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A NEW VERSION OF KRASNOSELSKII'S FIXED POINT THEOREM IN MODULAR SPACES

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Abstract. In this article, a new version of Krasnoselskii's fixed point theorem in modular spaces is presented.

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1. Introduction

The theory of modular space was initiated by Nakano [8] in 1950 in connection with the theory of order spaces and redefined and generalized by Musielak and Orlicz [7] in 1959. By defining a norm, particular Banach spaces of functions can be considered. Metric fixed theory for these Banach spaces of functions has been widely studied (see [2]). Another direction is based on considering and abstractly given functional which controls the growth of the functions. Even though a metric is not defined, many problems in fixed point theory for nonexpansive mappings can be reformulated in modular spaces.

Modular version of Krasnoselskii's fixed point theorem in modular space is proved in [3]. Using this, we show the existence of solution of a perturbed integral equation. In this article, a new version of Krasnoselskii's fixed point theorem in modular space is presented. In order to do this and for the sake of convenience, some definitions and notations are recalled from [2], [3], [4], [6], [7], [8] and [9].

Definition 1.1. Let X be an arbitrary vector space over $K (= \mathbb{R}$ or $\mathbb{C})$. A functional $\rho : X \to [0, +\infty)$ is called modular if:

1. $\rho(x) = 0$ if and only if $x = 0$.
2. $\rho(\alpha x) = \rho(x)$ for $\alpha \in K$ with $|\alpha| = 1$, for all $x, y \in X$.
3. $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ if $\alpha, \beta \geq 0, \alpha + \beta = 1$, for all $x, y \in X$.

Definition 1.2. If 3) in Definition 1.1 is replaced by:

$$\rho(\alpha x + \beta y) \leq \alpha^s \rho(x) + \beta^s \rho(y)$$

for $\alpha, \beta \geq 0, \alpha^s + \beta^s = 1$ with an $s \in (0, 1]$, then the modular ρ is called an s-convex modular, and if $s = 1$, ρ is called a convex modular.

Definition 1.3. A modular ρ defines a corresponding modular space, i.e. the space X_ρ is given by

$$X_\rho = \{x \in X |\ \rho(\lambda x) \to 0\ as\ \lambda \to 0\}.$$

Definition 1.4. Let X_ρ be a modular space.

1. A sequence $\{x_n\}_n$ in X_ρ is said to be:
   - (a) ρ-convergent to x if $\rho(x_n - x) \to 0$ as $n \to +\infty$.
   - (b) ρ-Cauchy if $\rho(x_n - x_m) \to 0$ as $n, m \to +\infty$.
2. X_ρ is ρ-complete if any ρ-Cauchy sequence is ρ-convergent.
3. A subset $B \subset X_\rho$ is said to be ρ-closed if for any sequence $\{x_n\}_n \subset B$ with $x_n \to x$, then $x \in B$. $\bar{B}^\rho$ denotes the closure of B in the sense of ρ.
4. A subset $B \subset X_\rho$ is called ρ-bounded if

$$\delta_\rho(B) = \sup_{x,y \in B} \rho(x - y) < +\infty,$$

   where $\delta_\rho(B)$ is called the ρ-diameter of B.
5. We say that ρ has Fatou property if:

$$\rho(x - y) \leq \liminf \rho(x_n - y_n)$$

whenever

$$x_n \xrightarrow{\rho} x \text{ and } y_n \xrightarrow{\rho} y.$$

(6) ρ is said to satisfy the Δ_2-condition if: $\rho(2x_n) \to 0$ as $n \to +\infty$ whenever $\rho(x_n) \to 0$ as $n \to +\infty$.

2. Main result

In this section a new generalization of Theorem 2.1 in [3] is presented. Due to do this, we recall some preliminaries from [3].

Definition 2.1. *If X is a normed space and $E \subseteq X$, a function $f : E \to X$ is said to be compact if f is continuous and $clf(A)$ is compact whenever A is a bounded subset of E.*

Lemma 2.2. *Let ρ be a convex modular and X_ρ be a modular space. If a subset B of X_ρ is ρ-bounded then B is $\|.\|_\rho$-bounded.*

Theorem 2.3. *Schauder's fixed point theorem Let E be a closed, bounded and convex subset of a normed space X. If $f : E \to X$ is a compact map such that $f(E) \subseteq E$ then, there exists $x \in E$ such that $f(x) = x$*

Following theorem is a new generalization of Theorem 2.1 in [3]

Theorem 2.4. *Let ρ be a convex modular that satisfies the Δ_2-condition, X_ρ be a ρ-complete modular space and B be a convex, ρ-closed, ρ-bounded subset of X_ρ. Assume that U and T are two applications from B into B such that U is ρ-completely continuous and there exists real number $c > 1$ such that*

$$\rho(c(Tx - Ty)) \leq \psi(\rho(x - y)),$$

where $\psi : \mathbb{R} \to (0, \infty)$ is upper semi continuous and satisfies $\psi(t) < t$. And $T(B) + U(B) \subset B$. Then the operator $S = T + U$ has a fixed point.

