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FIXED POINTS OF A SEQUENCE OF ASYMPTOTICALLY NONEXPANSIVE MAPPINGS

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Abstract. In a Banach space with uniformly Gâteaux differentiable norm and uniform normal structure, by constructing an iterative sequence $\{z_n\}$ for a family of asymptotically nonexpansive mappings $\{T_m\}_{m=1}^{\infty}$, the necessary and sufficient conditions for the strong convergence of $\{z_n\}$ to a common fixed point of $\{T_m\}_{m=1}^{\infty}$ are analyzed.
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1. Introduction

Let X be a Banach space, C a nonempty subset of X and $T : C \to C$ a mapping. Then T is said to be a *contraction* on C with contractive constant $\alpha \in (0, 1)$ if $\|T(x) - T(y)\| \leq \alpha\|x - y\|$, for all $x, y \in C$; T is said to be *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$, for all $x, y \in C$ and T is said to be *uniformly L-Lipschitzian* $(L > 0)$ if $\|T^n x - T^n y\| \leq L\|x - y\|$, for all $x, y \in C$ and for all $n \in \mathbf{N}$. If there exists a sequence $\{k_n\}$ of positive numbers with $\lim_{n\to\infty} k_n = 1$ such that $\|T^n x - T^n y\| \leq k_n\|x - y\|$, for all $x, y \in C$ and for all $n \in \mathbf{N}$, then T is said to be *asymptotically nonexpansive*. It is clear that an asymptotically nonexpansive mapping is uniformly L-Lipschitzian for some constant $L > 0$.

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The asymptotically nonexpansive mappings introduced by Goebel and Kirk are important generalizations of nonexpansive mappings. Goebel and Kirk [10] proved that if X is a uniformly convex Banach space and C is a nonempty bounded and closed convex subset of X, then any asymptotically nonexpansive mapping $T : C \to C$ has a fixed point in C. This result has been extended (see, e.g., [11, 20, 30]).

Let C be a nonempty closed convex subset of X, $u \in C$ and T a nonexpansive self-mapping of C. The following iteration scheme for $u = 0$ was first introduced by Halpern [16] in 1967:

$$x_0 \in C, \quad x_{n+1} = \alpha_n u + (1-\alpha_n)Tx_n, \qquad n \in \mathbf{N}. \tag{1}$$

He pointed out that the conditions $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$ are necessary for the strong convergence of the iterative sequence $\{x_n\}$ to a fixed point of T. In 1980, Reich gave the iteration scheme (1) in the case when X is uniformly smooth and $\alpha_n = n^{-\lambda}$ with $0 < \lambda < 1$. Henceforward, the iteration scheme (1) has been comprehensively studied (see, e.g., [27] and the references therein).

On the other hand, many nice results of the iterative methods for approximating fixed points of asymptotically nonexpansive mappings have been also established (see, e.g., [3, 6, 7, 8, 17, 18, 19, 23, 24, 25, 26, 28, 29, 32, 33, 34, 35]). In particular, Lim and Xu [18] proved the strong convergence of an approximating fixed point of an asymptotically nonexpansive mappings in uniformly smooth Banach spaces including the L_p spaces $(1 < p < \infty)$ which partially extended a celebrated convergence theorem due to Reich [22] for nonexpansive mappings. Their result was extended to Banach spaces with uniformly Gâteaux differentiable norm and uniform normal structure by Chidume, Li and Udomene [8].

Theorem(Chidume, Li and Udomene [8]**) 1.1.** *Let X be a Banach space with uniformly Gâteaux differentiable norm and uniform normal structure, C a nonempty bounded closed convex subset of X, $T : C \to C$ an asymptotically nonexpansive mapping with sequence $\{k_n\} \subset [1, \infty)$. Let $u \in C$ and let $\{t_n\}$ be a sequence in $(0, 1)$ such that $\lim_{n\to\infty} t_n = 1$ and $\lim_{n\to\infty}(k_n-1)/(k_n-t_n) = 0$. Then for each $n \in \mathbf{N}$ there is a unique point $x_n \in C$ such that*

$$x_n = \left(1 - \frac{t_n}{k_n}\right) u + T^n x_n.$$

Define the sequence $\{z_n\}$ *iteratively by* $z_1 \in C$,

$$z_{n+1} = \left(1 - \frac{t_n}{k_n}\right) u + \frac{t_n}{k_n} T^n z_n, \qquad n \in \mathbf{N}.$$

Suppose, in addition, that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$ *and* $\lim_{n\to\infty} \|z_n - Tz_n\| = 0$. *Then the sequences* $\{x_n\}$ *and* $\{z_n\}$ *both converge strongly to a fixed point of* T.

By using viscosity approximation methods for a finite family of nonexpansive mappings in uniformly smooth Banach spaces, the necessary and sufficient conditions for the iterative sequence to converge to a common fixed point of those mappings are recently obtained by Chang [7]. In [17], we not only extended Theorem 1.1 to a finite family of asymptotically nonexpansive mappings, but proved that the sufficient condition for the strong convergence of $\{x_n\}$ and $\{z_n\}$ is also necessary as well in a uniformly convex Banach space with uniformly Gâteaux differentiable norm satisfying Opial's condition. Besides, the strong limit of $\{x_n\}$ and $\{z_n\}$ is, in fact, the unique solution of a variational inequality.

Theorem(Huang [17]**) 1.2.** *Let* X *be a uniformly convex Banach space with uniformly Gâteaux differentiable norm satisfying Opial's condition,* C *a nonempty closed convex subset of* X, $f : C \to C$ *a contraction with contractive constant* $\beta \in (0,1)$, $\{T_i\}_{i=1}^m$ *a finite family of asymptotically nonexpansive self-mappings of* C *with sequences* $\{k_{i+m(j-1)}\}_{j=1}^{\infty} \subset [1,\infty)$ $(1 \le i \le m)$ *such that* $\bigcap_{i=1}^m \mathrm{Fix}(T_i) \neq \emptyset$. *Let* $\{t_n\}$ *be a sequence in* $(0,1)$ *such that* $\lim_{n\to\infty}(k_n - 1)/(k_n - t_n) = 0$. *Then for each* $n \in \mathbf{N}$ *there is a unique point* $x_n \in C$ *such that*

$$x_n = \left(1 - \frac{t_n}{k_n}\right) f(x_n) + \frac{t_n}{k_n} T_{\bar{n}}^{j_n} x_n,$$

where $\bar{n} \equiv n \bmod m$ *and* $j_n = (n - \bar{n})/m + 1$. *Define the sequence* $\{z_n\}$ *iteratively by* $z_1 \in C$,

$$z_{n+1} = \left(1 - \frac{t_n}{k_n}\right) f(z_n) + \frac{t_n}{k_n} T_{\bar{n}}^{j_n} z_n, \qquad n \in \mathbf{N}.$$

For any C, f *and* $\{T_i\}_{i=1}^m$ *as above, the sequences* $\{x_n\}$ *and* $\{z_n\}$ *both converge strongly to a point* $\hat{x} \in \bigcap_{i=1}^m \mathrm{Fix}(T_i)$ *which is the unique solution of the* *variational inequality in* $\bigcap_{i=1}^m \mathrm{Fix}(T_i)$*:*

$$\langle f(\hat{x}) - \hat{x}, J(y - \hat{x})\rangle \leq 0, \quad \textit{for all } y \in \bigcap_{i=1}^{m} \mathrm{Fix}(T_i)$$

if and only if the following conditions are satisfied:

(i) $\lim_{n\to\infty} t_n = 1$;

(ii) $\sum_{n=1}^{\infty} [\,1 - (t_n/k_n)\,] = \infty$;

(iii) $\lim_{n\to\infty} \|x_n - T_i x_n\| = 0$ *and* $\lim_{n\to\infty} \|z_n - T_i z_n\| = 0$, *for* $i = 1, \ldots, m$.

