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A MODIFIED CQ METHOD FOR EQUILIBRIUM PROBLEMS, FIXED POINTS AND VARIATIONAL INEQUALITY

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Abstract. In this paper, we introduce a modified CQ iterative scheme for finding a common element of the set of solutions of an equilibrium problem, the set of fixed points of an infinite family of nonexpansive mappings and the set of the variational inequality for an α-inverse strongly monotone mapping in a Hilbert space. We obtain a strong convergence theorem for three sequences generated by this process. Based on this result, we also get several new and interesting results which generalize and extend some well-known strong convergence theorems in the literature.

Key Words and Phrases: equilibrium problem, CQ method, nonexpansive mapping, variational inequality, strong convergence, fixed point.
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1. Introduction

Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Let F be a bifunction from $C \times C$ to R, where R is the set of

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real numbers. The equilibrium problem for $F : C \times C \to R$ is to find $x \in C$ such that

$$F(x, y) \geq 0 \text{ for all } y \in C. \tag{1.1}$$

The set of solutions of (1.1) is denoted by $EP(F)$. Numerous problems in physics, optimization and economics reduce to find a solution of (1.1), for more details, please see [1] and [2].

Recall that a mapping S of a closed convex subset C of H is nonexpansive [3] if there holds that $\|Sx - Sy\| \leq \|x - y\|$ for all $x, y \in C$.

We denote the set of fixed points of S by $Fix(S)$. It is known (see [3]) that $Fix(S)$ is closed convex, but possibly empty. A mapping A of C into H is called monotone if

$$\langle Ax - Ay, x - y\rangle \geq 0$$

for all $x, y \in C$. A mapping A of C into H is called α-inverse-strongly monotone if there exists a positive real number α such that

$$\langle x - y, Ax - Ay\rangle \geq \alpha\|Ax - Ay\|^2$$

for all $x, y \in C$.

Let the mapping $A : C \to H$ be α-inverse-strongly monotone. The variational inequality problem is to find a point $x \in C$ such that

$$\langle Ax, y - x\rangle \geq 0$$

for all $y \in C$. The set of solutions of the variational inequality problem is denoted by $VI(C, A)$.

Some methods have been proposed to solve the equilibrium problem; see, for instance, [1, 2, 4-6, 8-10] and the references therein]. Recently, Combettes and Hirstoaga [4] introduced an iterative scheme of finding the best approximation to the initial data when $EP(F)$ is nonempty and proved a strong convergence theorem. Takahashi and Takahashi [5] introduced the following iterative scheme by the viscosity approximation method for finding a common element of the set of solution (1.1) and the set of fixed points of a nonexpansive mapping in a Hilbert space. Starting with an arbitrary $x_1 \in H$, define sequences $\{x_n\}$ and $\{u_n\}$ by

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \qquad \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Su_n, \forall n \in N. \end{cases} \tag{1.2}$$

They proved that under certain appropriate conditions imposed on $\{\alpha_n\}$ and $\{r_n\}$, the sequences $\{x_n\}$ and $\{u_n\}$ generated by (1.2) converge strongly to $z \in Fix(S) \cap EP(F)$, where $z = P_{Fix(S)\cap EP(F)}f(z)$, where f is a contraction on H.

Tada and Takahashi [6] introduced the following iterative scheme by the CQ method for finding a common element of the set of solution (1.1) and the set of fixed points of a nonexpansive mapping in a Hilbert space. Starting with an arbitrary $x_1 \in H$, define sequences $\{x_n\}$ and $\{u_n\}$ by

$$\begin{cases} u_n \in C, F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \qquad \forall y \in C, \\ w_n = (1-\alpha_n)x_n + \alpha_n S u_n, \\ C_n = \{z \in H : \|w_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in H : \langle x_n - z, x - x_n\rangle \geq 0\}, x_{n+1} = P_{C_n \bigcap Q_n} x \end{cases} \tag{1.3}$$

They proved that under certain appropriate conditions imposed on $\{\alpha_n\}$ and $\{r_n\}$, the sequences $\{x_n\}$ and $\{u_n\}$ generated by (1.3) converge strongly to $P_{Fix(S)\cap EP(F)}x$. Nakajo and Takahashi [7] introduced an iterative schemes by the CQ method for finding an element of the set of fixed points of a nonexpansive mapping in a Hilbert space. Kikkawa and Takahashi [8] introduced an iterative schemes by the CQ method for finding a common element of the set of fixed points of a family of infinite of nonexpansive mapping in a Hilbert space. Su, Shang and Qin [9] introduced the following iterative scheme by the viscosity approximation method for finding a common element of the set of solution (1.1) and the set of fixed points of a nonexpansive mapping and the set of solutions of the variational inequality problem for an α-inverse strongly monotone mapping in a Hilbert space. Starting with an arbitrary $x_1 \in H$, define sequences $\{x_n\}$ and $\{u_n\}$ by

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \qquad \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) S P_C(u_n - \lambda_n A u_n), \forall n \in N. \end{cases} \tag{1.4}$$

They proved that under certain appropriate conditions imposed on $\{\alpha_n\}$, $\{r_n\}$ and $\{\lambda_n\}$, the sequences $\{x_n\}$ and $\{u_n\}$ generated by (1.4) converge strongly to $z \in Fix(S) \cap EP(F)$ $\cap VI(C, A)$, where

$$z = P_{Fix(S)\cap EP(F)\cap VI(C,A)} f(z).$$

Yao, Liou and Yao [10], Ceng and Yao [11] introduced some iterative viscosity approximation schemes for a common element of the set of solution (1.1) and the set of fixed points of infinitely nonexpansive mappings in a Hilbert space.

