

**LERAY-SCHAUDER, LEFSCHETZ AND
KRASNOSELSKII FIXED POINT THEORY IN FRÉCHET
SPACES FOR GENERAL CLASSES OF VOLTERRA
OPERATORS**

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Abstract. New fixed point theorems for maps (single-valued and multi-valued) between Fréchet spaces are presented. The proof relies on fixed point theory in Banach spaces and viewing a Fréchet space as the projective limit of a sequence of Banach spaces. In particular we obtain an applicable Leray-Schauder alternative, a Lefschetz fixed point theorem and a Krasnoselskii fixed point theorem in Fréchet spaces for general classes of maps.

Key Words and Phrases: fixed point theory, Fréchet space, single-valued and multivalued maps, projective limits.

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1. INTRODUCTION

This paper presents new fixed point theorems for maps (single-valued and multi-valued) between Fréchet spaces. In the literature [1, 3, 4] one usually assumes the map F is defined on a subset X of a Fréchet space E and its restriction (again called F) is well defined on $\overline{X_n}$ (see Section 2). In general, of course, for Volterra operators the restriction is always defined on X_n and in most applications it is in fact defined on $\overline{X_n}$ and usually even on E_n (see Section 2). In this paper we make use of the fact that the restriction is

well defined on X_n and we only assume it admits an extension (satisfying certain properties) on $\overline{X_n}$. We also show how easily one can extend fixed point theory in Banach spaces to fixed point theory in Fréchet spaces. In particular we obtain an applicable Leray-Schauder alternative, a Lefschetz fixed point theorem and a Krasnoselskii fixed point theorem in Fréchet spaces for Volterra type operators.

Suppose X and Y are Hausdorff topological spaces. Given a class \mathcal{X} of maps, $\mathcal{X}(X, Y)$ denotes the set of multi-valued maps $F : X \rightarrow 2^Y$ (where 2^Y denotes the family of all nonempty subsets of Y) belonging to \mathcal{X} , and \mathcal{X}_c the set of finite compositions of maps in \mathcal{X} . A class \mathcal{U} of maps is defined by the following properties:

- (i). \mathcal{U} contains the class \mathcal{C} of single-valued continuous functions;
 - (ii). each $F \in \mathcal{U}_c$ is upper semicontinuous and compact-valued;
- and
- (iii). for any polytope P , $F \in \mathcal{U}_c(P, P)$ has a fixed point, where the intermediate spaces of composites are suitably chosen for each \mathcal{U} .

Definition 1.1. $F \in \mathcal{U}_c^\kappa(X, Y)$ if for any compact subset K of X , there is a $G \in \mathcal{U}_c(K, Y)$ with $G(x) \subseteq F(x)$ for each $x \in K$.

Examples of \mathcal{U}_c^κ maps are the Kakutani maps, the acyclic maps, the approximable maps, and the maps admissible in the sense of Górniewicz.

Existence in Section 2 is based on a Leray-Schauder alternative [2] which we state here for the convenience of the reader.

Theorem 1.1. *Let E be a Banach space, U an open convex subset of E and $0 \in U$. Suppose $F \in \mathcal{U}_c^\kappa(\overline{U}, E)$ is a upper semicontinuous countably condensing map with $x \notin \lambda Fx$ for $x \in \partial U$ and $\lambda \in (0, 1)$. Then F has a fixed point in \overline{U} .*

Also, existence in Section 2 will be based on some Lefschetz type fixed point theory. Let X , Y and Γ be Hausdorff topological spaces. A continuous single valued map $p : \Gamma \rightarrow X$ is called a Vietoris map (written $p : \Gamma \rightrightarrows X$) if the following two conditions are satisfied:

- (i). for each $x \in X$, the set $p^{-1}(x)$ is acyclic;
- (ii). p is a proper map i.e. for every compact $A \subseteq X$ we have that $p^{-1}(A)$ is compact.

Let $D(X, Y)$ be the set of all pairs $X \xleftarrow{p} \Gamma \xrightarrow{q} Y$ where p is a Vietoris map and q is continuous. We will denote every such diagram by (p, q) . Given two

diagrams (p, q) and (p', q') , where $X \xleftarrow{p'} \Gamma \xrightarrow{q'} Y$, we write $(p, q) \sim (p', q')$ if there are maps $f : \Gamma \rightarrow \Gamma'$ and $g : \Gamma' \rightarrow \Gamma$ such that $q' \circ f = q$, $p' \circ f = p$, $q \circ g = q'$ and $p \circ g = p'$. The equivalence class of a diagram $(p, q) \in D(X, Y)$ with respect to \sim is denoted by

$$\phi = \{X \xleftarrow{p} \Gamma \xrightarrow{q} Y\} : X \rightarrow Y$$

or $\phi = [(p, q)]$ and is called a morphism from X to Y . We let $M(X, Y)$ be the set of all such morphisms. For any $\phi \in M(X, Y)$ a set $\phi(x) = qp^{-1}(x)$ where $\phi = [(p, q)]$ is called an image of x under a morphism ϕ .

Consider vector spaces over a field K . Let E be a vector space and $f : E \rightarrow E$ an endomorphism. Now let $N(f) = \{x \in E : f^{(n)}(x) = 0 \text{ for some } n\}$ where $f^{(n)}$ is the n^{th} iterate of f , and let $\tilde{E} = E \setminus N(f)$. Since $f(N(f)) \subseteq N(f)$ we have the induced endomorphism $\tilde{f} : \tilde{E} \rightarrow \tilde{E}$. We call f admissible if $\dim \tilde{E} < \infty$; for such f we define the generalized trace $Tr(f)$ of f by putting $Tr(f) = tr(\tilde{f})$ where tr stands for the ordinary trace.

Let $f = \{f_q\} : E \rightarrow E$ be an endomorphism of degree zero of a graded vector space $E = \{E_q\}$. We call f a Leray endomorphism if (i). all f_q are admissible and (ii). almost all \tilde{E}_q are trivial. For such f we define the generalized Lefschetz number $\Lambda(f)$ by

$$\Lambda(f) = \sum_q (-1)^q Tr(f_q).$$

Let H be the Čech homology functor with compact carriers and coefficients in the field of rational numbers K from the category of Hausdorff topological spaces and continuous maps to the category of graded vector spaces and linear maps of degree zero. Thus $H(X) = \{H_q(X)\}$ is a graded vector space, $H_q(X)$ being the q -dimensional Čech homology group with compact carriers of X . For a continuous map $f : X \rightarrow X$, $H(f)$ is the induced linear map $f_\star = \{f_{\star q}\}$ where $f_{\star q} : H_q(X) \rightarrow H_q(X)$.