This work is mainly devoted to construct iteration schemes for an infinite sequence of asymptotically nonexpansive mappings in a uniformly convex Banach space with uniformly Gâteaux differentiable norm satisfying Opial's condition and establish the necessary and sufficient conditions for the strong convergence of the iterative sequences to a common fixed point of those mappings. To this end, in Section 3, regarded as the generalization of Theorem 1.1, we consider iterative sequences $\{x_n\}$ and $\{z_n\}$ as in Theorem 1.2 for finitely many asymptotically nonexpansive mappings $\{T_i\}_{i=1}^m$ in Banach spaces with uniformly Gâteaux differentiable norm and uniform normal structure. The conditions (i)-(iii) in Theorem 1.2 are verified to be still necessary and sufficient for the strong convergence of $\{x_n\}$ and $\{z_n\}$ to a common fixed point of $\{T_i\}_{i=1}^m$. The approach, also cf. [8] and [13, pp. 46-47], is different from that of [17]. In Section 4, we provide the iterations for an infinite sequence of asymptotically nonexpansive mappings in uniformly convex Banach spaces with uniformly Gâteaux differentiable norm satisfying Opial's condition. Furthermore, the corresponding work in Section 3 is investigated for this infinite sequence of mappings. It is noteworthy that the results in [17] can be generalized to an infinite sequence of asymptotically nonexpansive mappings with the same iteration schemes and arguments as developed in Section 4. Consequently, we remark that Theorems 4.1 and 4.2 are improved by relaxing the boundedness on the set C.

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2. Preliminaries

Let X be a Banach space, X^* the dual space of X, and $S = \{x \in X : \|x\| = 1\}$ the unit sphere of X. Let $J : X \to 2^{X^*}$ be the *normalized duality mapping* defined by

$$J(x) = \{x^* \in X^* : \langle x, x^* \rangle = \|x\| \cdot \|x^*\|,\ \|x\| = \|x^*\|\}, \qquad x \in X,$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing. If J is single-valued, J is odd, that is, $J(-x) = -J(x)$, $x \in X$. A Banach space X is said to admit a *weakly sequentially continuous* normalized duality mapping $J : X \to 2^{X^*}$ if J is single-valued and weak-to-weak* continuous.

We say that X is *smooth* (or X has a *Gâteaux differentiable norm*) if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2}$$

exists for all $x, y \in S$; X is said to have a *uniformly Gâteaux differentiable norm* if for each $y \in X$ the limit (2) is attained uniformly in $x \in S$. If the limit (2) exists and is attained uniformly in $x, y \in S$, X is said to be *uniformly smooth*. It is well known that X is smooth if and only if the duality mapping J is single-valued [9]. The norm of X is uniformly Gâteaux differentiable is equivalent to the condition that J is single-valued and norm-to-weak* continuous, uniformly on bounded subsets of X [9].

Let C be a nonempty bounded subset of X and let $\operatorname{diam} C = \sup\{\|x - y\| : x, y \in C\}$ be the *diameter* of C. For each $x \in C$, let $r(x, C) = \sup\{\|x - y\| : y \in C\}$ so that the *Chebyshev radius* of C relative to itself is defined by $r(C) = \inf\{r(x, C) : x \in C\}$. The *normal structure coefficient* $N(X)$ of X, cf. [1, 4], is defined by

$$N(X) = \inf\left\{ \frac{\operatorname{diam} C}{r(C)} : C \text{ is a bounded and closed convex subset of } X \text{ with } \operatorname{diam} C > 0 \right\}.$$

A space X is said to have *uniform normal structure* if $N(X) > 1$.

A Banach space X is said to be *uniformly convex* [1] if for each $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that for $x, y \in X$ with $\|x\| \le 1$ and $\|y\| \le 1$,

$$\left\| \frac{x + y}{2} \right\| \le 1 - \delta(\epsilon), \qquad \text{whenever } \|x - y\| \ge \epsilon.$$

It is well known that every space with uniform normal structure is reflexive, cf. [12, 21], and that all uniformly convex or uniformly smooth Banach spaces have uniform normal structure [1].

A mapping $f : E \to X$, where $E \subset X$, is said to be *demiclosed* at $y \in X$ if, for any sequence $\{x_n\}$ in E, the conditions $x_n \to x \in E$ weakly and $f(x_n) \to y$ strongly together imply $f(x) = y$. Recall also that a Banach space X is said to satisfy *Opial's condition* [12] if whenever a sequence $\{x_n\}$ in X converges weakly to x, then $\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$, for $y \neq x$. Gossez and Lami Dozo have shown that a Banach space with weakly sequentially continuous normalized duality mapping must satisfy Opial's condition [14, Theorem 1]. If X is a reflexive Banach space which satisfies Opial's condition, C is a closed convex subset of X and $T : C \to C$ is a nonexpansive mapping, then $I - T$ is demiclosed, where I denotes the identity mapping of X [12, Theorem 10.3]. Significantly, Gornicki's demiclosedness principle asserts that if X is a uniformly convex Banach space satisfying Opial's condition, C is a nonempty closed convex subset of X and $T : C \to C$ is an asymptotically nonexpansive mapping, then $I - T$ is demiclosed at zero [15].

We denote a Banach limit by LIM. Recall that $\mathrm{LIM} \in (\ell^\infty)^*$ such that $\|\mathrm{LIM}\| = 1$ and for all $\{a_n\} \in (\ell^\infty)^*$, $\liminf_{n\to\infty} a_n \leq \mathrm{LIM}_n a_n \leq \limsup_{n\to\infty} a_n$ and $\mathrm{LIM}_n a_n = \mathrm{LIM}_n a_{n+1}$.

The following lemmas will be needed to prove our results.

Lemma 2.1. ([5]) *Let X be a real Banach space. Then*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y)\rangle,$$

for all $x, y \in X$ and $j(x + y) \in J(x + y)$.

Lemma 2.2. ([18]) *Let X be a real Banach space with uniform normal structure, C a nonempty bounded subset of X, $T : C \to C$ a uniformly k-Lipschitzian mapping with $k < N(X)^{1/2}$. Suppose that there exists a nonempty bounded and closed convex subset E of C with the following property (P):*

$$x \in E \quad \textit{implies} \quad \omega_w(x) \subset E, \tag{P}$$

where

$$\omega_w(x) = \{y \in X : y = \textit{weak-}\lim_j T^{n_j} x \textit{ for some } n_j \to \infty\}$$

is the weak ω-limit set of T at x. Then T has a fixed point in E.