Inspired by the above results, in the present paper, we introduce a modified CQ iterative process for finding the common element of the set of fixed points of a nonexpansive mapping, the set of an equilibrium problem and the set of solutions of the variational inequality problem for an α-inverse strongly monotone mapping in a Hilbert space. Then, we obtain a strong convergence theorem for three sequences generated by this process. Based on this result, we also get several new and interesting results which generalize and extend some well-known results in [6-8].

2. PRELIMINARIES

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let C be a nonempty closed convex subset of H. Let symbols $\to$ and $\rightharpoonup$ denote strong and weak convergence, respectively. In a real Hilbert space H, it is well known that

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2$$

for all $x, y \in H$ and $\lambda \in [0,1]$.

For any $x \in H$, there exists a unique nearest point in C, denoted by $P_C(x)$, such that $\|x - P_C(x)\| \leq \|x - y\|$ for all $y \in C$. The mapping P_C is called the metric projection of H onto C. We know that P_C is a nonexpansive mapping from H onto C and satisfies

$$\langle x - y, P_C x - P_C y \rangle \geq \|P_C x - P_C y\|^2 \tag{2.1}$$

It is also known that $P_C x \in C$ and

$$\langle x - P_C(x), P_C(x) - y \rangle \geq 0 \tag{2.2}$$

for all $x \in H$ and $y \in C$.

It is easy to see that (2.2) is equivalent to

$$\|x - y\|^2 \geq \|x - P_C(x)\|^2 + \|y - P_C(x)\|^2 \tag{2.3}$$

for all $x \in H$ and $y \in C$.

Let A be α-inverse-strongly monotone mapping of C into H. In the context of the variational inequality problem the characterization of projection (2.1) implies the following:

$$u \in VI(C, A) \Rightarrow u = P_C(u - \lambda Au), \lambda > 0,$$

and

$$u = P_C(u - \lambda Au) \text{ for some } \lambda > 0 \Rightarrow \; u \in VI(C, A).$$

It is known that if T is a nonexpansive mapping of C into itself and I is the identity mapping of H, then $A = I - T$ is $\frac{1}{2}$-inverse-strongly monotone and $VI(C, A) = Fix(T)$. If A is α-inverse-strongly monotone of C into H, then it is obvious that A is $\frac{1}{\alpha}$-Lipschitz continuous. We also have that for all $x, y \in C$ and $\lambda > 0$,

$$\begin{aligned} \|(I - \lambda A)x - (I - \lambda A)y\|^2 &= \|x - y - \lambda(Ax - Ay)\|^2 \\ &= \|x - y\|^2 - 2\lambda\langle x - y, Ax - Ay\rangle + \lambda^2\|Ax - Ay\|^2 \\ &\leq \|x - y\|^2 + \lambda(\lambda - 2\alpha)\|Ax - Ay\|^2. \end{aligned} \tag{2.4}$$

So, if $\lambda \leq 2\alpha$, then $I - \lambda A$ is a nonexpansive mapping of C into H.

It is also known that H satisfies the Opial's condition [12], i.e., for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

holds for every $y \in H$ with $x \neq y$.

A set-valued mapping $T : H \to 2^H$ is called monotone if for all $x, y \in H$, $f \in Tx$ and $g \in Ty$ imply $\langle x-y, f-g\rangle \geq 0$. A monotone mapping $T : H \to 2^H$ is maximal if its graph $G(T)$ of T is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal if and only if for $(x, f) \in H \times H, \langle x - y, f - g\rangle \geq 0$ for every $(y, g) \in G(T)$ implies $f \in Tx$. Let A is an α-inverse strongly monotone mapping of C into H and let N_Cv be normal cone to C at $v \in C$, i.e, $N_Cv = \{w \in H : \langle v - u, w\rangle \geq 0, \forall u \in C\}$. Define

$$Tv = \begin{cases} Av + N_Cv & \text{if } v \in C, \\ \emptyset & \text{if } v \notin C, \end{cases}$$

then T is maximal monotone and $0 \in Tv$ if and only if $v \in VI(C, A)$ (see [13]).

For solving the equilibrium problem, let us assume that the bifunction F satisfies the following conditions:

(A1) $F(x,x)=0$ for all $x\in C$;
(A2) F is monotone, i.e. $F(x,y)+F(y,x)\leq 0$ for any $x,y\in C$;
(A3) for each $x,y,z\in C$,

$$\lim_{t\downarrow 0} F(tz+(1-t)x,y)\leq F(x,y);$$

(A4) for each $x\in C, y\mapsto F(x,y)$ is convex and lower semicontinuous.

We recall some lemmas needed later.

Lemma 2.1. [2] Let C be a nonempty closed convex subset of H, let F be a bifunction from $C\times C$ to R satisfying (A1)-(A4). Let $r>0$ and $x\in H$. Then, there exists $z\in C$ such that $F(z,y)+\frac{1}{r}\langle y-z,z-x\rangle\geq 0$, for all $y\in C$.

Lemma 2.2. [4] Let C be a nonempty closed convex subset of H, let F be a bifunction from $C\times C$ to R satisfying (A1)-(A4). For $r>0$ and $x\in H$, define a mapping $T_r:H\to C$ as follows.

$$T_r(x)=\{z\in C: F(z,y)+\frac{1}{r}\langle y-z,z-x\rangle\geq 0,\forall y\in C\}$$

for all $x\in H$. Then, the following statements hold:
(1) T_r is single-valued;
(2) T_r is firmly nonexpansive, i.e, for any $x,y\in H$,

$$\|T_r(x)-T_r(y)\|^2\leq\langle T_r(x)-T_r(y),x-y\rangle;$$

(3) $F(T_r)=EP(F)$;
(4) $EP(F)$ is closed and convex.