Proof: At first, we show that $I - T$ is a bijection from B into $U(B)$, where I is the identity map. Let $x \in B$, and consider the sequence

$$y_{n+1} = Ty_n + Ux,$$

with $y_0 \in B$ a fixed element. Then, the sequence $\{y_n\}$ is Cauchy. Since $\rho(y_{m+n} - y_n) = \rho(Ty_{m+n-1} - Ty_{n-1})$ and $\{Ty_n\}$ is Cauchy. Indeed, at first we show that the sequence $\{\rho(c(Ty_n - Ty_{n-1}))\}$ converges to 0. For $n \in \mathbb{N}$ we have:

$$\begin{aligned}\rho(c(Ty_n - Ty_{n-1})) &\leq \psi(\rho(y_n - y_{n-1}))\\ &= \psi(\rho(Ty_{n-1} - Ty_{n-2}))\\ &< \rho((Ty_{n-1} - Ty_{n-2}))\\ &< \rho(c(Ty_{n-1} - Ty_{n-2})).\end{aligned}$$

Consequently, $\{\rho(c(Ty_n - Ty_{n-1}))\}$ is decreasing and bounded from below $(\rho(x) \geq 0)$. Therefore $\{\rho(c(Ty_n - Ty_{n-1}))\}$ converges to a.
Now, if $a \neq 0$

$$\begin{aligned}a = \lim_{n\to\infty} \rho(c(Ty_n - Ty_{n-1})) &\leq \lim_{n\to\infty} \psi(\rho(y_n - y_{n-1}))\\ &= \lim_{n\to\infty} \psi(\rho(Ty_{n-1} - Ty_{n-2}))\\ &\leq \lim_{n\to\infty} \psi(\rho(c(Ty_{n-1} - Ty_{n-2})))\end{aligned}$$

then

$$a \leq \psi(a)$$

which is a contradiction. So $a = 0$.
Now, we show that $\{Ty_n\}$ is a Cauchy sequence. Suppose that $\{Ty_n\}$ is not a Cauchy sequence. Then, there are an $\epsilon > 0$ and sequences of integers $\{m_k\}, \{n_k\}$, whit $m_k > n_k \geq k$, and such that

$$d_k = \rho(T^m x - T^n x) \geq \epsilon \quad \text{for } k = 1, 2, \cdots . \tag{1}$$

We can assume that

$$\rho(Ty_{m-1} - Ty_n) < \epsilon, \tag{2}$$

by choosing m_k to be the smallest number exceeding n_k for which (1) holds. Indeed, let

$$\Sigma_k = \{m_k \in \mathbb{N} |\ \exists n_k \in \mathbb{N}, \rho(Ty_{m_k} - Ty_{n_k}) \geq \epsilon\}.$$

Obviously, $\Sigma_k \neq \emptyset$ and since $\Sigma_k \subset \mathbb{N}$ then by Well Ordering Principle, Σ_k has the minimum element $m \in \mathbb{N}$, and clearly, (2) holds.
Now, let $\alpha \in \mathbb{R}^+$ be such that

$$\frac{1}{c} + \frac{1}{\alpha} = 1 \tag{3}$$

then, we have

$$\begin{aligned} d_k &= \rho(Ty_m - Ty_n) \\ &= \rho(\tfrac{c}{c}(Ty_m - Ty_{n+1} + Ty_{n+1} - Ty_n)) \\ &\leq \rho(c(Ty_m - Ty_{n+1})) + \rho(\alpha(Ty_{n+1} - Ty_n)) \\ &\leq \psi(\rho(y_m - y_{n+1})) + \rho(\alpha(Ty_{n+1} - Ty_n)) \\ &= \psi(\rho(Ty_{m-1} - Ty_n)) + \rho(\alpha(Ty_{n+1} - Ty_n)) \\ &\leq \rho(Ty_{m-1} - Ty_n) + \rho(\alpha(Ty_{n+1} - Ty_n)) \\ &\leq \epsilon + \rho(\alpha(Ty_{n+1} - Ty_n)). \end{aligned}$$

If k tends to infinity, then so does n, and by Δ_2-condition, $\rho(\alpha(Ty_{n+1} - Ty_n)) \to 0$.
Hence, $d_k \to \epsilon^+$, as $k \to \infty$.
But now,

$$\begin{aligned} d_k &= \rho(Ty_m - Ty_n) \\ &\leq \rho(c(Ty_{m+1} - Ty_{n+1})) + \rho(2\alpha(Ty_m - Ty_{m+1})) + \rho(2\alpha(Ty_{n+1} - Ty_n)) \\ &\leq \psi(\rho(Ty_m - Ty_n)) + \rho(2\alpha(Ty_m - Ty_{m+1})) + \rho(2\alpha(Ty_{n+1} - Ty_n)) \end{aligned}$$