Lemma 2.3. ([31]) *Let* $\{a_n\}$ *ba a sequence of nonnegative real numbers satisfying the following condition:*

$$a_{n+1} \leq (1-\alpha_n)a_n + \sigma_n, \qquad n \in \mathbf{N},$$

where each $0 < \alpha_n < 1$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$. *If either* $\sigma_n = o(\alpha_n)$, *or* $\limsup_{n\to\infty} \sigma_n \leq 0$, *then* $\lim_{n\to\infty} a_n = 0$.

3. The main results

This section is devoted to constructing an iterated sequence for a finite family of asymptotically nonexpansive mappings $\{T_i\}_{i=1}^m$ and establishing the necessary and sufficient conditions for this iterative sequence to converge to a common fixed point $\hat{x}$ of $\{T_i\}_{i=1}^m$. In fact, the fixed point $\hat{x}$ is the unique solution of the variational inequality:

$$\langle f(\hat{x}) - \hat{x}, J(y-\hat{x})\rangle \leq 0, \tag{3}$$

for all $y \in \bigcap_{i=1}^m \mathrm{Fix}(T_i)$, where $\mathrm{Fix}(T_i)$ denotes the fixed point set of T_i. Let X be a real Banach space, C a nonempty closed convex subset of X, $\{T_i\}_{i=1}^m$ a finite family of asymptotically nonexpansive self-mappings of C with sequences $\{k_{i+m(j-1)}\}_j$ $(1 \leq i \leq m)$, $f : C \to C$ a contraction with contractive constant $\beta \in (0,1)$. We may always assume that $k_n \geq 1$, for all $n \in \mathbf{N}$. Let $\{t_n\}$ be a sequence in $(0,1)$ such that $\lim_{n\to\infty} t_n = 1$. Throughout the rest of this paper, for each $n \in \mathbf{N}$, we denote $\bar{n} \equiv n \bmod m$ and write $n = \bar{n} + m(\hat{n}-1)$, for some $\hat{n} \in \mathbf{N}$. Thus $\hat{n} = 1$ if $1 \leq n \leq m$, $\hat{n} = 2$ if $m+1 \leq n \leq 2m$, and so on. Define a mapping $S_n : C \to C$ by

$$S_n(x) = \left(1 - \frac{t_n}{k_n}\right) f(x) + \frac{t_n}{k_n} T_{\bar{n}}^{\hat{n}} x.$$

Then for $x, y \in C$ we have

$$\begin{aligned}\|S_n(x) - S_n(y)\| &\leq \left(1 - \frac{t_n}{k_n}\right)\|f(x) - f(y)\| + \frac{t_n}{k_n}\|T_{\bar{n}}^{\hat{n}} x - T_{\bar{n}}^{\hat{n}} y\| \\ &\leq \left[\beta\left(1 - \frac{t_n}{k_n}\right) + t_n\right]\|x-y\|.\end{aligned}$$

In addition, suppose that $\lim_{n\to\infty}(k_n - 1)/(k_n - t_n) = 0$. Then

$$\lim_{n\to\infty} \frac{1-t_n}{1-t_n/k_n} = \lim_{n\to\infty} k_n\left(1 - \frac{k_n - 1}{k_n - t_n}\right) = 1 \tag{4}$$

and thus $\beta(1-t_n/k_n)+t_n<1$, for all sufficiently large n. By Banach Contraction Principle, S_n has a unique fixed point $x_n \in C$, that is,

$$x_n = \left(1-\frac{t_n}{k_n}\right) f(x_n) + \frac{t_n}{k_n} T_{\tilde{n}}^{\hat{n}} x_n. \tag{5}$$

If only one asymptotically nonexpansive mapping $T : C \to C$ is considered, i.e., $m = 1$, then (5) reduces to

$$x_n = \left(1-\frac{t_n}{k_n}\right) f(x_n) + \frac{t_n}{k_n} T^n x_n. \tag{6}$$

In particular, if we set $f \equiv u$, for some fixed $u \in C$, our iterative process (6) becomes the one studied by Lim and Xu in [18] and Chidume, Li and Udomene in [8]. Their results are extended as follows.

Theorem 3.1. *Let X be a real Banach space with uniformly Gâteaux differentiable norm and uniform normal structure, C a nonempty bounded closed convex subset of X, $T : C \to C$ an asymptotically nonexpansive mapping with sequence $\{k_n\} \subset [1,\infty)$, $f : C \to C$ a contraction with contractive constant $\beta \in (0,1)$. Let $\{t_n\}$ be a sequence in $(0,1)$ such that $\lim_{n\to\infty} t_n = 1$ and $\lim_{n\to\infty}(k_n-1)/(k_n-t_n) = 0$. The sequence $\{x_n\}$ defined by (6) converges strongly to a point $\hat{x} \in \mathrm{Fix}(T)$ which is the unique solution of the variational inequality (3) in $\mathrm{Fix}(T)$ if and only if $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$.*

Proof. Discarding a few terms if necessary, we can assume that $\beta(1-t_n/k_n)+t_n<1$, for all $n \in \mathbf{N}$. The set $\mathrm{Fix}(T)$ is nonempty by Lemma 2.2. Notice that the variational inequality (3) has at most one solution in $\mathrm{Fix}(T)$. For, if u and v are two solutions of (3) in $\mathrm{Fix}(T)$, we have

$$\langle f(u)-u, J(v-u)\rangle \le 0, \qquad \langle f(v)-v, J(u-v)\rangle \le 0.$$

Adding these two inequalities, we obtain

$$\langle [u-f(u)]-[v-f(v)], J(u-v)\rangle \le 0$$

which implies

$$(1-\beta)\|u-v\|^2 \le \langle [u-f(u)]-[v-f(v)], J(u-v)\rangle \le 0.$$

Therefore $u = v$.

If the sequence $\{x_n\}$ converges strongly to $\hat{x} \in \mathrm{Fix}(T)$, then

$$\|x_n - Tx_n\| \le \|x_n - \hat{x}\| + \|T\hat{x} - Tx_n\| \le (1+k_1)\|x_n - \hat{x}\|$$

and hence $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$.