Let $S_1,S_2,...$ be an infinite family of nonexpansive mappings of C into itself and let $\xi_1,\xi_2,...$ be real numbers such that $0\leq\xi_i\leq 1$ for every $i\in N$. For any $n\in N$, define a mapping W_n of C into C as follows:

$$U_{n,n+1}=I,$$

$$U_{n,n}=\xi_n T_n U_{n,n+1}+(1-\xi_n)I,$$

$$U_{n,n-1}=\xi_{n-1}T_{n-1}U_{n,n}+(1-\xi_{n-1})I,$$

$$...$$

$$U_{n,k}=\xi_k T_k U_{n,k+1}+(1-\xi_k)I,$$

$$U_{n,k-1}=\xi_{k-1}T_{k-1}U_{n,k}+(1-\xi_{k-1})I,$$

$$...$$

$$U_{n,2} = \xi_2 T_2 U_{n,3} + (1 - \xi_2)I,$$

$$W_n = U_{n,1} = \xi_1 T_1 U_{n,2} + (1 - \xi_1)I.$$

Such a mapping W_n is called the W-mapping generated by $S_n, S_{n-1}, ..., S_1$ and $\xi_n, \xi_{n-1}, ..., \xi_1$; see [14, 15].

Lemma 2.3. [15] Let C be a nonempty closed convex subset of a Banach space E. Let $S_1, S_2, ...$ be an infinite family of nonexpansive mappings of C into itself such that $\cap_{i=1}^{\infty} Fix(S_i)$ is nonempty, and let $\xi_1, \xi_2, ...$ be real numbers such that $0 < \xi_i \leq d < 1$ for every $i \in N$. For any $n \in N$, let W_n be the W-mapping of C into itself generated by $S_n, S_{n-1}, ..., S_1$ and $\xi_n, \xi_{n-1}, ..., \xi_1$. Then W_n is asymptotically regular and nonexpansive. Further, if E is strict convex, then $F(W_n) = \cap_{i=1}^{n} Fix(S_i)$.

Lemma 2.4. [14] Let C be a nonempty closed convex subset of a strictly convex Banach space E. Let $S_1, S_2, ...$ be an infinite family of nonexpansive mappings of C into itself such that $\cap_{i=1}^{\infty} Fix(S_i)$ is nonempty, and let $\xi_1, \xi_2, ...$ be real numbers such that $0 < \xi_i \leq d < 1$ for every $i \in N$. Then for every $x \in C$ and $k \in N$, the limit $lim_{n \to \infty} U_{n,k} x$ exists.

Remark 2.1. Using Lemma 2.4, one can define a mapping $U_{\infty,k}$ and W of C into itself as follows:

$$U_{\infty,k} x = lim_{n \to \infty} U_{n,k} x$$

and $Wx = lim_{n \to \infty} W_n x = lim_{n \to \infty} U_{n,1} x$ for every $x \in C$. Such a mapping W is called the W-mapping generated by $S_1, S_2, ...$ and $\xi_1, \xi_2, ...$ Since W_n is nonexpansive, $W : C \to C$ is also nonexpansive. Indeed, observe that for each $x, y \in C$

$$\|Wx - Wy\| = lim_{n \to \infty} \|W_n x - W_n y\| \leq \|x - y\|.$$

If $\{x_n\}$ is a bounded sequence in C, then we have

$$lim_{n \to \infty} \|Wx - W_n x\| = 0.$$

Lemma 2.5. [15] Let C be a nonempty closed convex subset of a strictly convex Banach space E. Let $S_1, S_2, ...$ be an infinite family of nonexpansive mappings of C into itself such that $\cap_{i=1}^{\infty} Fix(S_i)$ is nonempty, and let $\xi_1, \xi_2, ...$ be real numbers such that $0 < \xi_i \leq d < 1$ for every $i \in N$. Then $F(W) = \cap_{n=1}^{\infty} Fix(S_n)$.

3. Strong convergence theorem

In this section, we show a strong convergence theorem which solves the problem of finding a common element of the set of solutions of an equilibrium problem, the set of fixed points of a nonexpansive mapping and the set of the variational inequality for an α-inverse strongly monotone mapping in a Hilbert space.

Theorem 3.1 Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ to R satisfying (A1)-(A4) and let A be an α-inverse strongly monotone mapping of C into H. Let $S_1, S_2, ...$ be an infinite family of nonexpansive mappings of C into itself such that $\Omega = \cap_{i=1}^{\infty} Fix(S_i) \cap VI(C, A) \cap EP(F) \neq \emptyset$ and let $\xi_1, \xi_2, ...$ be real numbers such that $0 < \xi_i \leq d < 1$ for every $i \in N$. For any $n \in N$, let W_n be the W-mapping of C into itself generated by $S_n, S_{n-1}, ..., S_1$ and $\xi_n, \xi_{n-1}, ..., \xi_1$. Let $\{x_n\}$, $\{u_n\}$, $\{y_n\}$ and $\{z_n\}$ be sequences generated by

$$\begin{cases} x_1 = x \in H, \\ F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \qquad \forall y \in C, \\ z_n = \alpha_n x_n + (1 - \alpha_n) W_n P_C(u_n - \lambda_n A u_n), \\ C_n = \{z \in H : \|z_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in H : \langle x_n - z, x - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \bigcap Q_n} x \end{cases}$$

for every $n = 1, 2, ...$ If $\{\lambda_n\} \subset [a, b] \subset (0, 2\alpha)$, $\{\alpha_n\} \subset [0, c]$ for some $c \in [0, 1]$ and $\{r_n\} \subset (0, \infty)$ satisfies $\liminf_{n\to\infty} r_n > 0$. Then, $\{x_n\}$, $\{u_n\}$ and $\{z_n\}$ converge strongly to $w = P_\Omega(x)$.