The Čech homology functor can be extended to a category of morphisms (see [7], pp. 364) and also note the homology functor H extends over this category i.e. for a morphism

$$\phi = \{X \xleftarrow{p} \Gamma \xrightarrow{q} Y\} : X \rightarrow Y$$

we define the induced map

$$H(\phi) = \phi_* : H(X) \rightarrow H(Y)$$

by putting $\phi_* = q_* \circ p_*^{-1}$.

Let $\phi : X \rightarrow Y$ be a multi-valued map (note for each $x \in X$ we assume $\phi(x)$ is a nonempty subset of Y). A pair (p, q) of single valued continuous maps of the form $X \xrightarrow{p} \Gamma \xrightarrow{q} Y$ is called a selected pair of ϕ (written $(p, q) \subset \phi$) if the following two conditions hold:

(i). p is a Vietoris map

and

(ii). $q(p^{-1}(x)) \subset \phi(x)$ for any $x \in X$.

Definition 1.2. A upper semicontinuous compact map $\phi : X \rightarrow Y$ is said to be admissible (and we write $\phi \in Ad(X, Y)$) provided there exists a selected pair (p, q) of ϕ .

Definition 1.3. A upper semicontinuous map $\phi : X \rightarrow Y$ is said to be admissible in the sense of Górniewicz (and we write $\phi \in AD(X, Y)$) provided there exists a selected pair (p, q) of ϕ .

Definition 1.4. A map $\phi \in Ad(X, X)$ is said to be a Lefschetz map if for each selected pair $(p, q) \subset \phi$ the linear map $q_* p_*^{-1} : H(X) \rightarrow H(X)$ (the existence of p_*^{-1} follows from the Vietoris Theorem) is a Leray endomorphism.

If $\phi : X \rightarrow X$ is a Lefschetz map, we define the Lefschetz set $\Lambda(\phi)$ (or $\Lambda_X(\phi)$) by

$$\Lambda(\phi) = \{\Lambda(q_* p_*^{-1}) : (p, q) \subset \phi\}.$$

Definition 1.5. A Hausdorff topological space X is said to be a Lefschetz space provided every $\phi \in Ad(X, X)$ is a Lefschetz map and $\Lambda(\phi) \neq \{0\}$ implies ϕ has a fixed point.

Also, in Section 2 we present Krasnoselskii compression and expansion theorems in the Fréchet space setting. Let $E = (E, |\cdot|)$ be a normed linear space and $C \subseteq E$ a closed cone. For each $r > 0$ let us denote $BC(0, r) = \{x \in C : |x| \leq r\}$. Notice that $BC(0, r) = B(0, R) \cap C$ where $B(0, r) = \{x \in E : |x| \leq r\}$. Our next result, Theorem 1.2, was established in [12] and Theorem 1.3 can be found in [8].

Theorem 1.2. Let $E = (E, |\cdot|)$ be a normed linear space, $C \subseteq E$ a closed cone, r, R constants and $0 < r < R$. Suppose $F \in \mathcal{U}_c^k(B(0, R) \cap C, C)$ is

compact with

$$\left\{ \begin{array}{l} |y| \geq |x| \text{ for all } y \in F(x) \text{ and } x \in \partial BC(0, r) \\ \text{and} \\ |y| \leq |x| \text{ for all } y \in F(x) \text{ and } x \in \partial BC(0, R). \end{array} \right.$$

Then F has a fixed point in $BC_{r,R} = \{x \in C : r \leq \|x\| \leq R\}$.

Theorem 1.3. Let $E = (E, |\cdot|)$ be a normed linear space, $C \subseteq E$ a closed cone, r, R constants and $0 < r < R$. Suppose $F \in AD(C, C)$ is completely continuous with

$$\left\{ \begin{array}{l} |y| \leq |x| \text{ for all } y \in F(x) \text{ and } x \in \partial BC(0, r) \\ \text{and} \\ |y| \geq |x| \text{ for all } y \in F(x) \text{ and } x \in \partial BC(0, R). \end{array} \right.$$

Then F has a fixed point in $BC_{r,R}$.

Now let I be a directed set with order \leq and let $\{E_\alpha\}_{\alpha \in I}$ be a family of locally convex spaces. For each $\alpha \in I, \beta \in I$ for which $\alpha \leq \beta$ let $\pi_{\alpha,\beta} : E_\beta \rightarrow E_\alpha$ be a continuous map. Then the set

$$\left\{ x = (x_\alpha) \in \prod_{\alpha \in I} E_\alpha : x_\alpha = \pi_{\alpha,\beta}(x_\beta) \text{ for all } \alpha, \beta \in I, \alpha \leq \beta \right\}$$

is a closed subset of $\prod_{\alpha \in I} E_\alpha$ and is called the projective limit of $\{E_\alpha\}_{\alpha \in I}$ and is denoted by $\lim_{\leftarrow} E_\alpha$ (or $\lim_{\leftarrow} \{E_\alpha, \pi_{\alpha,\beta}\}$ or the generalized intersection ([9], pp. 439) $\cap_{\alpha \in I} E_\alpha$.) We also refer to [5] and [6] for the definition of the projective (or inverse) limit, as well as, for its properties with respect to fixed point theory.

