_n))) \rightarrow 0 \text{ as } n \rightarrow \infty,$$

imply that there exists  $x \in X$  such that

$$d(x_n, A^n(x)) \rightarrow 0 \text{ as } n \rightarrow \infty,$$

i.e., the operator  $A$  has the limit shadowing property;

(ix) if  $(x_n)_{n \in \mathbb{N}}$  is bounded sequence in  $X$ , then

$$A^n(x_n) \rightarrow x_A^* \text{ as } n \rightarrow \infty;$$

(x) if  $B : X \rightarrow X$  is such that there exists  $\eta > 0$  with

$$d(A(x), B(x)) \leq \eta, \forall x \in X,$$

then:

$$x_B^* \in F_B \Rightarrow d(x_A^*, x_B^*) \leq \frac{\eta}{1-\alpha};$$

(xi) if  $A_n : X \rightarrow X, A_n \xrightarrow{\text{unif.}} A, x_n^* \in F_{A_n}, n \in \mathbb{N}$ , then

$$x_n^* \rightarrow x_A^* \text{ as } n \rightarrow \infty;$$

(xii) if  $X$  is a Banach space, then  $1_X - A : X \rightarrow X$  is a topological isomorphism;

(xiii) if  $(X, d)$  is a bounded metric space, then

$$\bigcap_{n \in \mathbb{N}} A^n(X) = \{x_A^*\},$$

*i.e.*,  $A$  is a Janos operator.

**Remark 1.1.** For the above conclusions see:

- (i) W.A. Kirk [35], I.A. Rus [58], [62],...
- (ii) C. Bessaga [8], P.R. Meyers [42], V.I. Opoitsev [45], B.E. Rhoades [57], I.A. Rus [58], [60], I.A. Rus, A. Petruşel and M.A. Şerban [69],...
- (iii)+(iv): V. Berinde [5], K. Deimling [19], W.A. Kirk [35], M.A. Krasnoselskii and P.P. Zabrejko [37], I.A. Rus [58], P.P. Zabrejko [74], E. Zeidler [76],...
- (v)+(vi): L. d'Apuzzo [4], I.A. Rus [62],...
- (vii): F.S. De Blasi and J. Myjak [18], E. Matouskova, S. Reich and A.J. Zaslavski [41], S. Reich and A.J. Zaslavski [54], I.A. Rus [63],...
- (viii): A.M. Ostrowski [46], J. Jachymski [32], T. Eirola, O. Nevalina and S. Yu. Pilyugin [22], S. Yu. Pilyugin [49], V. Glăvan and V. Guţu [25], [26],...
- (ix): D. Reem, S. Reich and A.J. Zaslavski [53], S. Reich and A.J. Zaslavski [55],...
- (x): V. Berinde [5], S. Bernfeld and V. Lakshmikantham [7], M.A. Krasnoselskii and P. Zabrejko [37], I.A. Rus [58], [60], [62],...
- (xi): F.E. Bonsal [9], V. Berinde [5], I.A. Rus [58], [59], [64],...
- (xii): J. Dugundji and A. Granas [21], A. Granas and J. Dugundji [28], I.A. Rus [58], [59],...
- (xiii): L. Janos [33], W.A. Kirk [35], I.A. Rus [60], [62], I.A. Rus, A. Petruşel and M.A. Şerban [69],...

**Theorem 1.2.** (fibre contraction theorem; M.W. Hirsch and C.C. Pugh [30], I.A. Rus [62]). *Let  $(X, \rightarrow)$  be an  $L$ -space,  $(Y, d)$  be a complete metric space,  $B : X \rightarrow X$  and  $C : X \times Y \rightarrow Y$  be two operators. We suppose that:*

- (i)  $B$  is a PO;
- (ii) there exists  $\alpha \in (0, 1)$  such that  $C(x, \cdot) : Y \rightarrow Y$  is  $\alpha$ -contraction, for each  $x \in X$ ;
- (iii) the operator  $C$  is continuous.