Proof. It is obvious that C_n is closed and Q_n is closed and convex for every $n = 1, 2, ...$ As $C_n = \{z \in H : \|z_n - x_n\|^2 + 2\langle z_n - x_n, x_n - z\rangle \leq 0\}$, we also have that C_n is convex for every $n = 1, 2, ...$ As $Q_n = \{z \in H : \langle x_n - z, x - x_n\rangle \geq 0\}$, we have $\langle x_n - z, x - x_n\rangle \geq 0$ for all $z \in Q_n$ and, by (2.2), $x_n = P_{Q_n} x$. Put $t_n = P_C(u_n - \lambda_n A u_n)$ for every $n = 1, 2, ...$ Let $u \in \Omega$ and let $\{T_{r_n}\}$ be a sequence of mappings defined as in Lemma 2.2. Then $u = P_C(u - \lambda_n A u) = T_{r_n} u$. And from $u_n = T_{r_n} x_n \in C$, we have

$$\begin{aligned} \|u_n - u\| &= \|T_{r_n} x_n - T_{r_n} u\| \\ &\leq \|x_n - u\|. \end{aligned} \tag{3.1}$$

Since A is an α-inverse strongly monotone mapping, from (2.4), we have

$$\begin{aligned}\|t_n - u\| &= \|P_C(u_n - \lambda_n Au_n) - P_C(u - \lambda_n Au)\| \\ &\le \|u_n - \lambda_n Au_n - (u - \lambda_n Au)\| \\ &\le \|u_n - u\|. \qquad (3.2)\end{aligned}$$

Therefore from (3.1), (3.2), $z_n = \alpha_n x_n + (1-\alpha_n)W_n t_n$ and $u = W_n u$, we have

$$\begin{aligned}\|z_n - u\|^2 &= \|\alpha_n x_n + (1-\alpha_n)W_n t_n - u\|^2 \\ &\le \alpha_n\|x_n - u\|^2 + (1-\alpha_n)\|W_n t_n - u\|^2 \\ &\le \alpha_n\|x_n - u\|^2 + (1-\alpha_n)\|t_n - u\|^2 \\ &\le \alpha_n\|x_n - u\|^2 + (1-\alpha_n)\|u_n - u\|^2 \\ &\le \|x_n - u\|^2, \qquad (3.3)\end{aligned}$$

for every $n = 1, 2, \ldots$ and hence $u \in C_n$. So, $\Omega \subset C_n$ for every $n = 1, 2, \ldots$. Next, let us show by mathematical induction that $\{x_n\}$ is well defined and $\Omega \subset C_n \cap Q_n$ for every $n = 1, 2, \ldots$. For $n = 1$ we have $x_1 = x \in C$ and $Q_1 = H$. Hence we obtain $\Omega \subset C_1 \cap Q_1$. Suppose that x_k is given and $\Omega \subset C_k \cap Q_k$ for some $k \in N$. Since Ω is nonempty, $C_k \cap Q_k$ is a nonempty closed convex subset of H. So, there exists a unique element $x_{k+1} \in C_k \cap Q_k$ such that $x_{k+1} = P_{C_k \cap Q_k} x$. It is also obvious that there holds $\langle x_{k+1} - z, x - x_{k+1}\rangle \ge 0$ for every $z \in C_k \cap Q_k$. Since $\Omega \subset C_k \cap Q_k$, we have $\langle x_{k+1} - z, x - x_{k+1}\rangle \ge 0$ for every $z \in \Omega$ and hence $\Omega \subset Q_{k+1}$. Therefore, we obtain $\Omega \subset C_{k+1} \cap Q_{k+1}$.

Let $l_0 = P_\Omega x$. From $x_{n+1} = P_{C_n \cap Q_n} x$ and $l_0 \in \Omega \subset C_n \cap Q_n$, we have

$$\|x_{n+1} - x\| \le \|l_0 - x\| \qquad (3.4)$$

for every $n = 1, 2, \ldots$. Therefore, $\{x_n\}$ is bounded. From (3.1)-(3.3), we also obtain that $\{t_n\}$, $\{z_n\}$ and $\{u_n\}$ are bounded. Since $x_{n+1} \in C_n \cap Q_n \subset C_n$ and $x_n = P_{Q_n} x$, we have

$$\|x_n - x\| \le \|x_{n+1} - x\|$$

for every $n = 1, 2, \ldots$. Therefore, $\lim_{n\to\infty} \|x_n - x\|$ exists.

Since $x_n = P_{Q_n} x$ and $x_{n+1} \in Q_n$, using (2.3), we have

$$\|x_{n+1} - x_n\|^2 \le \|x_{n+1} - x\|^2 - \|x_n - x\|^2$$

for every $n = 1, 2, ...$ This implies that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0.$$

Since $x_{n+1} \in C_n$, we have $\|z_n - x_{n+1}\| \le \|x_n - x_{n+1}\|$ and hence

$$\|x_n - z_n\| \le \|x_n - x_{n+1}\| + \|x_{n+1} - z_n\| \le 2\|x_n - x_{n+1}\|$$

for every $n = 1, 2, ...$ From $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$, we have $\|x_n - z_n\| \to 0$.