2. FIXED POINT THEORY IN FRÉCHET SPACES

Let $E = (E, \{|\cdot|_n\}_{n \in N})$ be a Fréchet space with the topology generated by a family of seminorms $\{|\cdot|_n : n \in N\}$; here $N = \{1, 2, \dots\}$. We assume that the family of seminorms satisfies

$$|x|_1 \leq |x|_2 \leq |x|_3 \leq \dots \text{ for every } x \in E. \tag{2.1}$$

A subset X of E is bounded if for every $n \in N$ there exists $r_n > 0$ such that $|x|_n \leq r_n$ for all $x \in X$. For $r > 0$ and $x \in E$ we denote $B(x, r) = \{y \in E : |x - y|_n \leq r \text{ for all } n \in N\}$. To E we associate a sequence of Banach

spaces $\{(\mathbf{E}_n, |\cdot|_n)\}$ described as follows. For every $n \in N$ we consider the equivalence relation \sim_n defined by

$$x \sim_n y \text{ iff } |x - y|_n = 0. \tag{2.2}$$

We denote by $\mathbf{E}^n = (E / \sim_n, |\cdot|_n)$ the quotient space, and by $(\mathbf{E}_n, |\cdot|_n)$ the completion of \mathbf{E}^n with respect to $|\cdot|_n$ (the norm on \mathbf{E}^n induced by $|\cdot|_n$ and its extension to \mathbf{E}_n are still denoted by $|\cdot|_n$). This construction defines a continuous map $\mu_n : E \rightarrow \mathbf{E}_n$. Now since (2.1) is satisfied the seminorm $|\cdot|_n$ induces a seminorm on \mathbf{E}_m for every $m \geq n$ (again this seminorm is denoted by $|\cdot|_n$). Also (2.2) defines an equivalence relation on \mathbf{E}_m from which we obtain a continuous map $\mu_{n,m} : \mathbf{E}_m \rightarrow \mathbf{E}_n$ since \mathbf{E}_m / \sim_n can be regarded as a subset of \mathbf{E}_n . Now $\mu_{n,m} \mu_{m,k} = \mu_{n,k}$ if $n \leq m \leq k$ and $\mu_n = \mu_{n,m} \mu_m$ if $n \leq m$. We now assume the following condition holds:

$$\left\{ \begin{array}{l} \text{for each } n \in N, \text{ there exists a Banach space } (E_n, |\cdot|_n) \\ \text{and an isomorphism (between normed spaces) } j_n : \mathbf{E}_n \rightarrow E_n. \end{array} \right. \tag{2.3}$$

- Remark 2.1.** (i). For convenience the norm on E_n is denoted by $|\cdot|_n$.
 (ii). In our applications $\mathbf{E}_n = \mathbf{E}^n$ for each $n \in N$.
 (iii). Note if $x \in \mathbf{E}_n$ (or \mathbf{E}^n) then $x \in E$. However if $x \in E_n$ then x is not necessarily in E and in fact E_n is easier to use in applications (even though E_n is isomorphic to \mathbf{E}_n). For example if $E = C[0, \infty)$, then \mathbf{E}^n consists of the class of functions in E which coincide on the interval $[0, n]$ and $E_n = C[0, n]$.

Finally we assume

$$\left\{ \begin{array}{l} E_1 \supseteq E_2 \supseteq \dots \text{ and for each } n \in N, \\ |j_n \mu_{n,n+1} j_{n+1}^{-1} x|_n \leq |x|_{n+1} \text{ for all } x \in E_{n+1}. \end{array} \right. \tag{2.4}$$

Let $\lim_{\leftarrow} E_n$ (or $\cap_1^\infty E_n$ where \cap_1^∞ is the generalized intersection [9]) denote the projective limit of $\{E_n\}_{n \in N}$ (note $\pi_{n,m} = j_n \mu_{n,m} j_m^{-1} : E_m \rightarrow E_n$ for $m \geq n$) and note $\lim_{\leftarrow} E_n \cong E$. For convenience we write $E = \lim_{\leftarrow} E_n$.

For each $X \subseteq E$ and each $n \in N$ we set $X_n = j_n \mu_n(X)$, and we let $\overline{X_n}$, $\text{int } X_n$ and ∂X_n denote respectively the closure, the interior and the boundary of X_n with respect to $|\cdot|_n$ in E_n . Also the pseudo-interior of X is defined by

$$\text{pseudo-int}(X) = \{x \in X : j_n \mu_n(x) \in \overline{X_n} \setminus \partial X_n \text{ for every } n \in N\}.$$

The set X is pseudo-open if $X = pseudo - int(X)$. For $r > 0$ and $x \in E_n$ we denote $B_n(x, r) = \{y \in E_n : |x - y|_n \leq r\}$.

Let $M \subseteq E$ and consider the map $F : M \rightarrow 2^E$. Assume for each $n \in N$ and $x \in M$ that $j_n \mu_n F(x)$ is closed. Let $n \in N$ and $M_n = j_n \mu_n(M)$. Since we only consider Volterra type operators we assume

$$\text{if } x, y \in E \text{ with } |x - y|_n = 0 \text{ then } H_n(F(x), F(y)) = 0; \tag{2.5}$$

here H_n denotes the appropriate generalized Hausdorff distance (alternatively we could assume for all $n \in N$ and for all $x, y \in M$ if $j_n \mu_n x = j_n \mu_n y$ then $j_n \mu_n F(x) = j_n \mu_n F(y)$). Of course here we do not need to assume that $j_n \mu_n F(x)$ is closed for each $n \in N$ and $x \in M$). Now (2.5) guarantees that we can define (a well defined) F_n on M_n as follows:

For $y \in M_n$ there exists a $x \in M$ with $y = j_n \mu_n(x)$ and we let

$$F_n(y) = j_n \mu_n F x$$

(we could of course call it $F(y)$ since it is clear in the situation we use it); note $F_n : M_n \rightarrow C(E_n)$ and note if there exists a $z \in M$ with $y = j_n \mu_n(z)$ then $j_n \mu_n F(x) = j_n \mu_n F(z)$ from (2.5) (here $C(E_n)$ denotes the family of nonempty closed subsets of E_n). In this paper we assume F_n will be defined on $\overline{M_n}$ i.e. we assume the F_n described above admits an extension (again we call it F_n) $F_n : \overline{M_n} \rightarrow 2^{E_n}$ (we will assume certain properties on the extension).

We now show how easily one can extend fixed point theory in Banach spaces to applicable fixed point theory in Fréchet spaces.