*Then, the operator  $A := (B, C) : X \times Y \rightarrow X \times Y$ ,  $(x, y) \mapsto (B(x), C(x, y))$ , is a PO.*

**Remark 1.2.** In the case of weakly Picard operators on an ordered  $L$ -space we have (see [62], [65]):

**Abstract Gronwall Lemma.** *Let  $(X, \rightarrow, \leq)$  be an ordered  $L$ -space and  $A : X \rightarrow X$  be an operator. We suppose that:*

- (i)  $A$  is Picard operator ( $F_A = \{x_A^*\}$ );
- (ii)  $A$  is increasing.

Then:

- (a)  $x \leq A(x) \Rightarrow x \leq x_A^*$ ;
- (b)  $x \geq A(x) \Rightarrow x \geq x_A^*$ .

**Abstract Comparison Theorem.** Let  $(X, \rightarrow, \leq)$  be an ordered  $L$ -space and  $A, B, C : X \rightarrow X$  be three operators. We suppose that:

- (i)  $A \leq B \leq C$ ;
- (ii)  $A, B$  and  $C$  are weakly Picard operators;
- (iii) the operator  $B$  is increasing.

Then:

$$x \leq y \leq z \Rightarrow A^\infty(x) \leq B^\infty(y) \leq C^\infty(z).$$

From this abstract results and from Theorem 1.1 (i) we have:

**Gronwall Lemma for Contractions.** Let  $(X, d, \leq)$  be an ordered complete metric space and  $A : X \rightarrow X$  be an operator. We suppose that:

- (i)  $A$  is a contraction ( $F_A = \{x_A^*\}$ );
- (ii)  $A$  is increasing.

Then:

- (a)  $x \leq A(x) \Rightarrow x \leq x_A^*$ ;
- (b)  $x \geq A(x) \Rightarrow x \geq x_A^*$ .

**Comparison Theorem for Contractions.** Let  $(X, d, \leq)$  be an ordered complete metric space and  $A, B, C : X \rightarrow X$  be three operators. We suppose that:

- (i)  $A \leq B \leq C$ ;
- (ii)  $A, B$  and  $C$  are contractions ( $F_A = \{x_A^*\}, F_B = \{x_B^*\}, F_C = \{x_C^*\}$ );
- (iii) the operator  $B$  is increasing.

Then:

$$x_A^* \leq x_B^* \leq x_C^*.$$

## 2. INFINITE MATRICES

Let  $X$  be a nonempty set,

$$s(X) := \{(x_n)_{n \in \mathbb{N}^*} \mid x_n \in X, n \in \mathbb{N}^*\}$$

and

$$M(X) := \{(x_{ij})_1^\infty \mid x_{ij} \in X, i, j \in \mathbb{N}^*\}$$

where

$$(x_{ij})_1^\infty := \begin{pmatrix} x_{11} & x_{12} & x_{13} & \dots \\ x_{21} & x_{22} & x_{23} & \dots \\ x_{31} & x_{32} & x_{33} & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}$$

is an infinite matrix.

If  $X = \mathbb{R}$ , then  $(s(\mathbb{R}), +, \mathbb{R}, \leq, \rightarrow)$  and  $(M(\mathbb{R}), +, \mathbb{R}, \leq, \rightarrow)$  are linear ordered L-spaces, where  $\rightarrow$  is termwise convergence.

For  $A \in M(\mathbb{R})$  we denote  $|A| := \sup_{1 \leq i \leq \infty} \sum_{j \in \mathbb{N}^*} |a_{ij}|$ .

The functional

$$|\cdot| : M(\mathbb{R}) \rightarrow \mathbb{R}_+ \cup \{+\infty\}, \quad A \mapsto |A|$$

is a generalized norm on  $M(\mathbb{R})$ .

**Definition 2.1.** A matrix  $S \in M(\mathbb{R})$  is called row-column-finite if there are only a finite number of nonzero elements in each row and each column.

**Definition 2.2.** A matrix  $S \in M(\mathbb{R})$  is called Neumann matrix if  $S^n$  is defined for all  $n \in \mathbb{N}$  and  $\sum_{n \in \mathbb{N}} S^n$  termwise converges.