For $u \in \Omega$, we have, from Lemma 2.2,

$$\|u_n - u\|^2 = \|T_{r_n}x_n - T_{r_n}u\|^2 \le \langle T_{r_n}x_n - T_{r_n}u, x_n - u\rangle$$
$$= \langle u_n - u, x_n - u\rangle = \frac{1}{2}\{\|u_n - u\|^2 + \|x_n - u\|^2 - \|x_n - u_n\|^2\}.$$

Hence,

$$\|u_n - u\|^2 \le \|x_n - u\|^2 - \|x_n - u_n\|^2.$$

Then, by (3.3) and (3.2),

$$\begin{aligned}\|z_n - u\|^2 &\le \alpha_n\|x_n - u\|^2 + (1 - \alpha_n)\|t_n - u\|^2\\ &\le \alpha_n\|x_n - u\|^2 + (1 - \alpha_n)\|u_n - u\|^2\\ &\le \alpha_n\|x_n - u\|^2 + (1 - \alpha_n)[\|x_n - u\|^2 - \|x_n - u_n\|^2]\\ &= \|x_n - u\|^2 - (1 - \alpha_n)\|x_n - u_n\|^2.\end{aligned}$$

Hence,

$$\begin{aligned}(1 - c)\|x_n - u_n\|^2 &\le (1 - \alpha_n)\|x_n - u_n\|^2\\ &\le \|x_n - u\|^2 - \|z_n - u\|^2\\ &= (\|x_n - u\| + \|z_n - u\|)(\|x_n - u\| - \|z_n - u\|)\\ &\le (\|x_n - u\| + \|z_n - u\|)\|x_n - z_n\|.\end{aligned}$$

Since $\|x_n - z_n\| \to 0$ and the sequences $\{x_n\}$ and $\{z_n\}$ are bounded, we obtain $\|x_n - u_n\| \to 0$.

For $u \in \Omega$, from (3.3), (3.1) and (2.4), we have

$$\|z_n - u\|^2 \le \alpha_n\|x_n - u\|^2 + (1 - \alpha_n)\|t_n - u\|^2$$
$$\le \alpha_n\|x_n - u\|^2 + (1 - \alpha_n)\|P_C(u_n - \lambda_n Au_n) - P_C(u - \lambda_n Au)\|^2$$
$$\le \alpha_n\|x_n - u\|^2 + (1 - \alpha_n)\|u_n - \lambda_n Au_n - (u - \lambda_n Au)\|^2$$
$$\le \alpha_n\|x_n - u\|^2 + (1 - \alpha_n)[\|u_n - u\|2 + \lambda_n(\lambda_n - 2\alpha)\|Au_n - Au\|^2]$$
$$\le \|x_n - u\|^2 + (1 - \alpha_n)\lambda_n(\lambda_n - 2\alpha)\|Au_n - Au\|^2$$

and hence

$$\begin{aligned}&(1-c)a(2\alpha\ -b)\|Au_n-Au\|^2\\ &\leq (1-\alpha_n)\lambda_n(2\alpha-\lambda_n)\|Au_n-Au\|^2\\ &\leq \|x_n-u\|^2-\|z_n-u\|^2\\ &\leq (\|x_n-u\|+\|z_n-u\|)\|x_n-z_n\|\end{aligned}$$

Since $\|x_n-z_n\|\to 0$ and the sequences $\{x_n\}$ and $\{z_n\}$ are bounded, we obtain $\|Au_n-Au\|\to 0$.

From (2.1) and (2.4), we have

$$\begin{aligned}\|t_n-u\|^2&=\|P_C(u_n-\lambda_nAu_n)-P_C(u-\lambda_nAu)\|^2\\ &\leq \langle(u_n-\lambda_nAu_n)-(u-\lambda_nAu),t_n-u\rangle\\ &=\frac{1}{2}\{\|(u_n-\lambda_nAu_n)-(u-\lambda_nAu)\|^2+\|t_n-u\|^2\\ &\quad-\|[(u_n-\lambda_nAu_n)-(u-\lambda_nAu)]-(t_n-u)\|^2\}\\ &\leq \frac{1}{2}\{\|u_n-u\|^2+\|t_n-u\|^2-\|(u_n-t_n)-\lambda_n(Au_n-Au)\|^2\}\\ &=\frac{1}{2}\{\|u_n-u\|^2+\|t_n-u\|^2-\|u_n-t_n\|^2+2\lambda_n\langle u_n-t_n,Au_n-Au\rangle-\lambda_n^2\|Au_n-Au\|^2\}.\end{aligned}$$

Hence,

$$\|t_n-u\|^2\leq\|u_n-u\|^2-\|u_n-t_n\|^2+2\lambda_n\langle u_n-t_n,Au_n-Au\rangle-\lambda_n^2\|Au_n-Au\|^2$$

From (3.3) and (3.1), we have

$$\begin{aligned}\|z_n-u\|^2&\leq\alpha_n\|x_n-u\|^2+(1-\alpha_n)\|t_n-u\|^2\\ &\leq\alpha_n\|x_n-u\|^2+(1-\alpha_n)[\|u_n-u\|^2-\|u_n-t_n\|^2\\ &\quad+2\lambda_n\langle u_n-t_n,Au_n-Au\rangle-\lambda_n^2\|Au_n-Au\|^2]\\ &\leq\|x_n-u\|^2-(1-\alpha_n)\|u_n-t_n\|^2\\ &\quad+2\lambda_n(1-\alpha_n)\langle u_n-t_n,Au_n-Au\rangle-(1-\alpha_n)\lambda_n^2\|Au_n-Au\|^2\end{aligned}$$

and hence

$$\begin{aligned}&(1-c)\|u_n-t_n\|^2\leq(1-\alpha_n)\|u_n-t_n\|^2\\ &\leq\|x_n-u\|^2-\|z_n-u\|^2+2\lambda_n(1-\alpha_n)\langle u_n-t_n,Au_n-Au\rangle-(1-\alpha_n)\lambda_n^2\|Au_n-Au\|^2\\ &\leq(\|x_n-u\|+\|z_n-u\|)\|x_n-z_n\|\\ &\quad+2\lambda_n(1-\alpha_n)\langle u_n-t_n,Au_n-Au\rangle-(1-\alpha_n)\lambda_n^2\|Au_n-Au\|^2.\end{aligned}$$

Since $\|x_n - z_n\| \to 0$ and $\|Au_n - Au\| \to 0$, we obtain $\|u_n - t_n\| \to 0$. It follows from the Lipschitz-continuity of A that $\|Au_n - At_n\| \to 0$.