Conversely, suppose that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. For each $m \in \mathbf{N}$, since

$$\begin{aligned}\|x_n - T^m x_n\| &\le \|x_n - Tx_n\| + \|Tx_n - T^2 x_n\| + \cdots + \|T^{m-1}x_n - T^m x_n\| \\ &\le (1 + k_1 + \cdots + k_{m-1})\|x_n - Tx_n\|,\end{aligned}$$

it follows that $\lim_{n\to\infty} \|x_n - T^m x_n\| = 0$. Let $u \in \mathrm{Fix}(T)$ so that for $n \in \mathbf{N}$,

$$\begin{aligned}\langle x_n - T^n x_n, J(x_n - u)\rangle &= \langle x_n - u, J(x_n - u)\rangle + \langle u - T^n x_n, J(x_n - u)\rangle \\ &\ge \|x_n - u\|^2 - \|u - T^n x_n\| \cdot \|x_n - u\| \\ &\ge -(k_n - 1)\|x_n - u\|^2 \\ &\ge -(k_n - 1)(\operatorname{diam} C)^2.\end{aligned}$$

By (6) we have

$$x_n - T^n x_n = \frac{k_n - t_n}{t_n}[f(x_n) - x_n],$$

and so

$$\langle x_n - f(x_n), J(x_n - u)\rangle \le \frac{t_n(k_n - 1)}{k_n - t_n}(\operatorname{diam} C)^2, \tag{7}$$

where $t_n(k_n - 1)/(k_n - t_n) \to 0$ as $n \to \infty$. Define a mapping $\varphi : C \to [0, \infty)$ by

$$\varphi(x) = \mathrm{LIM}_n \|x_n - x\|^2, \quad x \in C.$$

Since φ is continuous and convex, $\varphi(x) \to \infty$ as $\|x\| \to \infty$. Corollary 5.87 in [2] implies that φ is weakly lower semicontinus. Since X is reflexive, φ attains its infimum over C. Hence the set

$$E = \{x \in C : \varphi(x) = \inf_{z\in C} \varphi(z)\}$$

is nonempty, closed and convex. Moreover, E satisfies the property (P). In fact, if $x \in E$ and $y = \text{weak-}\lim_j T^{m_j}x$ is in the weak ω-limit set $\omega_w(x)$ of T at x, then from the weak lower semicontinuity of φ and $\lim_{n\to\infty} \|x_n - T^m x_n\| = 0$, for $m \in \mathbf{N}$, we have

$$\begin{aligned}\varphi(y) &\le \liminf_j \varphi(T^{m_j}x) \le \limsup_m \varphi(T^m x) \\ &= \limsup_m(\mathrm{LIM}_n\|x_n - T^m x\|^2) \le \limsup_m(\mathrm{LIM}_n\|T^m x_n - T^m x\|^2) \\ &\le \limsup_m k_m^2 \mathrm{LIM}_n\|x_n - x\|^2 = \mathrm{LIM}_n\|x_n - x\|^2 \\ &= \inf_{z\in C} \varphi(z).\end{aligned}$$

Therefore $y \in E$ and E satisfies the property (P). By Lemma 2.2, T has a fixed point $\hat{x} \in E$. On the other hand, if $m \in \mathbf{N}$ and $z \in C$, then by Lemma 2.1

$$\begin{aligned}\|x_n - \hat{x}\|^2 &= \|x_n - (t_m/k_m)\hat{x} - (1 - t_m/k_m)z + (1 - t_m/k_m)(z - \hat{x})\|^2 \\ &\geq \|x_n - (t_m/k_m)\hat{x} - (1 - t_m/k_m)z\|^2 \\ &\quad + [2(k_m - t_m)/k_m] \left\langle z - \hat{x}, J\big(x_n - (t_m/k_m)\hat{x} - (1 - t_m/k_m)z\big) \right\rangle.\end{aligned} \tag{8}$$

Given $\epsilon > 0$, since J is norm-to-weak* uniformly continuous on C, there exists $m_0 \in \mathbf{N}$ such that for all $n \in \mathbf{N}$,

$$\left|\left\langle z-\hat{x}, J(x_n-\hat{x})-J\big(x_n-(t_m/k_m)\hat{x}-(1-t_m/k_m)z\big) \right\rangle\right| < \epsilon, \text{ whenever } m \geq m_0.$$

By (8) this inequality implies that

$$\begin{aligned}&\langle z - \hat{x}, J(x_n - \hat{x})\rangle \\ &< \epsilon + \left\langle z - \hat{x}, J\big(x_n - (t_m/k_m)\hat{x} - (1 - t_m/k_m)z\big) \right\rangle \\ &\leq \epsilon + [k_m/2(k_m - t_m)] \left[\|x_n - \hat{x}\|^2 - \|x_n - (t_m/k_m)\hat{x} - (1 - t_m/k_m)z\|^2 \right],\end{aligned}$$

whenever $n \in \mathbf{N}$ and $m \geq m_0$. Since $\hat{x}$ is a minimizer of φ over C, the above inequality asserts that

$$\mathrm{LIM}_n\langle z - \hat{x}, J(x_n - \hat{x})\rangle \leq 0, \quad \text{for all } z \in C.$$

In particular,

$$\mathrm{LIM}_n\langle f(x_n) - \hat{x}, J(x_n - \hat{x})\rangle \leq 0. \tag{9}$$

Choosing $u = \hat{x}$ in (7) and combining (7) with (9), we conclude that

$$\mathrm{LIM}_n\|x_n - \hat{x}\|^2 = \mathrm{LIM}_n\langle x_n - \hat{x}, J(x_n - \hat{x})\rangle \leq 0.$$

Therefore there is a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ converging strongly to $\hat{x}$. It follows from (7) that

$$\langle \hat{x} - f(\hat{x}), J(\hat{x} - y)\rangle \leq 0, \quad \text{for all } y \in \mathrm{Fix}(T),$$

and so $\hat{x}$ is a solution of the variational inequality (3). To complete the proof, suppose that there is another subsequence of $\{x_n\}$ converging strongly to a point, say $\bar{x}$. Then $\bar{x}$ is a fixed point of T because $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. Again, from (7) we obtain

$$\langle \bar{x} - f(\bar{x}), J(\bar{x} - y)\rangle \leq 0, \quad \text{for all } y \in \mathrm{Fix}(T),$$

which shows that $\bar{x}$ is also a solution of the variational inequality (3); hence $\bar{x} = \hat{x}$ by the uniqueness of the solution. This assures the strong convergence of $\{x_n\}$ to $\hat{x}$. □

Theorem 3.2. *Let X be a real Banach space with uniformly Gâteaux differentiable norm and uniform normal structure, C a nonempty bounded closed convex subset of X, $T : C \to C$ an asymptotically nonexpansive mapping with sequence $\{k_n\} \subset [1, \infty)$, $f : C \to C$ a contraction with contractive constant $\beta \in (0, 1)$. Let $\{t_n\}$ be a sequence in $(0, 1)$ such that $\lim_{n\to\infty}(k_n - 1)/(k_n - t_n) = 0$, and $\{x_n\}$ a sequence defined by (6). Define the sequence $\{z_n\}$ iteratively by $z_1 \in C$,*

$$z_{n+1} = \left(1 - \frac{t_n}{k_n}\right) f(z_n) + \frac{t_n}{k_n} T^n z_n, \qquad n \in \mathbf{N}. \tag{10}$$

For any C, T and f as above, the sequences $\{x_n\}$ and $\{z_n\}$ both converge strongly to a point $\hat{x} \in \mathrm{Fix}(T)$ which is the unique solution of the variational inequality (3) in $\mathrm{Fix}(T)$ if and only if the following conditions are satisfied:

(i) $\lim_{n\to\infty} t_n = 1$;

(ii) $\sum_{n=1}^{\infty} [\,1 - (t_n/k_n)\,] = \infty$, *that is,* $\prod_{n=1}^{\infty} t_n/k_n = 0$;

(iii) $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$ *and* $\lim_{n\to\infty} \|z_n - Tz_n\| = 0$.