**Remark 2.1.** In general, product of matrices is not associative. Row-finite and column-finite matrices form associative fields (see [16]).

In this paper we need the following results:

**Lemma 2.1.** Let  $S \in M(\mathbb{R})$  be a matrix. We suppose that:

- (i)  $S$  is row-column-finite matrix;
- (ii)  $|S| < 1$ .

Then:

- (a)  $S$  is Neumann matrix;
- (b)  $(E - S)^{-1} = \sum_{n \in \mathbb{N}} S^n$ .

**Proof.** The proof follows from a theorem by P. Dienes (see [16]; see also [40]).

**Lemma 2.2.** Let  $S, T_n \in M(\mathbb{R}_+)$ ,  $n \in \mathbb{N}$ . We suppose that:

- (i)  $S$  is as in Lemma 2.1;
- (ii)  $|T_n| \rightarrow 0$  as  $n \rightarrow \infty$ .

Then:

$$\sum_{k=0}^n S^{n-k} T_k \xrightarrow{|\cdot|} 0 \text{ as } n \rightarrow \infty.$$

**Proof.** We have

$$\left| \sum_{k=0}^n S^{n-k} T_k \right| \leq \sum_{k=0}^n |S^{n-k}| \cdot |T_k| \leq \sum_{k=0}^n |S|^{n-k} |T_k| \rightarrow 0 \text{ as } n \rightarrow \infty,$$

by a Cauchy Lemma (see [76]).

### 3. GENERALIZED METRIC SPACES

Let  $X$  be a nonempty set and  $d : X \times X \rightarrow s(\mathbb{R})$ . The functional  $d$  is a generalized metric iff:

- (i)  $d(x, y) \geq 0$ ,  $\forall x, y \in X$ ;  $d(x, y) = 0 \Leftrightarrow x = y$ ;
- (ii)  $d(x, y) = d(y, x)$ ,  $\forall x, y \in X$ ;
- (iii)  $d(x, y) \leq d(x, z) + d(z, y)$ ,  $\forall x, y, z \in X$ .

**Remark 3.1.** A functional  $d : X \times X \rightarrow s(\mathbb{R}_+)$ ,  $(x, y) \mapsto (d_k(x, y))_{k \in \mathbb{N}^*}$  is a generalized metric on  $X$  iff:

- (a)  $d_k$  is a pseudometric,  $\forall k \in \mathbb{N}^*$ ;
- (b)  $\forall x, y \in X$ ,  $x \neq y$ ,  $\exists k \in \mathbb{N}^*$ :  $d_k(x, y) \neq 0$ .

**Example 3.1.**  $X := \mathbb{R}^p$ ,  $d(x, y) := (|x_1 - y_1|, \dots, |x_p - y_p|, 0, \dots, 0)$ .

**Example 3.2.**  $X := C(\mathbb{R}, \mathbb{R}^p)$ ,  $d_k(x, y) := \max_{-k \leq t \leq k} \|x(t) - y(t)\|_{\mathbb{R}^p}$ ,  $d := (d_k)_{k \in \mathbb{N}^*}$ .

**Example 3.3.**  $X := s(\mathbb{R})$ ,  $d(x, y) := (|x_1 - y_1|, \dots, |x_n - y_n|, \dots)$ .

**Example 3.4.**  $X := C([a, b], s(\mathbb{R}))$ ,

$$d(x, y) := \left( \max_{t \in [a, b]} (|x_1(t) - x_2(t)| e^{-\tau(t-a)}), \dots, \max_{t \in [a, b]} (|x_n(t) - y_n(t)| e^{-\tau(t-a)}), \dots \right).$$

**Definition 3.1.** A generalized metric space  $(X, d)$  is complete (in the Weierstrass sense) iff

$$x_n \in X, \sum_{n \in \mathbb{N}^*} d(x_n, x_{n+1}) \text{ converges} \Rightarrow (x_n)_{n \in \mathbb{N}^*} \text{ converges}.$$

**Definition 3.2.** Let  $(X, d)$  be a generalized metric space,  $A : X \rightarrow X$  and  $S \in M(\mathbb{R}_+)$ . The operator  $A$  is  $S$ -contraction iff:

- (i)  $S$  is row and column finite;
- (ii)  $S$  is a Neumann matrix;

- (iii)  $\sum_{n \in \mathbb{N}} S^n d(x, y)$  converges,  $\forall x, y \in X$ ;
- (iv)  $d(A(x), A(y)) \leq Sd(x, y)$ ,  $\forall x, y \in X$ .