From $\|z_n - t_n\| \leq \|z_n - x_n\| + \|x_n - u_n\| + \|u_n - t_n\|$ we have $\|z_n - t_n\| \to 0$. From $\|t_n - x_n\| \leq \|t_n - u_n\| + \|x_n - u_n\|$ we also have $\|t_n - x_n\| \to 0$.

Since $z_n = \alpha_n x_n + (1 - \alpha_n) W_n t_n$, we have $(1 - \alpha_n)(W_n t_n - t_n) = \alpha_n (t_n - x_n) + (z_n - t_n)$. Then

$$(1 - c)\|W_n t_n - t_n\| \leq (1 - \alpha_n)\|W_n t_n - t_n\| \leq \alpha_n \|t_n - x_n\| + \|z_n - t_n\|$$

and hence $\|W_n t_n - t_n\| \to 0$. At the same time, observe that

$$\|W t_n - t_n\| \leq \|W t_n - W_n t_n\| + \|W_n t_n - t_n\|. \tag{3.5}$$

It follows from (3.5) and Remark 2.2, we have $\lim_{n\to\infty} \|W t_n - t_n\| = 0$.

As $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{ni} \rightharpoonup w$. From $\|x_n - u_n\| \to 0$, we obtain that $u_{ni} \rightharpoonup w$. From $\|u_n - t_n\| \to 0$, we obtain also that $t_{ni} \rightharpoonup w$. Since $\{u_{ni}\} \subset C$ and C is closed and convex, we obtain $w \in C$.

First, we show $w \in EP(F)$. By $u_n = T_{r_n} x_n$, we know that

$$F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \forall y \in C.$$

It follows from (A2) that

$$\frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq F(y, u_n), \forall y \in C.$$

Hence,

$$\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\rangle \geq F(y, u_{n_i}), \forall y \in C.$$

It follows from $\frac{u_{n_i} - x_{n_i}}{r_{n_i}} \to 0$, $u_{n_i} \rightharpoonup w$ and (A4) that

$$f(y, w) \leq 0, \forall y \in C.$$

For t with $0 < t \leq 1$ and $y \in C$, let $y_t = ty + (1 - t)w$. Since $y \in C$ and $w \in C$, we obtain $y_t \in C$ and hence $F(y_t, w) \leq 0$. So

$$0 = F(y_t, y_t) \leq tF(y_t, y) + (1 - t)F(y_t, w) \leq tF(y_t, y)$$

Dividing by t, we get

$$F(y_t, y) \geq 0.$$

Letting $t \to 0$, it follows from (A3) that

$$F(w, y) \geq 0$$

for all $y \in C$ and hence $w \in EP(F)$. We next show that $w \in Fix(W)$. Assume $w \notin Fix(W)$. Since $t_{n_i} \rightharpoonup w$ and $w \neq Ww$, from the Opial theorem [12] we have

$$\begin{aligned} \liminf_{i\to\infty} \|t_{n_i} - w\| &< \liminf_{i\to\infty} \|t_{n_i} - Ww\| \\ &\leq \liminf_{i\to\infty} \{\|t_{n_i} - Wt_{n_i}\| + \|Wt_{n_i} - Ww\|\} \\ &\leq \liminf_{i\to\infty} \|t_{n_i} - w\| \end{aligned}$$

This is a contradiction. So, we get $w \in Fix(W) = \cap_{i=1}^{\infty} Fix(S_i)$.

We shall show $w \in VI(C, A)$. Let

$$Tv = \begin{cases} Av + N_C v & \text{if } v \in C, \\ \emptyset & \text{if } v \notin C. \end{cases}$$

where $N_C v$ is the normal cone to C at $v \in C$. We have already mentioned that in this case the mapping T is maximal monotone, and $0 \in Tv$ if and only if $v \in VI(C, A)$. Let $(v, g) \in G(T)$. Then $Tv = Av + N_C v$ and hence $g - Av \in N_C v$. So, we have $\langle v - t, g - Av\rangle \geq 0$ for all $t \in C$. On the other hand, from $t_n = P_C(u_n - \lambda_n A u_n)$ and $v \in C$ we have

$$\langle u_n - \lambda_n A u_n - t_n, t_n - v\rangle \geq 0$$

and hence

$$\langle v - t_n, \frac{t_n - u_n}{\lambda_n} + Au_n\rangle \geq 0.$$

Therefore, we have

$$\begin{aligned} \langle v - t_{n_i}, g\rangle &\geq \langle v - t_{n_i}, Av\rangle \\ &\geq \langle v - t_{n_i}, Av\rangle - \langle v - t_{n_i}, \frac{t_{n_i} - u_{n_i}}{\lambda_{n_i}} + Au_{n_i}\rangle \\ &= \langle v - t_{n_i}, Av - Au_{n_i} - \frac{t_{n_i} - u_{n_i}}{\lambda_{n_i}}\rangle \\ &= \langle v - t_{n_i}, Av - At_{n_i} + At_{n_i} - Au_{n_i} - \frac{t_{n_i} - u_{n_i}}{\lambda_{n_i}}\rangle \\ &= \langle v - t_{n_i}, Av - At_{n_i}\rangle + \langle v - t_{n_i}, At_{n_i} - Au_{n_i}\rangle - \langle v - t_{n_i}, \frac{t_{n_i} - u_{n_i}}{\lambda_{n_i}}\rangle \end{aligned}$$

$$\geq \langle v - t_{n_i}, At_{n_i} - Au_{n_i}\rangle - \langle v - t_{n_i}, \frac{t_{n_i} - u_{n_i}}{\lambda_{n_i}}\rangle$$

Hence we obtain $\langle v - w, g\rangle \geq 0$ as $i \to \infty$. Since T is maximal monotone, we have $w \in T^{-1}0$ and hence $w \in VI(C, A)$. This implies $w \in \Omega$.