Theorem 2.1. *Let E and E_n be as described above and let $F : X \rightarrow 2^E$ where X is a convex subset of E . Also assume for each $n \in N$ and $x \in X$ that $j_n \mu_n F(x)$ is closed and also for each $n \in N$ that $F_n : \overline{X_n} \rightarrow 2^{E_n}$ is as described above. Suppose the following conditions are satisfied:*

$$0 \in pseudo - int(X) \tag{2.6}$$

$$\left\{ \begin{array}{l} \text{for each } n \in N, F_n \in \mathcal{U}_c^k(\overline{int X_n}, E_n) \text{ is a} \\ \text{upper semicontinuous countably condensing map} \end{array} \right. \tag{2.7}$$

$$\left\{ \begin{array}{l} \text{for each } n \in N, y \notin \lambda F_n(y) \text{ in } E_n \text{ for all} \\ \lambda \in (0, 1] \text{ and } y \in \partial int X_n \end{array} \right. \tag{2.8}$$

and

$$\begin{cases} \text{for each } n \in \{2, 3, \dots\} \text{ if } y \in \text{int } X_n \text{ solves } y \in F_n(y) \\ \text{in } E_n \text{ then } j_k \mu_{k,n} j_n^{-1}(y) \in \text{int } X_k \text{ for } k \in \{1, \dots, n-1\}. \end{cases} \quad (2.9)$$

Then F has a fixed point in E .

Proof. Fix $n \in N$. We would like to apply Theorem 1.1. To do so we need to show

$$\text{int } X_n \text{ is convex.} \quad (2.10)$$

To see this let $\hat{x}, \hat{y} \in \mu_n(X)$ and $\lambda \in [0, 1]$. Then for every $x \in \mu_n^{-1}(\hat{x})$ and $y \in \mu_n^{-1}(\hat{y})$ we have $\lambda x + (1 - \lambda)y \in X$ since X is convex and so $\lambda \hat{x} + (1 - \lambda)\hat{y} = \lambda \mu_n(x) + (1 - \lambda)\mu_n(y)$. It is easy to check that $\lambda \mu_n(x) + (1 - \lambda)\mu_n(y) = \mu_n(\lambda x + (1 - \lambda)y)$ so as a result

$$\lambda \hat{x} + (1 - \lambda)\hat{y} = \mu_n(\lambda x + (1 - \lambda)y) \in \mu_n(X),$$

and so $\mu_n(X)$ is convex. Now since j_n is linear we have $X_n = j_n(\mu_n(X))$ is convex and as a result $\text{int } X_n$ is convex, so (2.10) holds. From Theorem 1.1 for each $n \in N$ there exists $y_n \in \text{int } X_n$ with $y_n \in F_n(y_n)$ (we apply Theorem 1.1 with $U = \text{int } X_n$ and note $0 = j_n \mu_n(0) \in \overline{X_n} \setminus \partial X_n = \text{int } X_n$). Lets look at $\{y_n\}_{n \in N}$. Notice $y_1 \in \text{int } X_1$ and $j_1 \mu_{1,k} j_k^{-1}(y_k) \in \text{int } X_1$ for $k \in N \setminus \{1\}$ from (2.9). Note $j_1 \mu_{1,n} j_n^{-1}(y_n) \in F_1(j_1 \mu_{1,n} j_n^{-1}(y_n))$ in E_1 ; to see note for $n \in N$ fixed there exists a $x \in E$ with $y_n = j_n \mu_n(x)$ so $j_n \mu_n(x) \in F_n(y_n) = j_n \mu_n F(x)$ on E_n so on E_1 we have

$$\begin{aligned} j_1 \mu_{1,n} j_n^{-1}(y_n) &= j_1 \mu_{1,n} j_n^{-1} j_n \mu_n(x) \in j_1 \mu_{1,n} j_n^{-1} j_n \mu_n F(x) \\ &= j_1 \mu_{1,n} \mu_n F(x) = j_1 \mu_1 F(x) = F_1(j_1 \mu_1(x)) \\ &= F_1(j_1 \mu_{1,n} j_n^{-1} j_n \mu_n(x)) = F_1(j_1 \mu_{1,n} j_n^{-1}(y_n)). \end{aligned}$$

As a result $j_1 \mu_{1,n} j_n^{-1}(y_n) \in F_1(j_1 \mu_{1,n} j_n^{-1}(y_n))$ in E_1 , $j_1 \mu_{1,n} j_n^{-1}(y_n) \in \text{int } X_1$ for $n \in N$, together with (2.7) implies there is a subsequence (note the countably condensing map F_1 is hemicompact) N_1^* of N and a $z_1 \in \overline{\text{int } X_1}$ with $j_1 \mu_{1,n} j_n^{-1}(y_n) \rightarrow z_1$ in E_1 as $n \rightarrow \infty$ in N_1^* and $z_1 \in F_1 z_1$ since F_1 is upper semicontinuous. Also (2.8) implies $z_1 \in \text{int } X_1$. Let $N_1 = N_1^* \setminus \{1\}$. Now $j_2 \mu_{2,n} j_n^{-1}(y_n) \in \text{int } X_2$ for $n \in N_1$ together with (2.7) guarantees that there exists a subsequence N_2^* of N_1 and a $z_2 \in \overline{\text{int } X_2}$ with $j_2 \mu_{2,n} j_n^{-1}(y_n) \rightarrow z_2$ in E_2 as $n \rightarrow \infty$ in N_2^* and $z_2 \in F_2(z_2)$. Also (2.8) implies $z_2 \in \text{int } X_2$. Note from (2.4) and the

uniqueness of limits that $j_1 \mu_{1,2} j_2^{-1} z_2 = z_1$ in E_1 since $N_2^* \subseteq N_1$ (note $j_1 \mu_{1,n} j_n^{-1}(y_n) = j_1 \mu_{1,2} j_2^{-1} j_2 \mu_{2,n} j_n^{-1}(y_n)$ for $n \in N_2^*$). Let $N_2 = N_2^* \setminus \{2\}$. Proceed inductively to obtain subsequences of integers

$$N_1^* \supseteq N_2^* \supseteq \dots, \quad N_k^* \subseteq \{k, k + 1, \dots\}$$

and $z_k \in \overline{\text{int } X_k}$ with $j_k \mu_{k,n} j_n^{-1}(y_n) \rightarrow z_k$ in E_k as $n \rightarrow \infty$ in N_k^* and $z_k \in F_k z_k$. Also (2.8) implies $z_k \in \text{int } X_k$. Note $j_k \mu_{k,k+1} j_{k+1}^{-1} z_{k+1} = z_k$ in E_k for $k \in \{1, 2, \dots\}$. Also let $N_k = N_k^* \setminus \{k\}$.