Proof. Suppose that the sequences $\{x_n\}$ and $\{z_n\}$ both converge strongly to a point $\hat{x}$ of $\mathrm{Fix}(T)$ which is the unique solution of the variational inequality (3) in $\mathrm{Fix}(T)$. Then

$$\|x_n - Tx_n\| \le \|x_n - \hat{x}\| + \|T\hat{x} - Tx_n\| \le (1 + k_1)\|x_n - \hat{x}\| \to 0 \qquad \text{as } n \to \infty,$$

and similarly

$$\|z_n - Tz_n\| \le \|z_n - \hat{x}\| + \|T\hat{x} - Tz_n\| \le (1 + k_1)\|z_n - \hat{x}\| \to 0 \qquad \text{as } n \to \infty.$$

Hence condition (iii) holds.

Since $\|T^n z_n - \hat{x}\| = \|T^n z_n - T^n \hat{x}\| \le k_n \|z_n - \hat{x}\|$, for $n \in \mathbf{N}$, it follows that $\lim_{n\to\infty} T^n z_n = \hat{x}$. Letting $f \equiv u \in C$ with $u \notin \mathrm{Fix}(T)$, by (10) we have

$$\liminf_{n\to\infty} \frac{t_n}{k_n} \|\hat{x} - u\| = \liminf_{n\to\infty} \frac{t_n}{k_n} \|T^n z_n - u\| = \liminf_{n\to\infty} \|z_{n+1} - u\| = \|\hat{x} - u\|$$

which implies that $\lim_{n\to\infty} t_n/k_n = 1$ and condition (i) holds.

To prove condition (ii) is satisfied, we may take $C = \{x \in E : \|x\| \le 1\}$ to be the closed unit ball and $z_1 \neq 0$. Set $f \equiv 0$ and $T = -I : C \to C$ so that by (10) it follows that

$$z_{n+1} = (-1)^n \cdot \frac{t_n}{k_n} \cdot z_n$$

$$= (-1)^{n+(n-1)} \cdot \frac{t_n}{k_n} \cdot \frac{t_{n-1}}{k_{n-1}} \cdot z_{n-1} = \cdots\cdots = (-1)^{n(n+1)/2} \prod_{m=1}^{n} \frac{t_m}{k_m} \cdot z_1.$$

Since 0 is the only fixed point of T, it follows that $z_n \to \hat{x} = 0$ and

$$0 = \lim_{n\to\infty} \|z_{n+1} - 0\| = \lim_{n\to\infty} \prod_{m=1}^{n} \frac{t_m}{k_m} \cdot \|z_1 - 0\|;$$

hence $\prod_{n=1}^{\infty} t_n/k_n = 0$, that is, $\sum_{n=1}^{\infty}[\,1 - (t_n/k_n)\,] = \infty$.

From the above discussion, conditions (i), (ii) and (iii) are seen to be necessary. So we proceed to show their sufficiency. First, Theorem 3.1 assures that the sequence $\{x_n\}$ converges strongly to a point $\hat{x} \in \mathrm{Fix}(T)$ which is the unique solution of the variational inequality (3) in $\mathrm{Fix}(T)$. Next, we shall prove that

$$\limsup_{n\to\infty} \langle f(\hat{x}) - \hat{x}, J(z_n - \hat{x})\rangle \le 0. \tag{11}$$

By (6),

$$x_m - z_n = \left(1 - \frac{t_m}{k_m}\right)[f(x_m) - z_n] + \frac{t_m}{k_m}(T^m x_m - z_n).$$

Apply Lemma 2.1 to obtain

$$\begin{aligned}
\|x_m - z_n\|^2 &\le \frac{t_m^2}{k_m^2}\|T^m x_m - z_n\|^2 + 2\left(1 - \frac{t_m}{k_m}\right)\langle f(x_m) - z_n, J(x_m - z_n)\rangle \\
&\le \frac{t_m^2}{k_m^2}\left[\,\|T^m x_m - T^m z_n\| + \|T^m z_n - z_n\|\,\right]^2 \\
&\quad +2\left(1 - \frac{t_m}{k_m}\right)[\,\langle\, f(x_m) - x_m, J(x_m - z_n)\,\rangle + \|x_m - z_n\|^2\,] \\
&\le \frac{t_m^2}{k_m^2}\left[\,k_m\|x_m - z_n\| + \|T^m z_n - z_n\|\,\right]^2 \\
&\quad +2\left(1 - \frac{t_m}{k_m}\right)[\,\langle f(x_m) - x_m, J(x_m - z_n)\rangle + \|x_m - z_n\|^2\,] \\
&= \left[t_m^2 + 2\left(1 - \frac{t_m}{k_m}\right)\right]\|x_m - z_n\|^2 + \frac{2t_m^2}{k_m}\|x_m - z_n\| \cdot \|T^m z_n - z_n\|
\end{aligned}$$

$$+\frac{t_m^2}{k_m^2}\|T^m z_n - z_n\|^2 + 2\left(1 - \frac{t_m}{k_m}\right)\langle f(x_m) - x_m, J(x_m - z_n)\rangle,$$

hence we have

$$\begin{aligned}
&\limsup_{n\to\infty}\langle f(x_m) - x_m, J(z_n - x_m)\rangle \\
&\le \left[1 + \frac{k_m(t_m^2-1)}{2(k_m - t_m)}\right]\limsup_{n\to\infty}\|x_m - z_n\|^2 \\
&\quad + \frac{t_m^2}{k_m - t_m}\limsup_{n\to\infty}\|x_m - z_n\|\cdot\|T^m z_n - z_n\| \\
&\quad + \frac{t_m^2}{2k_m(k_m - t_m)}\limsup_{n\to\infty}\|T^m z_n - z_n\|^2. \qquad (12)
\end{aligned}$$

Observe that

$$\lim_{m\to\infty}\left[1 + \frac{k_m(t_m^2-1)}{2(k_m - t_m)}\right] = \lim_{m\to\infty}\left[1 - \frac{k_m(1+t_m)}{2}\cdot\left(1 - \frac{k_m - 1}{k_m - t_m}\right)\right] = 0.$$

Moreover, for each $m \in \mathbf{N}$,

$$\begin{aligned}
\|z_n - T^m z_n\| &\le \|z_n - Tz_n\| + \|Tz_n - T^2 z_n\| + \cdots + \|T^{m-1}z_n - T^m z_n\| \\
&\le (1 + k_1 + \cdots + k_{m-1})\|z_n - Tz_n\| \to 0 \qquad \text{as } n \to \infty.
\end{aligned}$$

Since the sequences $\{x_m\}$ and $\{z_n\}$ are bounded, (12) implies that

$$\limsup_{m\to\infty}\limsup_{n\to\infty}\langle f(x_m) - x_m, J(z_n - x_m)\rangle \le 0. \qquad (13)$$