**Remark 3.2.** For more details on  $S$ -contractions see [71], [70], [23], [27], [29]. For the inverse of an infinite matrix see [16], [27] and [40].

**Remark 3.3.** In the case of E. Tarafdar’s contractions ([68]),  $S = (s_{ij})_1^\infty$ ,  $s_{ij} = 0$  if  $i \neq j$  and  $s_{ii} = \lambda_i < 1$ , while in the case of I. Colojoarǎ’s contractions ([15]),  $s_{ij} = 0$ ,  $j \neq \varphi(i)$ ,  $A_{i\varphi(i)} = \lambda_i < 1$ , where  $\varphi : \mathbb{N}^* \rightarrow \mathbb{N}^*$ . For other results in connection with  $S$ -contraction see [2], [3], [20], [23], [24], [34], [35], [36], [56], [64], [69], [70], [71],...

#### 4. CONTRACTION PRINCIPLE IN A GENERALIZED METRIC SPACES

We have:

**Theorem 4.1.** *Let  $(X, d)$  be a complete generalized metric space and  $A : X \rightarrow X$  be an  $S$ -contraction. Then:*

- (i)  $F_A = \{x_A^*\}$  and  $A^n(x) \xrightarrow{d} x_A^*$  as  $n \rightarrow \infty$ , i.e.,  $A$  is a PO;
- (ii)  $F_A = F_{A^n} = \{x_A^*\}$ ,  $\forall n \in \mathbb{N}^*$ , i.e.,  $A$  is a Bessaga operator;
- (iii)  $d(A^n(x), x_A^*) \leq (E - S)^{-1} S^n d(x, A(x))$ ,  $\forall x \in X$  and  $n \in \mathbb{N}^*$ ;
- (iv)  $d(x, x_A^*) \leq (E - S)^{-1} d(x, A(x))$ ,  $\forall x \in X$ , i.e.,  $A$  is  $(E - S)^{-1} - PO$ ;
- (v)  $\sum_{n \in \mathbb{N}} d(A^n(x), A^{n+1}(x)) \leq (E - S)^{-1} d(x, A(x))$ ,  $\forall x \in X$ , i.e.,  $A$  is a good PO;
- (vi)  $\sum_{n \in \mathbb{N}} d(A^n(x), x_A^*) \leq (E - S)^{-1} d(x, x_A^*)$ ,  $\forall x \in X$ , i.e.,  $A$  is a special PO;
- (vii)  $F_A = \{x_A^*\}$  and if  $x_n \in X$ ,  $n \in \mathbb{N}$  are such that

$$d(x_n, A(x_n)) \rightarrow 0 \text{ as } n \rightarrow \infty,$$

then

$$x_n \rightarrow x_A^* \text{ as } n \rightarrow \infty,$$

i.e., the fixed point problem for the operator  $A$  is well posed;

- (viii) We suppose that  $|S| < 1$ . Then,  $x_n \in X$ ,  $n \in \mathbb{N}$  and

$$d(x_{n+1}, A(x_n)) \rightarrow 0 \text{ as } n \rightarrow \infty,$$

imply that there exists  $x \in X$  such that

$$d(x_n, A^n(x)) \rightarrow 0 \text{ as } n \rightarrow \infty,$$

i.e., the operator  $A$  has the limit shadowing property;