Fix $k \in N$. Now $z_k \in F_k(z_k)$ in E_k . Note as well that

$$\begin{aligned} z_k &= j_k \mu_{k,k+1} j_{k+1}^{-1} z_{k+1} = j_k \mu_{k,k+1} j_{k+1}^{-1} j_{k+1} \mu_{k+1,k+2} j_{k+2}^{-1} z_{k+2} \\ &= j_k \mu_{k,k+2} j_{k+2}^{-1} z_{k+2} = \dots = j_k \mu_{k,m} j_m^{-1} z_m = \pi_{k,m} z_m \end{aligned}$$

for every $m \geq k$. We can do this for each $k \in N$. As a result $y = (z_k) \in \lim_{\leftarrow} E_n = E$ and also note $y \in X$ since $z_k \in \text{int } X_k$ for each $k \in N$. Thus for each $k \in N$ we have

$$j_k \mu_k(y) = z_k \in F_k(z_k) = j_k \mu_k F(y) \text{ in } E_k$$

so $y \in F(y)$ in E . \square

Remark 2.2. Usually in our applications we have $\partial X_n = \partial \text{int } X_n$ (so $\overline{X_n} = \overline{\text{int } X_n}$). If X is pseudo-open then X_n is open in E_n so $\text{int } X_n = X_n$. To see this note $X_n \subseteq \overline{X_n} \setminus \partial X_n$ since if $y \in X_n$ then there exists $x \in X$ with $y = j_n \mu_n(x)$ and this together with $X = \text{pseudo-int } X$ yields $j_n \mu_n(x) \in \overline{X_n} \setminus \partial X_n$ i.e. $y \in \overline{X_n} \setminus \partial X_n$. In addition notice

$$\overline{X_n} \setminus \partial X_n = (\text{int } X_n \cup \partial X_n) \setminus \partial X_n = \text{int } X_n \setminus \partial X_n = \text{int } X_n$$

since $\text{int } X_n \cap \partial X_n = \emptyset$. Consequently

$$X_n \subseteq \overline{X_n} \setminus \partial X_n = \text{int } X_n, \text{ so } X_n = \text{int } X_n.$$

Remark 2.3. We can replace (2.9) in Theorem 2.1 with

$$\begin{cases} \text{for each } n \in \{2, 3, \dots\} \text{ if } y \in \text{int } X_n \text{ solves } y \in F_n(y) \\ \text{in } E_n \text{ then } j_k \mu_{k,n} j_n^{-1}(y) \in X_k \text{ for } k \in \{1, \dots, n - 1\} \end{cases}$$

provided (2.7) is changed to

$$\begin{cases} \text{for each } n \in N, F_n \in \mathcal{U}_c^k(\overline{X_n}, E_n) \text{ is a} \\ \text{upper semicontinuous countably condensing map} \end{cases} \tag{2.7}^*$$

Remark 2.4. It is possible to replace $\lambda \in (0, 1]$ in (2.8) with $\lambda \in (0, 1)$ provided in this case we take X to be a closed convex subset of E and (2.9) is changed to

$$\begin{cases} \text{for each } n \in \{2, 3, \dots\} \text{ if } y \in \overline{\text{int } X_n} \text{ solves } y \in F_n(y) \\ \text{in } E_n \text{ then } j_k \mu_{k,n} j_n^{-1}(y) \in \overline{\text{int } X_k} \text{ for } k \in \{1, \dots, n-1\}. \end{cases} \quad (2.9)^*$$

The proof follows as in Theorem 2.1 except in this case $y_n \in \overline{\text{int } X_n}$ and $z_k \in \overline{\text{int } X_k}$. Also from $y = (z_k) \in \lim_{\leftarrow} E_n = E$ and $\pi_{k,m}(y_m) \rightarrow z_k$ in E_k as $m \rightarrow \infty$ we can conclude that $y \in \overline{X} = X$ (note $q \in \overline{X}$ if and only if for every $k \in N$ there exists $(x_{k,m}) \in X$, $x_{k,m} = \pi_{k,n}(x_{n,m})$ for $n \geq k$ with $x_{k,m} \rightarrow j_k \mu_k(q)$ in E_k as $m \rightarrow \infty$). Thus $z_k = j_k \mu_k(y) \in X_k$ and so $j_k \mu_k(y) \in j_k \mu_k F(y)$ in E_k as before.

Note here also that (2.9)* could be replaced by

$$\begin{cases} \text{for each } n \in \{2, 3, \dots\} \text{ if } y \in \overline{\text{int } X_n} \text{ solves } y \in F_n(y) \\ \text{in } E_n \text{ then } j_k \mu_{k,n} j_n^{-1}(y) \in \overline{X_k} \text{ for } k \in \{1, \dots, n-1\} \end{cases}$$

provided (2.7) is changed to (2.7)* .

Example 2.1. Consider

$$y(t) \in h(t) + \int_0^t k(t, s) F(s, y(s)) ds \text{ for } t \in [0, T] \quad (2.11)$$

where $0 < T \leq \infty$ is fixed and $F : [0, T] \times \mathbf{R}^m \rightarrow CK(\mathbf{R}^m)$ where $CK(\mathbf{R}^m)$ denotes the family of nonempty convex compact subsets of \mathbf{R}^m . In addition we assume $h : [0, T] \rightarrow \mathbf{R}^m$ and the matrix valued function $k : \{(s, t) : 0 \leq s \leq t \leq t_n\} \rightarrow L_{m \times m}[0, t_n]$ for each $n \in N = \{1, 2, \dots\}$; here $t_n \in [0, T]$ with $t_n \uparrow T$.

We will now show that (2.11) has a solution if F is bounded by a L^1_{loc} -Carathéodory function g and if for each $n \in N$, the ordinary differential equation

$$\begin{cases} v'(t) = \sup_{t \in [0, t_n]} k(t) g(t, v(t)) \text{ a.e. } t \in [0, t_n] \\ v(0) = |h|_n \end{cases}$$

has a maximal solution. Recall a function $g : [0, t_n] \times \mathbf{R} \rightarrow \mathbf{R}$ ($n \in N$ fixed) is a L^1 -Carathéodory function if

- (a). the map $t \mapsto g(t, y)$ is measurable for all $y \in \mathbf{R}$,
- (b). the map $y \mapsto g(t, y)$ is continuous for a.e. $t \in [0, t_n]$,

and

(c). for any $r > 0$, $\exists \mu_r \in L^1[0, t_n]$ such that $|y| \leq r$ implies $|g(t, y)| \leq \mu_r(t)$ for a.e. $t \in [0, t_n]$.