From $l_0 = P_\Omega x$, $w \in \Omega$ and (3.4), we have

$$\|l_0 - x\| \leq \|w - x\| \leq \liminf_{i\to\infty} \|x_{n_i} - x\| \leq \limsup_{i\to\infty} \|x_{n_i} - x\| \leq \|l_0 - x\|.$$

So, we obtain

$$\lim_{i\to\infty} \|x_{n_i} - x\| = \|w - x\|.$$

From $x_{n_i} - x \rightharpoonup w - x$ we have $x_{n_i} - x \to w - x$ and hence $x_{n_i} \to w$. Since $x_n = P_{Q_n} x$ and $l_0 \in \Omega \subset C_n \cap Q_n \subset Q_n$, we have

$$-\|l_0 - x_{n_i}\|^2 = \langle l_0 - x_{n_i}, x_{n_i} - x\rangle + \langle l_0 - x_{n_i}, x - l_0\rangle \geq \langle l_0 - x_{n_i}, x - l_0\rangle.$$

As $i \to \infty$, we obtain $-\|l_0 - x_{n_i}\|^2 \geq \langle l_0 - w, x - l_0\rangle \geq 0$ by $l_0 = P_\Omega x$ and $w \in \Omega$. Hence we have $w = l_0$. This implies that $x_n \to l_0$. It is easy to see $u_n \to l_0$ and $z_n \to l_0$.

By Theorem 3.1, we can obtain the following new and interesting strong convergence theorems in a real Hilbert space.

Corollary 4.1 Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ to R satisfying (A1)-(A4) and let A be an α-inverse strongly monotone mapping of C into H such that $VI(C, A) \cap EP(F) \neq \emptyset$. Let $\{x_n\}$, $\{u_n\}$, $\{y_n\}$ and $\{z_n\}$ be sequences generated by

$$\begin{cases} x_1 = x \in H, \\ F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \qquad \forall y \in C, \\ z_n = \alpha_n x_n + (1 - \alpha_n) P_C(u_n - \lambda_n A u_n), \\ C_n = \{z \in H : \|z_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in H : \langle x_n - z, x - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \bigcap Q_n} x \end{cases}$$

for every $n = 1, 2, ...$ If $\{\lambda_n\} \subset [a, b] \subset (0, 2\alpha)$, $\{\alpha_n\} \subset [0, c]$ for some $c \in [0, 1]$ and $\{r_n\} \subset (0, \infty)$ satisfies $\liminf_{n\to\infty} r_n > 0$. Then, $\{x_n\}$, $\{u_n\}$ and $\{z_n\}$ converge strongly to $w = P_{VI(C,A)\cap EP(F)}(x)$.

Proof. Putting $S_n = I$ and $\alpha_n = 0$ for every $n = 1, 2, ...$, by Theorem 3.1 we obtain the desired result.

Corollary 4.2 Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ to R satisfying (A1)-(A4) and let $S_1, S_2, ...$ be an infinite family of nonexpansive mappings of C into itself such that $\cap_{i=1}^{\infty} Fix(S_i) \cap EP(F) \neq \emptyset$ and let $\xi_1, \xi_2, ...$ be real numbers such that $0 < \xi_i \leq d < 1$ for every $i \in N$. For any $n \in N$, let W_n be the W-mapping of C into itself generated by $S_n, S_{n-1}, ..., S_1$ and $\xi_n, \xi_{n-1}, ..., \xi_1$. Let $\{x_n\}$, $\{u_n\}$, $\{y_n\}$ and $\{z_n\}$ be sequences generated by

$$\begin{cases} x_1 = x \in H, \\ F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \qquad \forall y \in C, \\ z_n = \alpha_n x_n + (1-\alpha_n) W_n u_n, \\ C_n = \{z \in H : \|z_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in H : \langle x_n - z, x - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \bigcap Q_n} x \end{cases}$$

for every $n = 1, 2, ...$ If $\{\alpha_n\} \subset [0, c]$ for some $c \in [0, 1]$ and $\{r_n\} \subset (0, \infty)$ satisfies $\liminf_{n\to\infty} r_n > 0$. Then, $\{x_n\}$, $\{u_n\}$ and $\{z_n\}$ converge strongly to $w = P_{\cap_{i=1}^{\infty} Fix(S_i) \cap EP(F)}(x)$.

Proof. Putting $A = 0$, by Theorem 3.1 we obtain the desired result.