Since $\{x_m\}$ converges strongly to $\hat{x}$ and J is norm-to-weak* uniformly continuous on bounded sets, there exists $l \in \mathbf{N}$ such that for all $m, n \ge l$,

$$\begin{aligned}
|\,\langle f(x_m) - x_m, J(z_n - x_m)\rangle - \langle f(\hat{x}) - \hat{x}, J(z_n - x_m)\rangle\,| &\le \epsilon/2, \\
|\,\langle f(\hat{x}) - \hat{x}, J(z_n - x_m)\rangle - \langle f(\hat{x}) - \hat{x}, J(z_n - \hat{x})\rangle\,| &\le \epsilon/2.
\end{aligned}$$

Therefore for all $m, n \ge l$

$$\begin{aligned}
&|\,\langle f(x_m) - x_m, J(z_n - x_m)\rangle - \langle f(\hat{x}) - \hat{x}, J(z_n - \hat{x})\rangle\,| \\
&\le |\,\langle f(x_m) - x_m, J(z_n - x_m)\rangle - \langle f(\hat{x}) - \hat{x}, J(z_n - x_m)\rangle\,| \\
&\quad + |\,\langle f(\hat{x}) - \hat{x}, J(z_n - x_m)\rangle - \langle f(\hat{x}) - \hat{x}, J(z_n - \hat{x})\rangle\,| \\
&\le \epsilon,
\end{aligned}$$

and so by (13)

$$\limsup_{n\to\infty}\langle f(\hat{x}) - \hat{x}, J(z_n - \hat{x})\rangle \le \limsup_{m\to\infty}\limsup_{n\to\infty}\langle f(x_m) - x_m, J(z_n - x_m)\rangle + \epsilon \le \epsilon.$$

Since ϵ is arbitrary, this proves (11).

To prove $\{z_n\}$ converges strongly to $\hat{x}$, let $\gamma_n = 1 - t_n/k_n$, for $n \in \mathbf{N}$ so that $\gamma_n < 1/2$ for all sufficiently large n. Thus $1 - \beta\gamma_n > 1/2$ for sufficiently large n. It follows from (10) and Lemma 2.1 that

$$\begin{aligned}
\|z_{n+1} - \hat{x}\|^2 &\leq (1-\gamma_n)^2 \|T^n z_n - \hat{x}\|^2 + 2\gamma_n \langle f(z_n) - \hat{x}, J(z_{n+1} - \hat{x})\rangle \\
&\leq k_n^2 (1-\gamma_n)^2 \|z_n - \hat{x}\|^2 + 2\gamma_n \langle\, f(z_n) - f(\hat{x}) + f(\hat{x}) - \hat{x}, J(z_{n+1} - \hat{x})\,\rangle \\
&\leq k_n^2 (1-\gamma_n)^2 \|z_n - \hat{x}\|^2 + 2\beta\gamma_n \|z_n - \hat{x}\| \cdot \|z_{n+1} - \hat{x}\| \\
&\quad + 2\gamma_n \langle f(\hat{x}) - \hat{x}, J(z_{n+1} - \hat{x})\rangle \\
&\leq k_n^2 (1-\gamma_n)^2 \|z_n - \hat{x}\|^2 + \beta\gamma_n \left[\, \|z_n - \hat{x}\|^2 + \|z_{n+1} - \hat{x}\|^2 \,\right] \\
&\quad + 2\gamma_n \langle f(\hat{x}) - \hat{x}, J(z_{n+1} - \hat{x})\rangle.
\end{aligned}$$

Therefore

$$\begin{aligned}
\|z_{n+1} - \hat{x}\|^2 \leq \frac{1-(2-\beta)\gamma_n}{1-\beta\gamma_n}\|z_n - \hat{x}\|^2 + \frac{(k_n^2 - 1)(1-2\gamma_n) + k_n^2\gamma_n^2}{1-\beta\gamma_n}\|z_n - \hat{x}\|^2 \\
+ \frac{2\gamma_n}{1-\beta\gamma_n}\langle f(\hat{x}) - \hat{x}, J(z_{n+1} - \hat{x})\rangle. \qquad (14)
\end{aligned}$$

Since

$$\frac{1-(2-\beta)\gamma_n}{1-\beta\gamma_n} = 1 - \frac{2(1-\beta)\gamma_n}{1-\beta\gamma_n} < 1 - 2(1-\beta)\gamma_n,$$

by (14) we have

$$\|z_{n+1} - \hat{x}\|^2 \leq (1-\alpha_n)\|z_n - \hat{x}\|^2 + \sigma_n,$$

where $\alpha_n = 2(1-\beta)\gamma_n$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, and

$$\sigma_n = \frac{(k_n^2 - 1)(1-2\gamma_n) + k_n^2\gamma_n^2}{1-\beta\gamma_n}\|z_n - \hat{x}\|^2 + \frac{2\gamma_n}{1-\beta\gamma_n}\langle f(\hat{x}) - \hat{x}, J(z_{n+1} - \hat{x})\rangle.$$

The boundedness of $\{z_n\}$ and the inequality (11) imply that $\limsup_{n\to\infty} \sigma_n \leq 0$. By Lemma 2.3, we conclude that $\lim_{n\to\infty} \|z_n - \hat{x}\| = 0$. This completes the proof. □

Applying Theorem 3.1 and Theorem 3.2, we can further consider the case of any finite family of asymptotically nonexpansive mappings with nonempty common fixed point set. The corresponding results are established as follows.

Theorem 3.3. *Let X be a real Banach space with uniformly Gâteaux differentiable norm and uniform normal structure, C a nonempty bounded closed convex subset of X, $f : C \to C$ a contraction, $\{T_i\}_{i=1}^m$ a finite family of asymptotically nonexpansive self-mappings of C with sequences $\{k_{i+m(j-1)}\}_{j=1}^\infty \subset [1, \infty)$ $(1 \le i \le m)$ such that $\bigcap_{i=1}^m \mathrm{Fix}(T_i) \neq \emptyset$. Let $\{t_n\}$ be a sequence in $(0, 1)$ such that $\lim_{n\to\infty} t_n = 1$ and $\lim_{n\to\infty}(k_n - 1)/(k_n - t_n) = 0$. The sequence $\{x_n\}$ defined by (5) converges strongly to a common fixed point $\hat{x}$ of $\{T_i\}_{i=1}^m$ which is the unique solution of the variational inequality (3) in $\bigcap_{i=1}^m \mathrm{Fix}(T_i)$ if and only if $\lim_{n\to\infty} \|x_n - T_i x_n\| = 0$, for $i = 1, \dots, m$.*