A function $g : [0, T) \times \mathbf{R} \rightarrow \mathbf{R}$ is a L^1_{loc} -Carathéodory function if (a), (b) and (c) above hold when g is restricted to $[0, t_n] \times \mathbf{R}$ for any $n \in N$. For notational purposes if $u \in C[0, T)$ then for every $n \in \{1, 2, \dots\} = N$ we define the seminorms $\rho_n(u)$ by

$$\rho_n(u) = \sup_{t \in [0, t_n]} |u(t)|$$

where $t_n \uparrow T$. Note $C[0, T)$ is a locally convex linear topological space. The topology on $C[0, T)$, induced by the seminorms $\{\rho_n\}_{n \in N}$, is the topology of uniform convergence on every compact interval of $[0, T)$.

In particular if

$$\left\{ \begin{array}{l} \text{for each } n \in N, x \mapsto F(t, x) \text{ is upper} \\ \text{semicontinuous for a.e. } t \in [0, t_n] \end{array} \right. \quad (2.12)$$

$$\left\{ \begin{array}{l} \text{for each } n \in N, \text{ for each measurable } u : [0, t_n] \rightarrow \mathbf{R}^m \\ \text{the map } t \mapsto F(t, u(t)) \text{ has measurable single} \\ \text{valued selections} \end{array} \right. \quad (2.13)$$

$$\left\{ \begin{array}{l} \text{there exists a } L^1_{loc}[0, T) - \text{Carathéodory function} \\ g : [0, T) \times [0, \infty) \rightarrow [0, \infty) \text{ such that} \\ |F(t, x)| \leq g(t, |x|) \text{ for a.e. } t \in [0, T) \\ \text{and all } x \in \mathbf{R}^m \end{array} \right. \quad (2.14)$$

$$h \in C([0, T), \mathbf{R}^m) \quad (2.15)$$

$$g(t, x) \text{ is nondecreasing in } x \text{ for a.e. } t \in [0, T) \quad (2.16)$$

$$\left\{ \begin{array}{l} \text{for each } n \in N, \text{ for each } t \in [0, t_n] \text{ we have} \\ \text{that } k(t, s) \text{ is measurable on } [0, t] \text{ and } k(t) \\ = \text{ess sup } |k(t, s)|, 0 \leq s \leq t, \text{ is bounded on } [0, t_n] \end{array} \right. \quad (2.17)$$

$$\left\{ \begin{array}{l} \text{for each } n \in N, \text{ the map } t \mapsto k_t \text{ is continuous} \\ \text{from } [0, t_n] \text{ to } L^\infty([0, t_n], L_{m \times m}[0, t_n]); \\ \text{here } k_t(s) = k(t, s) \end{array} \right. \quad (2.18)$$

and

$$\left\{ \begin{array}{l} \text{for each } n \in N, \text{ the problem} \\ \left\{ \begin{array}{l} v'(t) = \left(\sup_{t \in [0, t_n]} k(t) \right) g(t, v(t)) \text{ a.e. } t \in [0, t_n] \\ v(0) = |h|_n \end{array} \right. \\ \text{has a maximal solution } r_n(t) \text{ on } [0, t_n] \text{ (here } r_n \in C[0, t_n]). \end{array} \right. \quad (2.19)$$

hold, then (2.11) has at least one solution $y \in C[0, T]$.

To see this note here that $E = C[0, T]$, \mathbf{E}^k consists of the class of functions in E which coincide on the interval $[0, t_k]$, $E_k = C[0, t_k]$ with of course $\pi_{n,m} = j_n \mu_{n,m} j_m^{-1} : E_m \rightarrow E_n$ for $m \geq n$ defined by $\pi_{n,m}(x) = x|_{[0, t_n]}$. We will apply Theorem 2.1 with

$$X = \{u \in C[0, T] : |u|_n \leq w_n \text{ for each } n \in N\};$$

here $|u|_n = \sup_{t \in I_n} |u(t)|$, where $I_n = [0, t_n]$, and $w_n = \sup_{t \in I_n} r_n(t) + 1$.

On any interval $I_n = [0, t_n]$ ($n \in N$) we let \mathcal{F}_n on $C(I_n)$ be defined by

$$\mathcal{F}_n(y) = \{u \in L^1(I_n) : u(t) \in F(t, y(t)) \text{ for a.e. } t \in I_n\}$$

and we let A_n on $L^1(I_n)$ be defined by

$$A_n y(t) = h(t) + \int_0^t k(t, s) y(s) ds,$$

and finally we let $F_n = A_n \circ \mathcal{F}_n$.

Now fix $n \in N$ and note

$$X_n (= \overline{X_n}) = \{y \in C(I_n) : |y|_n \leq w_n\}.$$

Firstly a standard argument from the literature [11] guarantees that

$$F_n : \overline{X_n} \rightarrow CK(E_n) \text{ is upper semicontinuous and compact,}$$

so (2.7) is true.

To see (2.8) fix $n \in N$ and let $y \in C[0, t_n]$ be such that $y \in \lambda F_n y$ for $\lambda \in (0, 1]$. Then

$$|y(t)| \leq |h|_n + \left(\sup_{t \in [0, t_n]} k(t) \right) \int_0^t g(s, |y(s)|) ds \equiv v(t)$$

for $t \in [0, t_n]$. Now (2.16) implies

$$v'(t) = \left(\sup_{t \in [0, t_n]} k(t) \right) g(t, |y(t)|) \leq \left(\sup_{t \in [0, t_n]} k(t) \right) g(t, v(t))$$

almost everywhere, so

$$\begin{cases} v'(t) \leq \left(\sup_{t \in [0, t_n]} k(t)\right) g(t, v(t)) & \text{for a.e. } t \in [0, t_n] \\ v(0) = |h|_n. \end{cases}$$

Now [10] (see Theorem 1.10.2) guarantees that $v(t) \leq r_n(t)$ for $t \in [0, t_n]$, so $|y|_n < w_n$ and (2.8) is satisfied.