(ix) if  $(x_n)_{n \in \mathbb{N}}$  is a bounded sequence in  $(X, d)$ , then

$$A^n(x_n) \rightarrow x_A^* \text{ as } n \rightarrow \infty;$$

(x) if  $B : X \rightarrow X$  is such that there exists  $\eta \in s(\mathbb{R}_+)$  with

$$d(A(x), B(x)) \leq \eta, \forall x \in X,$$

then

$$x_B^* \in F_B \Rightarrow d(x_A^*, x_B^*) \leq (E - S)^{-1}\eta;$$

(xi) if  $A_n : X \rightarrow X$ ,  $A_n \xrightarrow{\text{unif.}} A$ ,  $x_n^* \in F_{A^n}$ ,  $n \in \mathbb{N}$ , then

$$x_n^* \rightarrow x_A^* \text{ as } n \rightarrow \infty;$$

(xii) if  $X$  is linear space and  $d(x, y) = \|x - y\|$ , where  $\|\cdot\| : X \rightarrow s(\mathbb{R}_+)$  is a generalized norm on  $X$ , then,  $1_X - f : X \rightarrow X$  is a topological isomorphism;

(xiii) if  $(X, d)$  is bounded, then

$$\bigcap_{n \in \mathbb{N}} A^n(X) = \{x_A^*\},$$

i.e.,  $A$  is a Janos operator.

**Proof.** (i). This  $S$ -contraction principle is in fact:

- a generalization of Perov's fixed point theorem ([5], [47], [59]),
- a particular case of some fixed point theorems in gauge spaces ([75], [76], [23], [20], [2], [3], ...).

First of all we remark that  $\text{card}F_A \leq 1$ . Indeed, if  $x^*, y^* \in F_A$ , then

$$d(x^*, y^*) = d(A^n(x^*), A^n(y^*)) \leq S^n d(x^*, y^*) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

On the other hand

$$\sum_{n \in \mathbb{N}} d(A^n(x), A^{n+1}(x)) \leq \sum_{n \in \mathbb{N}} S^n d(x, A(x)) < +\infty.$$

This implies that  $(A^n(x))_{n \in \mathbb{N}}$  converges. Let  $x_A^*$  be its limit. From the continuity of  $A$  it follows that  $x_A^* \in F_A$ . So,  $F_A = \{x_A^*\}$  and  $A^n(x) \rightarrow x_A^*$ , for all  $x \in X$ . By definition (see [59])  $A$  is a PO.

(ii). Let  $y \in F_{A^n}$ ,  $n > 1$ . By (i)  $A^k(y) \rightarrow x_A^*$  as  $k \rightarrow \infty$ . Since  $A^{kn}(y) = y$ ,  $k \in \mathbb{N}^*$ , we get  $y = x_A^*$ .

(iii). We have

$$d(A^n(x), A^{n+p}(x)) \leq d(A^n(x), A^{n+1}(x)) + \dots + d(A^{n+p-1}(x), A^{n+p}(x))$$

$$\begin{aligned} &\leq S^n d(x, A(x)) + \cdots + S^{n+p-1} d(x, A(x)) \\ &\leq (S^n + \cdots + S^{n+p-1} + \dots) d(x, A(x)) = (E - S)^{-1} S^n d(x, A(x)). \end{aligned}$$

Thus,

$$d(A^n(x), x_A^*) \leq (E - S)^{-1} S^n d(x, A(x)).$$

(iv). From

$$\begin{aligned} d(x, x_A^*) &\leq d(x, A(x)) + d(A(x), x_A^*) \\ &\leq d(x, A(x)) + S d(x, x_A^*), \end{aligned}$$

it follows that

$$d(x, x_A^*) \leq (E - S)^{-1} d(x, A(x)), \quad \forall x \in X.$$

By definition ([59])  $A$  is  $(E - S)^{-1} - PO$ .

$$\begin{aligned} \text{(v). } \sum_{n \in \mathbb{N}} d(A^n(x), A^{n+1}(x)) &\leq \sum_{n \in \mathbb{N}} S^n d(x, A(x)) \\ &= \left( \sum_{n \in \mathbb{N}} S^n \right) d(x, A(x)) = (E - S)^{-1} d(x, A(x)). \end{aligned}$$

Since,  $\sum_{n \in \mathbb{N}} d(A^n(x), A^{n+1}(x)) < +\infty$ , by definition (see [4] or [59]),  $A$  is good

a PO.