Proof. The necessity follows directly from the strong convergence of $\{x_n\}$. To prove the sufficiency, let $k \in \{1, 2, \dots, m\}$ so that by Theorem 3.1, the sequence $\{x_{k+m(j-1)}\}_{j=1}^\infty$ converges strongly to a point $\hat{x}_k$ of $\mathrm{Fix}(T_k)$ which is the unique solution of the variational inequality (3) in $\mathrm{Fix}(T_k)$. For any $i \neq k$, since $\lim_{n\to\infty} \|x_n - T_i x_n\| = 0$ and $\{x_{k+m(j-1)}\}_{j=1}^\infty$ is a subsequence of $\{x_n\}$, it follows that $\hat{x}_k$ is also a fixed point of T_i. Consequently, the points $\hat{x}_1, \hat{x}_2, \dots, \hat{x}_m$ are the solutions of the variational inequality (3) in $\bigcap_{i=1}^m \mathrm{Fix}(T_i)$; hence all $\hat{x}_k$'s are reduced to one point, say $\hat{x}$, of $\bigcap_{i=1}^m \mathrm{Fix}(T_i)$. This assures the strong convergence of $\{x_n\}$ to $\hat{x}$. □

Theorem 3.4. *Let X be a real Banach space with uniformly Gâteaux differentiable norm and uniform normal structure, C a nonempty bounded closed convex subset of X, $f : C \to C$ a contraction, $\{T_i\}_{i=1}^m$ a finite family of asymptotically nonexpansive self-mappings of C with sequences $\{k_{i+m(j-1)}\}_{j=1}^\infty \subset [1, \infty)$ $(1 \le i \le m)$ such that $\bigcap_{i=1}^m \mathrm{Fix}(T_i) \neq \emptyset$. Let $\{t_n\}$ be a sequence in $(0, 1)$ such that $\lim_{n\to\infty}(k_n - 1)/(k_n - t_n) = 0$ and let $\{x_n\}$ be a sequence defined by (5). Define the sequence $\{z_n\}$ iteratively by $z_1 \in C$,*

$$z_{n+1} = \left(1 - \frac{t_n}{k_n}\right) f(z_n) + \frac{t_n}{k_n} T_{\bar{n}}^{\hat{n}} z_n, \qquad n \in \mathbf{N}. \tag{15}$$

For any C, f and $\{T_i\}_{i=1}^m$ as above, the sequences $\{x_n\}$ and $\{z_n\}$ both converge strongly to a point $\hat{x} \in \bigcap_{i=1}^m \mathrm{Fix}(T_i)$ which is the unique solution of the variational inequality (3) in $\bigcap_{i=1}^m \mathrm{Fix}(T_i)$ if and only if the following conditions are satisfied for $i = 1, \dots, m$:

(i) $\lim\limits_{j\to\infty} t_{i+m(j-1)} = 1$;

(ii) $\sum_{j=1}^{\infty}[\,1-(t_{i+m(j-1)}/k_{i+m(j-1)})\,]=\infty$;

(iii) $\lim_{n\to\infty}\|x_n-T_ix_n\|=0$ *and* $\lim_{n\to\infty}\|z_n-T_iz_n\|=0$.

Proof. This is an immediate consequence of Theorem 3.2 and Theorem 3.3. □

4. Applications

This section is to utilize all the previous results to establish the necessary and sufficient conditions for the strong convergence of an iterative sequence to a common fixed point of an infinite family of asymptotically nonexpansive mappings in uniformly convex Banach spaces with uniformly Gâteaux differentiable norm satisfying Opial's condition.

Given X, f and C as in Section 3, let $\{T_m\}_{m=1}^{\infty}$ be an infinite family of asymptotically nonexpansive self-mappings of C with sequences $\{k_n^{(m)}\}_{n=1}^{\infty}\subset[1,\infty)$ $(m\in\mathbf{N})$ such that $F=\bigcap_{m=1}^{\infty}\operatorname{Fix}(T_m)\neq\emptyset$. For $m\in\mathbf{N}$, let $\{t_n^{(m)}\}_{n=1}^{\infty}$ be a sequence in $(0,1)$ such that $\lim_{n\to\infty}t_n^{(m)}=1$ and $\lim_{n\to\infty}(k_n^{(m)}-1)/(k_n^{(m)}-t_n^{(m)})=0$. For any fixed $m\in\mathbf{N}$, define a mapping $S_n^{(m)}:C\to C$ by

$$S_n^{(m)}(x)=\left(1-\frac{t_n^{(m)}}{k_n^{(m)}}\right)f(x)+\frac{t_n^{(m)}}{k_n^{(m)}}T_m^nx.$$

Then $S_n^{(m)}$ has a fixed point $x_n^{(m)}\in C$. Hence

$$x_n^{(m)}=\left(1-\frac{t_n^{(m)}}{k_n^{(m)}}\right)f(x_n^{(m)})+\frac{t_n^{(m)}}{k_n^{(m)}}T_m^nx_n^{(m)},\qquad n\in\mathbf{N}.\tag{16}$$

For any fixed $m\in\mathbf{N}$, define the sequence $\{z_n^{(m)}\}_{n=1}^{\infty}$ iteratively by $z_1^{(m)}\in C$,

$$z_{n+1}^{(m)}=\left(1-\frac{t_n^{(m)}}{k_n^{(m)}}\right)f(z_n^{(m)})+\frac{t_n^{(m)}}{k_n^{(m)}}T_m^nz_n^{(m)},\qquad n\in\mathbf{N}.\tag{17}$$

Note that for each $m\in\mathbf{N}$, it is seen from Theorem 3.1 that the sequence $\{x_n^{(m)}\}_{n=1}^{\infty}$, converges strongly to a point $\hat{x}_m\in\operatorname{Fix}(T_m)$ which is the unique solution of (3) in $\operatorname{Fix}(T_m)$ if and only if $\lim_{n\to\infty}\|x_n^{(m)}-T_mx_n^{(m)}\|=0$.

Let X be a Banach space with uniformly Gâteaux differentiable norm. If X satisfies Opial's condition, then the normalized duality mapping $J:X\to X^*$ is weakly sequentially continuous at zero [14, Theorem 2]. Therefore, we can apply Theorems 3.1 to obtain the following result.

Theorem 4.1. *Let X be a uniformly convex Banach space with uniformly Gâteaux differentiable norm satisfying Opial's condition, C a nonempty bounded closed convex subset of X, $f : C \to C$ a contraction with contractive constant $\beta \in (0,1)$, $\{T_m\}_{m=1}^{\infty}$ a family of asymptotically nonexpansive self-mappings of C with sequences $\{k_n^{(m)}\}_{n=1}^{\infty} \subset [1,\infty)$ $(m \in \mathbf{N})$ such that $F = \bigcap_{m=1}^{\infty} \mathrm{Fix}(T_m) \neq \emptyset$. For $m \in \mathbf{N}$, let $\{t_n^{(m)}\}_{n=1}^{\infty}$ be a sequence in $(0,1)$ such that $\lim_{n\to\infty} t_n^{(m)} = 1$ and $\lim_{n\to\infty}(k_n^{(m)} - 1)/(k_n^{(m)} - t_n^{(m)}) = 0$. For each $m \in \mathbf{N}$, the sequence $\{x_n^{(m)}\}_{n=1}^{\infty}$ defined by (16) converges strongly to a point $\hat{x}_m$ of $\mathrm{Fix}(T_m)$ such that the sequence $\{\hat{x}_m\}_{m=1}^{\infty}$ converges strongly to a point of F which is the unique solution of the variational inequality (3) in F if and only if $\lim_{n\to\infty} \|x_n^{(m)} - T_m x_n^{(m)}\| = 0$, for $m \in \mathbf{N}$, and $\lim_{m\to\infty} \|\hat{x}_m - T_i\hat{x}_m\| = 0$, for $i \in \mathbf{N}$.*