To show (2.9) fix $n \in \{2, 3, \dots\}$ and suppose $y \in \text{int } X_n$ is a solution of $y(t) \in h(t) + \int_0^t k(t, s) F(s, y(s)) ds$ for $t \in [0, t_n]$. Next fix $k \in \{1, \dots, n - 1\}$. We must show $y \in \text{int } X_k$. Now since $t_n \uparrow T$ notice $[0, t_k] \subseteq [0, t_n]$ so as a result

$$y(t) \in h(t) + \int_0^t k(t, s) F(s, y(s)) ds \quad \text{for } t \in [0, t_k],$$

so essentially the same reasoning as above yields $|y|_k < w_k$, so $y \in \text{int } X_k$ and (2.9) holds.

The result now follows immediately from Theorem 1.1.

Now we present some Lefschetz type theorems in Fréchet spaces. Let E and E_n be as described above.

Definition 2.1. A set $A \subseteq E$ is said to be PRLS if for each $n \in N$, $A_n \equiv j_n \mu_n (A)$ is a Lefschetz space.

Definition 2.2. A set $A \subseteq E$ is said to be CPRLS if for each $n \in N$, $\overline{A_n}$ is a Lefschetz space.

Theorem 2.2. Let E and E_n be as described above, $C \subseteq E$ is an PRLS and $F : C \rightarrow 2^E$. Also assume for each $n \in N$ and $x \in C$ that $j_n \mu_n F(x)$ is closed and also for each $n \in N$ that $F_n : \overline{C_n} \rightarrow 2^{E_n}$ is as described above. Suppose the following conditions are satisfied:

$$\text{for each } n \in N, F_n \in \text{Ad}(C_n, C_n) \tag{2.20}$$

$$\text{for each } n \in N, \Lambda_{C_n}(F_n) \neq \{0\} \tag{2.21}$$

and

$$\begin{cases} \text{for each } n \in \{2, 3, \dots\} \text{ if } y \in C_n \text{ solves } y \in F_n(y) \text{ in } E_n \\ \text{then } j_k \mu_{k,n} j_n^{-1}(y) \in C_k \text{ for } k \in \{1, \dots, n - 1\}. \end{cases} \tag{2.22}$$

Then F has a fixed point in E .

Proof. For each $n \in N$ there exists $y_n \in C_n$ with $y_n \in F_n(y_n)$ in E_n . Now (2.20) guarantees (see Theorem 2.1) that there exists is a subsequence N_1^* of N and a $z_1 \in C_1$ (note $F_n : C_n \rightarrow 2^{C_n}$ is a compact map) with

$j_1 \mu_{1,n} j_n^{-1}(y_n) \rightarrow z_1$ in E_1 as $n \rightarrow \infty$ in N_1^* and $z_1 \in F_1 z_1$. Let $N_1 = N_1^* \setminus \{1\}$. Proceed inductively to obtain subsequences of integers

$$N_1^* \supseteq N_2^* \supseteq \dots, \quad N_k^* \subseteq \{k, k+1, \dots\}$$

and $z_k \in C_k$ with $j_k \mu_{k,n} j_n^{-1}(y_n) \rightarrow z_k$ in E_k as $n \rightarrow \infty$ in N_k^* and $z_k \in F_k z_k$. Note $j_k \mu_{k,k+1} j_{k+1}^{-1} z_{k+1} = z_k$ in E_k for $k \in \{1, 2, \dots\}$. Also let $N_k = N_k^* \setminus \{k\}$. Now essentially the same reasoning as in Theorem 2.1 guarantees the result. \square

Essentially the same reasoning as in Theorem 2.2 (with Remark 2.4) yields the following result.

Theorem 2.3. *Let E and E_n be as described above, $C \subseteq E$ is an CPRLS and $F : C \rightarrow 2^E$. Also assume C is a closed subset of E , for each $n \in N$ and $x \in C$ that $j_n \mu_n F(x)$ is closed and also for each $n \in N$ that $F_n : \overline{C_n} \rightarrow 2^{E_n}$ is as described above. Suppose the following conditions are satisfied:*

$$\text{for each } n \in N, F_n \in \text{Ad}(\overline{C_n}, \overline{C_n}) \quad (2.23)$$

$$\text{for each } n \in N, \Lambda_{\overline{C_n}}(F_n) \neq \{0\} \quad (2.24)$$

and

$$\begin{cases} \text{for each } n \in \{2, 3, \dots\} \text{ if } y \in \overline{C_n} \text{ solves } y \in F_n(y) \text{ in } E_n \\ \text{then } j_k \mu_{k,n} j_n^{-1}(y) \in \overline{C_k} \text{ for } k \in \{1, \dots, n-1\}. \end{cases} \quad (2.25)$$

Then F has a fixed point in E .

Remark 2.5. Note we can remove the assumption in Theorem 2.3 that C is a closed subset of E if we assume $F : Y \rightarrow 2^E$ with $C \subseteq Y$ and $\overline{C_n} \subseteq Y_n$ for each $n \in N$.

Remark 2.6. Of course there are analogue results for compact morphisms (see the ideas here and in [3]).

Next we present some Krasnoselskii results in the Fréchet space setting.

Theorem 2.4. *Let E and E_n be as described in the beginning of Section 2, C a closed cone in E , r and R are constants with $0 < r < R$, and $F : Y \rightarrow 2^E$ with $Y \subseteq E$, and $\overline{W_n} = B_n(0, R) \cap \overline{C_n} \subseteq Y_n$ for each $n \in N$; here $W_n = \overline{C_n} \cap V_n$ where $V_n = \{x \in E_n : |x|_n < R\}$. Also assume for each $n \in N$ and $x \in Y$ that $j_n \mu_n F(x)$ is closed and also for each $n \in N$ that $F_n : \overline{W_n} \rightarrow 2^{E_n}$ is as described above. Suppose the following conditions are*

satisfied (here $\Omega_n = U_n \cap \overline{C_n}$ with $U_n = \{x \in E_n : |x|_n < r\}$):

$$\left\{ \begin{array}{l} \text{for each } n \in N, F_n \in \mathcal{U}_c^k(B_n(0, R) \cap \overline{C_n}, \overline{C_n}) \\ \text{is a upper semicontinuous compact map} \end{array} \right. \quad (2.26)$$