$$\begin{aligned} \text{(vi) } \sum_{n \in \mathbb{N}} d(A^n(x), x_A^*) &= \sum_{n \in \mathbb{N}} d(A^n(x), A^n(x_A^*)) \\ &\leq \sum_{n \in \mathbb{N}} S^n d(x, x_A^*) = (E - S)^{-1} d(x, x_A^*). \end{aligned}$$

Since,  $\sum_{n \in \mathbb{N}} d(A^n(x), x_A^*) < +\infty$ , by definition (see [4] or [59])  $A$  is a special PO.

(vii). Since

$$\begin{aligned} d(x_n, x_A^*) &\leq d(x_n, A(x_n)) + d(A(x_n), x_A^*) \\ &\leq d(x_n, A(x_n)) + S d(x_n, x_A^*), \end{aligned}$$

hence

$$d(x_n, x_A^*) \leq (E - S)^{-1} d(x_n, A(x_n)) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

(viii). We have

$$\begin{aligned} d(x_n, x_A^*) &\leq d(x_n, A(x_{n-1})) + d(A(x_{n-1}), x_A^*) \\ &\leq d(x_n, A(x_{n-1})) + Sd(x_{n-1}, A(x_{n-2})) + \cdots + \\ &\quad + S^{n-1}d(x_1, A(x_0)) + S^n d(x_0, x^*). \end{aligned}$$

From Lemma 2.2 we have that

$$d(x_n, x_A^*) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Hence,

$$d(x_n, A^n(x)) \leq d(x_n, x_A^*) + d(x_A^*, A^n(x)) \rightarrow 0 \text{ as } n \rightarrow \infty, \text{ for all } x \in X.$$

(ix). Let  $\eta \in s(\mathbb{R}_+)$  be such that

$$d(x_n, x_m) \leq \eta, \forall n, m \in \mathbb{N}^*.$$

We have

$$\begin{aligned} d(x_n, A(x_n)) &\leq d(x_n, x_A^*) + Sd(x_n, x_A^*) \\ &\leq (E + S)[\eta + d(x_1, x_A^*)]. \end{aligned}$$

From (iii) it follows that

$$d(A^n(x_n), x_A^*) \leq (E - S)^{-1} S^n (E + S)[\eta + d(x_1, x_A^*)].$$

So,

$$d(A^n(x_n), x_A^*) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

(x). From (iv) it follows that

$$\begin{aligned} d(x_B^*, x_A^*) &\leq (E - S)^{-1} d(x_B^*, A(x_B^*)) \\ &= (E - S)^{-1} d(B(x_B^*), A(x_B^*)) \\ &\leq (E - S)^{-1} \eta. \end{aligned}$$

(xi).  $A_n \xrightarrow{\text{unif.}} A$  as  $n \rightarrow \infty$  implies that there exists  $\eta_n \in s(\mathbb{R}_+)$ ,  $\eta_n \rightarrow 0$ , such that

$$d(A_n(x), A(x)) \leq \eta_n, \forall x \in X.$$

From (x) we have that

$$d(x_n^*, x_A^*) \leq (E - S)^{-1} \eta_n \rightarrow 0 \text{ as } n \rightarrow \infty.$$

(xii). Let  $y \in X$  and  $A_y : X \rightarrow X$  be defined by  $A_y(x) := A(x) + y$ . It is clear that  $A_y$  is a  $S$ -contraction for all  $y \in X$ . By (i),  $A_y$  has a unique fixed point, i.e.,  $1_X - A$  is a bijection. From (x),  $(1_X - A)^{-1}$  is continuous. Thus,  $1_X - A$  is a topological isomorphism.