Thus, as $k \to \infty$, we obtain $\epsilon \leq \psi(\epsilon)$, which is a contradiction for $\epsilon > 0$. Therefore, $\{Ty_n\}$ is a Cauchy sequence, and with respect to this point that X_ρ is complete, there is a $y \in B$ such that $\rho(y_n - y) \longrightarrow 0$ as $n \to +\infty$. And $y = Ty + Ux$, since

$$\begin{aligned} \rho(\tfrac{y-Ty-Ux}{2}) &= \rho(\tfrac{y-y_n+y_n-Ty-Ux}{2}) \\ &= \rho(\tfrac{y-y_n+Ty_{n-1}-Ty}{2}) \\ &\leq \rho(y - y_n) + \rho(Ty_{n-1} - Ty). \end{aligned}$$

Since T is continuous, the right hand side tends to zero, this implies $y - Ty = Ux$.
It means that for any $x \in B$ there exists $y \in B$ such that $(I - T)y = Ux$. So $(I - T)B \subset U(B)$, and the operator $I - T$ is surjective from B into $U(B)$. Now, let $y_1, y_2 \in B$ which $(I - T)y_1 = (I - T)y_2$, then $y_1 - y_2 = Ty_1 - Ty_2$ and

$$\begin{aligned} \rho(y_1 - y_2) &= \rho(Ty_1 - Ty_2) \\ &< \rho(c(Ty_1 - Ty_2)) \\ &\leq \psi(\rho(y_1 - y_2)) \\ &< \rho(y_1 - y_2). \end{aligned}$$

This is contradiction, so $\rho(y_1 - y_2) = 0$ and $y_1 = y_2$. Therefore, $I - T$ is injective. Consequently, $I - T$ is a bijective operator from B into $U(B)$.
Claim: $(I - T)^{-1}$ is continuous.
Let $\{x_n\}$ be a sequence in $U(B)$ such that $x_n \to x \in U(B)$. Consider the sequence $z_n = (I - T)^{-1}(x_n)$, then $\{z_n\}$ is ρ-Cauchy. Indeed,

$$\begin{aligned} z_{m+n} - z_n &= z_{m+n} - Tz_{m+n} + Tz_{m+n} - Tz_n + Tz_n - z_n \\ &= x_{m+n} + Tz_{m+n} - Tz_n - x_n \\ &= x_{m+n} - x_n + Tz_{m+n} - Tz_n. \end{aligned}$$

Now, by choosing α in relation (3), we have:

$$\begin{aligned} \rho(z_{m+n} - z_n) &= \rho(\tfrac{1}{c}(c(Tz_{m+n} - Tz_n)) + \tfrac{1}{\alpha}\alpha(x_{m+n} - x_n)) \\ &\leq \tfrac{1}{c}\rho(c(Tz_{m+n} - Tz_n)) + \tfrac{1}{\alpha}\rho(\alpha(x_{m+n} - x_n)) \\ &\leq \tfrac{1}{c}\psi(\rho(z_{m+n} - z_n)) + \tfrac{1}{\alpha}(x_{m+n} - x_n)) \\ &< \tfrac{1}{c}\rho(z_{m+n} - z_n) + \tfrac{1}{\alpha}(x_{m+n} - x_n)). \end{aligned}$$

Therefore

$$\rho(z_{m+n} - z_n) < \frac{c}{c-1} \times \frac{1}{\alpha}\rho(\alpha(x_{m+n} - x_n)).$$

And since $\rho(x_{m+n} - x_n) \to 0$ then by Δ_2-condition $\rho(\alpha(x_{m+n} - x_n)) \to 0$. Therefore $\rho(z_{m+n} - z_n) \to 0$. So by hypothesis X_ρ is complete, $\{z_n\}$ converges to an element $z \in B$. Then $x_n = z_n - T(z_n)$ is convergent to $x = z - Tz$. Indeed,

$$\rho(\frac{z_n - T(z_n) - z + Tz}{2}) \leq \rho(z_n - z) + \rho(Tz_n - Tz) \to 0,$$

and by Δ_2-condition $\rho(z_n - Tz_n - (z - Tz)) \to 0$. Therefore $(I - T)^{-1}(x_n)$ converges to $(I - T)^{-1}(x)$ which implies that $(I - T)^{-1}$ is continuous.
Now, consider

$$f(x) = (I - T)^{-1}U(x).$$

Since U is ρ-completely continuous and $(I - T)^{-1}$ is ρ-continuous, by Δ_2-condition, U is $\|.\|_\rho$-completely continuous, and $(I - T)^{-1}$ is $\|.\|_\rho$-continuous. Consequently, f is $\|.\|_\rho$-completely continuous from B into B. By Δ_2-condition, B is $\|.\|_\rho$-closed. Then by Lemma 2.2 and Schauder's fixed point theorem, f has a fixed point x_0. Then

$$f(x_0) = x_0,$$

and this means that

$$x_0 = (I - T)^{-1}U(x_0)$$

or

$$x_0 = (T + U)(x_0)$$

and the proof is complete.

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