$$\left\{ \begin{array}{l} |y|_n \geq |x|_n \text{ for all } y \in F_n(x) \text{ and } x \in \partial\Omega_n \\ \text{and} \\ |y|_n \leq |x|_n \text{ for all } y \in F_n(x) \text{ and } x \in \partial W_n \end{array} \right. \quad (2.27)$$

$$\left\{ \begin{array}{l} \text{for every } k \in N \text{ and any subsequence } A \subseteq \{k, k + 1, \dots\} \\ \text{if } x \in \overline{C_n} \text{ is such that } x \in \overline{W_n} \setminus \Omega_n \text{ for some } n \in A \\ \text{then there exists a } \gamma > 0 \text{ with } |j_k \mu_{k,n} j_n^{-1} x|_k \geq \gamma \end{array} \right. \quad (2.28)$$

and

$$\left\{ \begin{array}{l} \text{for each } n \in \{2, 3, \dots\} \text{ if } y \in B_n(0, R) \cap \overline{C_n} \\ \text{solves } y \in F_n(y) \text{ in } E_n \text{ then } j_k \mu_{k,n} j_n^{-1}(y) \in B_k(0, R) \cap \overline{C_k} \\ \text{for } k \in \{1, \dots, n - 1\}. \end{array} \right. \quad (2.29)$$

Then F has a fixed point in E .

Remark 2.7. Note (2.28) is only needed to guarantee that the fixed point y satisfies $|j_k \mu_k(y)|_k \geq \gamma$ for $k \in N$. If we assume all the conditions in Theorem 2.4 except (2.28) then again F has a fixed point in E but the above property is not guaranteed.

Remark 2.8. Note in Theorem 2.4 if $x \in \overline{W_n}$ then $x \in Y_n$ so there exists a $y \in Y$ with $x = j_n \mu_n(y)$ and so $F_n(x) = j_n \mu_n F(y)$.

Proof. Fix $n \in N$. Now $\overline{C_n}$ is a cone (the argument in Theorem 2.1 guarantees that C_n is convex and as a result $\overline{C_n}$ is convex and similarly it is easy to see show $t\hat{x} \in \mu_n(C)$ for every $t \geq 0$, so $\overline{C_n}$ is a cone). Theorem 1.2 guarantees that there exists $y_n \in \overline{W_n} \setminus \Omega_n$ with $y_n \in F_n(y_n)$ in E_n . Lets look at $\{y_n\}_{n \in N}$. Notice $y_n \in \overline{W_1}$ for each $n \in N$ from (2.29). Now (2.26) guarantees (see Theorem 2.1) that there exists is a subsequence N_1^* of N and a $z_1 \in \overline{W_1}$ with $j_1 \mu_{1,n} j_n^{-1}(y_n) \rightarrow z_1$ in E_1 as $n \rightarrow \infty$ in N_1^* and $z_1 \in F_1(z_1)$. Also $y_n \in \overline{W_n} \setminus \Omega_n$ together with (2.28) yields $|j_1 \mu_{1,n} j_n^{-1}(y_n)|_1 \geq \gamma$ for $n \in N$ and so $|z_1|_1 \geq \gamma$. Let $N_1 = N_1^* \setminus \{1\}$. Proceed inductively to obtain subsequences of integers

$$N_1^* \supseteq N_2^* \supseteq \dots, \quad N_k^* \subseteq \{k, k + 1, \dots\}$$

and $z_k \in \overline{W_k}$ with $j_k \mu_{k,n} j_n^{-1}(y_n) \rightarrow z_k$ in E_k as $n \rightarrow \infty$ in N_k^* and $z_k \in F_k(z_k)$. Note $j_k \mu_{k,k+1} j_{k+1}^{-1} z_{k+1} = z_k$ in E_k for $k \in \{1, 2, \dots\}$ and $|z_k|_k \geq \gamma$. Also let $N_k = N_k^* \setminus \{k\}$. Now essentially the same reasoning as in Theorem 2.1 guarantees the result. \square

Remark 2.11. In Theorem 2.4 it is possible (see Remark 2.4) to replace $\overline{W_n} \subseteq Y_n$ with $\overline{W_n}$ a subset of the closure of Y_n in E_n provided Y is a closed subset of E so in this case we could have $Y = B(0, R) \cap C$ if $\overline{W_n}$ is a subset of the closure of $j_n \mu_n (B(0, R) \cap C)$ in E_n .

If we apply Theorem 1.3 instead of Theorem 1.2 we have the following result.

Theorem 2.5. *Let E and E_n be as described in the beginning of Section 2, C a closed cone in E , r and R are constants with $0 < r < R$, and $F : Y \rightarrow 2^E$ with $Y \subseteq E$, and $\overline{C_n} \subseteq Y_n$ for each $n \in N$. Also assume for each $n \in N$ and $x \in Y$ that $j_n \mu_n F(x)$ is closed and also for each $n \in N$ that $F_n : \overline{C_n} \rightarrow 2^{E_n}$ is as described above. Suppose the following conditions are satisfied (here $W_n = \overline{C_n} \cap V_n$ where $V_n = \{x \in E_n : |x|_n < R\}$ and $\Omega_n = U_n \cap \overline{C_n}$ with $U_n = \{x \in E_n : |x|_n < r\}$):*

$$\left\{ \begin{array}{l} \text{for each } n \in N, F_n \in AD(\overline{C_n}, \overline{C_n}) \\ \text{is a completely continuous map} \end{array} \right. \quad (2.30)$$

and

$$\left\{ \begin{array}{l} |y|_n \leq |x|_n \text{ for all } y \in F_n(x) \text{ and } x \in \partial\Omega_n \\ \text{and} \\ |y|_n \geq |x|_n \text{ for all } y \in F_n(x) \text{ and } x \in \partial W_n. \end{array} \right. \quad (2.31)$$

In addition assume (2.28) and (2.29) hold. Then F has a fixed point in E .

Remark 2.9. In Theorem 2.5 it is possible (see Remark 2.4) to replace $\overline{C_n} \subseteq Y_n$ with $\overline{C_n}$ a subset of the closure of Y_n in E_n , provided Y is a closed subset of E and so in this case we could have $Y = C$.

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