(xiii). Let  $\eta \in s(\mathbb{R}_+)$  be such that,  $d(x, y) \leq \eta, \forall x, y \in X$ . We remark that,

$$x_A^* \in \bigcap_{n \in \mathbb{N}} A^n(X).$$

On the other hand

$$d(A^n(x), A^n(y)) \leq S^n d(x, y) \leq S^n \eta \rightarrow 0 \text{ as } n \rightarrow \infty.$$

So,

$$\bigcap_{n \in \mathbb{N}} A^n(X) = \{x_A^*\}.$$

**Remark 4.1.** From Remark 1.2. and Theorem 4.1. (i) we have:

**Gronwall Lemma for  $S$ -Contractions.** *Let  $(X, d, \leq)$  be an ordered complete generalized metric space and  $A : X \rightarrow X$  be an operator. We suppose that:*

- (i)  $A$  is an  $S$ -contraction ( $F_A = \{x_A^*\}$ );
- (ii)  $A$  is increasing.

Then:

- (a)  $x \leq A(x) \Rightarrow x \leq x_A^*$ ;
- (b)  $x \geq A(x) \Rightarrow x \geq x_A^*$ .

**Comparison Theorem for  $S$ -Contractions.** *Let  $(X, d, \leq)$  be an ordered complete generalized metric space and  $A, B, C : X \rightarrow X$  be three operators.*

We suppose that:

- (i)  $A \leq B \leq C$ ;
- (ii)  $A, B$  and  $C$  are  $S$ -contractions ( $F_A = \{x_A^*\}, F_B = \{x_B^*\}, F_C = \{x_C^*\}$ );
- (iii) the operator  $B$  is increasing.

Then:

$$x_A^* \leq x_B^* \leq x_C^*.$$

## 5. GRAPHIC CONTRACTION PRINCIPLE

**Definition 5.1.** Let  $(X, d)$  be a complete generalized metric space,  $A : X \rightarrow X$  and  $S \in M(\mathbb{R}_+)$ . The operator  $A$  is a graphic  $S$ -contraction iff:

- (i)  $S$  is row and column finite;
- (ii)  $S$  is Neumann matrix;
- (iii)  $\sum_{n \in \mathbb{N}} S^n d(x, y)$  converges,  $\forall x, y \in X$ ;
- (iv)  $d(A^2(x), A(x)) \leq Sd(x, A(x))$ ,  $\forall x \in X$ .

**Remark 5.1.** For other details concerning the graphic contraction principle in a metric space see [59] (I.A. Rus (1972), S. Kasahara (1975), H.L. Hicks-B.E. Rhoades (1979),...). See also [68].

We have:

**Theorem 5.1.** Let  $(X, d)$  be a complete generalized metric space and  $A : X \rightarrow X$  be an orbitally continuous graphic  $S$ -contraction. Then:

(i)  $F_A \neq \emptyset$  and  $A^n(x) \rightarrow x^*(x) =: A^\infty(x) \in F_A$ ,  $\forall x \in X$ , i.e.,  $A$  is a weakly Picard operator;

(ii)  $F_A = F_{A^n} \neq \emptyset$ ,  $\forall n \in \mathbb{N}^*$ ;

(iii)  $d(A^n(x), A^\infty(x)) \leq (E - S)^{-1} S^n d(x, A(x))$ ,  $\forall x \in X$  and  $n \in \mathbb{N}^*$ ;

(iv)  $d(x, A^\infty(x)) \leq (E - S)^{-1} d(x, A(x))$ ,  $\forall x \in X$ , i.e.,  $A$  is a  $(E - S)^{-1} -$ WPO.

(v)  $\sum_{n \in \mathbb{N}} d(A^n(x), A^{n+1}(x)) \leq (E - S)^{-1} d(x, A(x))$ ,  $\forall x \in X$ , i.e.,  $A$  is a good WPO;

(vi)  $\sum_{n \in \mathbb{N}} d(A^n(x), A^\infty(x)) \leq (E - S)^{-1} d(x, A^\infty(x))$ ,  $\forall x \in X$ , i.e.,  $A$  is a special WPO;

(vii) the fixed point problem for  $A|_{(BA)_A(x^*)} : (BA)_A(x^*) \rightarrow (BA)_A(x^*)$  is well-posed,  $\forall x^* \in F_A$ , where

$$(BA)_A(x^*) := \{x \in X \mid A^n(x) \rightarrow x^* \text{ as } n \rightarrow \infty\}$$

is the basin of attraction of  $A$  w.r.t.  $x^*$ ;

(viii)  $\forall x^* \in F_A$ , the operator  $A|_{(BA)_A(x^*)} : (BA)_A(x^*) \rightarrow (BA)_A(x^*)$  has the limit shadowing property;

(ix) Let  $B : X \rightarrow X$  be such that:

(a)  $B$  is  $(E - S)^{-1} -$ WPO;

(b) there exists  $\eta \in s(\mathbb{R}_+) : d(A(x), B(x)) \leq \eta$ ,  $\forall x \in X$ .