Corollary 4.3 Let C be a nonempty closed convex subset of a real Hilbert space H. Let A be an α-inverse strongly monotone mapping of C into H. Let $S_1, S_2, ...$ be an infinite family of nonexpansive mappings of C into itself such that $\cap_{i=1}^{\infty} Fix(S_i) \cap VI(C, A) \neq \emptyset$ and let $\xi_1, \xi_2, ...$ be real numbers such that $0 < \xi_i \leq d < 1$ for every $i \in N$. For any $n \in N$, let W_n be the W-mapping of C into itself generated by $S_n, S_{n-1}, ..., S_1$ and $\xi_n, \xi_{n-1}, ..., \xi_1$. Let $\{x_n\}$, $\{u_n\}$, $\{y_n\}$ and $\{z_n\}$ be sequences generated by

$$\begin{cases} x_1 = x \in H, \\ \langle y - u_n, u_n - x_n\rangle \geq 0, \qquad \forall y \in C, \\ z_n = \alpha_n x_n + (1-\alpha_n) W_n P_C(u_n - \lambda_n A u_n), \\ C_n = \{z \in H : \|z_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in H : \langle x_n - z, x - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \bigcap Q_n} x \end{cases}$$

for every $n = 1, 2, ...$ If $\{\lambda_n\} \subset [a, b] \subset (0, 2\alpha)$ and $\{\alpha_n\} \subset [0, c]$ for some $c \in [0, 1]$. Then, $\{x_n\}$, $\{u_n\}$ and $\{z_n\}$ converge strongly to $w = P_{\cap_{i=1}^{\infty} Fix(S_i) \cap VI(C,A)}(x)$.

Proof. Putting $F(x, y) = 0$ for all $x, y \in C$ and $r_n = 1$ for all $n \in N$, by Theorem 3.1 we obtain the desired result.

Corollary 4.4 Let C be a nonempty closed convex subset of a real Hilbert space H. Let $S_1, S_2, ...$ be an infinite family of nonexpansive mappings of C into itself such that $\cap_{i=1}^{\infty} Fix(S_i) \neq \emptyset$ and let $\xi_1, \xi_2, ...$ be real numbers such that $0 < \xi_i \leq d < 1$ for every $i \in N$. For any $n \in N$, let W_n be the W-mapping of C into itself generated by $S_n, S_{n-1}, ..., S_1$ and $\xi_n, \xi_{n-1}, ..., \xi_1$. Let $\{x_n\}$, $\{u_n\}$, $\{y_n\}$ and $\{z_n\}$ be sequences generated by

$$\begin{cases} x_1 = x \in H, \\ z_n = \alpha_n x_n + (1 - \alpha_n) W_n u_n, \\ C_n = \{z \in H : \|z_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in H : \langle x_n - z, x - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \bigcap Q_n} x \end{cases}$$

for every $n = 1, 2, ...$ If $\{\alpha_n\} \subset [0, c]$ for some $c \in [0, 1]$, then, $\{x_n\}$, $\{u_n\}$ and $\{z_n\}$ converge strongly to $w = P_{\cap_{i=1}^{\infty} Fix(S_i) \cap VI(C,A)}(x)$.

Proof. We know that $P_H = I$. Putting $A = 0$ and $u_n = x_n$ for all $n = 1, 2, ...$, by Theorem 4.4 we obtain the desired result.

Remark 4.1. Corollary 4.2 generalizes and extends Theorem 3.1 in [6]. Corollary 4.4 generalizes and extends Theorem 3.4 in [7]. If $\alpha_n = 1$ for every $n = 1, 2, ...$, by Corollary 4.4, we recover Theorem 3.1 in [8].

REFERENCES

[1] S. D. Flam, A. S. Antipin, *Equilibrium programming using proximal-like algorithms,* Math. Program, **78**(1997) 29-41.

[2] E. Blum, W. Oettli, *From optimization and variational inequalities to equilibrium problems*, Math. Stud., **63**(1994) 123-145.

[3] K. Goebel, W. A. Kirk, *Topics on Metric Fixed-Point Theory,* Cambridge University Press, Cambridge, England, 1990.

[4] P. L. Combettes, S. A. Hirstoaga, *Equilibrium programming in Hilbert spaces*, J. Nonlinear Convex Anal., **6**(2005), 117-136.

[5] S. Takahashi, and W. Takahashi, *Viscosity approximation methods for equilibrium problems and fixed point problems in Hilbert spaces*, J. Math. Anal. Appl., **331**(2006), 506-515.

[6] A. Tada, and W. Takahashi, *Weak and Strong Convergence Theorems for a Nonexpansive Mapping and an Equilibrium Problem*, J. Optim. Theory Appl., **133**(2007), 359-370.

[7] K. Nakajo and W. Takahashi, *Strong convergence theorems for nonexpansive mappings and nonexpansive semigroups*, J. Math. Anal. Appl., **279**(2003), 372-379.

[8] M. Kikkawa and W. Takahashi, *Approximating Fixed Points of Infinite Nonexpansive Mappings by the Hybrid Method*, J. Optim. Theory Appl., **117**(2003), 93-101.

[9] Y. Su, M. Shang and X. Qin, *An iterative method of solutions for equilibrium and optimization problems*, Nonlinear Analysis (2007), doi:10.1016/j.na.2007.08.045.

[10] L.C. Zeng, J.C. Yao, *Strong convergence theorem by an extragradient method for fixed point problems and variational inequality problems,* Taiwan. J. Math., **10**(2006), 1293-1303.

[11] Y. Yao, J.-C. Yao, *On modified iterative method for nonexpansive mappings and monotone mappings*, Appl. Math. Comput., **186**(2)(2007), 1551-1558.

[12] Z. Opial, *Weak convergence of the sequence of successive approximation for nonexpansive mappings*, Bull. Amer. Math. Soc., **73**(1967), 561-597.

[13] R. T. Rockafellar, *On the maximality of sums of nonlinear monotone operators*, Trans. Amer. Math. Soc., **149** (1970) 75-88.

[14] K. Shimoji and W. Takahashi, *Strong convergence to common fixed points of infinite nonexpansive mappings and applications*, Taiwanese Journal of Mathematics, **5**(2)(2001), 387-404.

[15] W. Takahashi and K. Shimoji, *Convergence Theorems for Nonexpansive Mappings and Feasibility Problems*, Mathematical and Computer Modelling, **32**(2000), 1463-1471.

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