Then

$$H_d(F_A, F_B) \leq (E - S)^{-1}\eta.$$

Here,  $H_d := (H_{d_1}, \dots, H_{d_n}, \dots)$  stands for Pompeiu-Hausdorff functional.

(x) Let  $A_n : X \rightarrow X$ ,  $n \in \mathbb{N}^*$ , be such that:

(a)  $A_n$ ,  $n \in \mathbb{N}$ , are  $(E - S)^{-1}$ -WPOs;

(b)  $A_n \xrightarrow{\text{unif.}} A$  as  $n \rightarrow \infty$ .

Then,  $H_d(F_{A_n}, F_A) \rightarrow 0$  as  $n \rightarrow \infty$ .

**Proof.** The proof is similar to that of Theorem 4.1.  $\square$

**Remark 5.1.** From Remark 1.2. and Theorem 5.1. (i) we have:

**Gronwall Lemma for Graphic  $S$ -Contractions.** Let  $(X, d, \leq)$  be an ordered complete generalized metric space and  $A : X \rightarrow X$  be an operator. We suppose that:

(i)  $A$  is an graphic  $S$ -contraction ( $F_A = \{x_A^*\}$ );

(ii)  $A$  is orbitally continuous;

(iii)  $A$  is increasing.

Then:

(a)  $x \leq A(x) \Rightarrow x \leq x_A^\infty$ ;

(b)  $x \geq A(x) \Rightarrow x \geq x_A^\infty$ .

**Comparison Theorem for Graphic  $S$ -Contractions.** Let  $(X, d, \leq)$  be an ordered complete generalized metric space and  $A, B, C : X \rightarrow X$  be three operators. We suppose that:

(i)  $A \leq B \leq C$ ;

(ii)  $A, B$  and  $C$  are graphic  $S$ -contractions ( $F_A = \{x_A^*\}$ ,  $F_B = \{x_B^*\}$ ,  $F_C = \{x_C^*\}$ );

(iii)  $A, B$  and  $C$  are orbitally continuous;

(iv) the operator  $B$  is increasing.

Then:

$$x \leq y \leq z \Rightarrow A^\infty(x) \leq B^\infty(y) \leq C^\infty(z).$$

## 6. OPEN PROBLEMS

The above considerations give rise to the following problems:

**Problem 6.1.** For which metrical fixed point theorems we have similar results as above?

**Problem 6.2.** There are many types of generalized metric spaces. For which generalized metric spaces we have similar results?

**Problem 6.3.** Let  $(X, d)$  be a generalized metric space,  $Y \subset X$  be a subset of  $X$  and  $f : Y \rightarrow X$  be an operator. The problem is to extend the results of this paper to the case of nonself operators.

## 7. APPLICATIONS

For some applications of the metrical fixed point theory see: V.G. Angelov [2], V. Berinde [5], S. Bernfeld and V. Lakshmikantham [7], A. Buică [11], T.A. Burton [12] and [13], C. Corduneanu [17], M. Frigon [23], M.A. Krasnoselskii and P. Zabrejko [37], V. Mureşan [43], D. O'Regan and R. Precup [47], I.A. Rus [56]-[59], I.A. Rus and E. Egri [63], P.P. Zabrejko and T.A. Makarevich [70], E. Zeidler [71], D. Otrocol and I.A. Rus [74], V. Lakshmikantham and S. Leela [38], V. Niemytzki [44], V. Radu [51], I.A. Rus, A. Petruşel and G. Petruşel [64].

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