Proof. The uniqueness of the solution of (3) in F can be referred to Theorem 3.1 for the proof. For each $m \in \mathbf{N}$, if the sequence $\{x_n^{(m)}\}_{n=1}^{\infty}$ converges strongly to a point $\hat{x}_m$ of $\mathrm{Fix}(T_m)$ such that the sequence $\{\hat{x}_m\}_{m=1}^{\infty}$ converges strongly to a point of F, then by Theorem 3.1, for $m \in \mathbf{N}$, $\lim_{n\to\infty} \|x_n^{(m)} - T_m x_n^{(m)}\| = 0$, and for $i \in \mathbf{N}$, since

$$\|\hat{x}_m - T_i\hat{x}_m\| \leq \|\hat{x}_m - \hat{x}\| + \|T_i\hat{x} - T_i\hat{x}_m\| \leq (1 + k_1^{(i)})\|\hat{x}_m - \hat{x}\|,$$

we have $\lim_{n\to\infty} \|\hat{x}_m - T_i\hat{x}_m\| = 0$.

Conversely, for $m \in \mathbf{N}$, since $\lim_{n\to\infty} \|x_n^{(m)} - T_m x_n^{(m)}\| = 0$, by Theorem 3.1 the sequence $\{x_n^{(m)}\}_{n=1}^{\infty}$ converges strongly to a point $\hat{x}_m$ of $\mathrm{Fix}(T_m)$ which is the unique solution of the variational inequality (3) in $\mathrm{Fix}(T_m)$:

$$\langle f(\hat{x}_m) - \hat{x}_m, J(y - \hat{x}_m)\rangle \leq 0, \qquad \text{for all } y \in \mathrm{Fix}(T_m). \tag{18}$$

Given any subsequence $\{\hat{x}_{m_j}\}$ of $\{\hat{x}_m\}$, since X is reflexive, there is a subsequence of $\{\hat{x}_{m_j}\}$, still denoted by $\{\hat{x}_{m_j}\}$, converging weakly to a point $\hat{x} \in C$. Since each $I - T_i$ is demiclosed at zero and $\lim_{m_j\to\infty} \|\hat{x}_{m_j} - T_i\hat{x}_{m_j}\| = 0$, we have $\hat{x} - T_i\hat{x} = 0$ and so $\hat{x} \in F = \bigcap_{i=1}^{\infty} \mathrm{Fix}(T_i)$. For $m \in \mathbf{N}$, we obtain from (16) that

$$\begin{aligned}\|x_n^{(m)} - \hat{x}\|^2 &= \left\langle \left(1 - \frac{t_n^{(m)}}{k_n^{(m)}}\right)[f(x_n^{(m)}) - \hat{x}] + \frac{t_n^{(m)}}{k_n^{(m)}}(T_m^n x_n^{(m)} - T_m^n\hat{x}), J(x_n^{(m)} - \hat{x})\right\rangle \\ &\leq \left(1 - \frac{t_n^{(m)}}{k_n^{(m)}}\right) \|f(x_n^{(m)}) - f(\hat{x})\| \cdot \|x_n^{(m)} - \hat{x}\| + \frac{t_n^{(m)}}{k_n^{(m)}} \cdot k_n^{(m)}\|x_n^{(m)} - \hat{x}\|^2\end{aligned}$$

$$+\left(1-\frac{t_n^{(m)}}{k_n^{(m)}}\right)\left\langle f(\hat{x})-\hat{x}, J(x_n^{(m)}-\hat{x})\right\rangle$$

$$\leq\left[\beta\left(1-\frac{t_n^{(m)}}{k_n^{(m)}}\right)+t_n^{(m)}\right]\|x_n^{(m)}-\hat{x}\|^2+\left(1-\frac{t_n^{(m)}}{k_n^{(m)}}\right)\langle f(\hat{x})-\hat{x}, J(x_n^{(m)}-\hat{x})\rangle,$$

and so for sufficiently large n,

$$\|x_n^{(m)}-\hat{x}\|^2\leq\left[\frac{1-t_n^{(m)}}{1-(t_n^{(m)}/k_n^{(m)})}-\beta\right]^{-1}\langle f(\hat{x})-\hat{x}, J(x_n^{(m)}-\hat{x})\rangle. \tag{19}$$

Since $\{x_n^{(m)}\}_{n=1}^{\infty}$ converges strongly to $\hat{x}_m$, by (4) the inequality (19) implies that

$$\|\hat{x}_m-\hat{x}\|^2\leq\frac{1}{1-\beta}\langle f(\hat{x})-\hat{x}, J(\hat{x}_m-\hat{x})\rangle \quad \text{for } m\in\mathbf{N}.$$

Therefore the weak convergence of $\{\hat{x}_{m_j}\}$ to $\hat{x}$ and the weakly sequential continuity of J at zero ensure that

$$\limsup_{m_j\to\infty}\|\hat{x}_{m_j}-\hat{x}\|^2\leq\lim_{m_j\to\infty}\frac{1}{1-\beta}\langle f(\hat{x})-\hat{x}, J(\hat{x}_{m_j}-\hat{x})\rangle=0;$$

hence $\{\hat{x}_{m_j}\}$ converges strongly to $\hat{x}$.

Indeed, $\hat{x}$ is a solution of the variational inequality (3) in F. For, if $y\in F$, observe that

$$\begin{aligned}&|\langle\hat{x}_m-f(\hat{x}_m), J(\hat{x}_m-y)\rangle-\langle\hat{x}-f(\hat{x}), J(\hat{x}-y)\rangle|\\ &=|\langle[\hat{x}_m-f(\hat{x}_m)]-[\hat{x}-f(\hat{x})], J(\hat{x}_m-y)\rangle+\langle\hat{x}-f(\hat{x}), J(\hat{x}_m-y)-J(\hat{x}-y)\rangle|\\ &\leq\|[\hat{x}_m-f(\hat{x}_m)]-[\hat{x}-f(\hat{x})]\|\cdot\|\hat{x}_m-y\|+|\langle\hat{x}-f(\hat{x}), J(\hat{x}_m-y)-J(\hat{x}-y)\rangle|.\end{aligned} \tag{20}$$

Since $\{\hat{x}_{m_j}\}$ converges strongly to $\hat{x}$ and J is norm-to-weak* continuous, it follows from (18) and (20) that

$$\langle f(\hat{x})-\hat{x}, J(y-\hat{x})\rangle=\lim_{m_j\to\infty}\langle f(\hat{x}_{m_j})-\hat{x}_{m_j}, J(y-\hat{x}_{m_j})\rangle\leq 0.$$

This shows that $\hat{x}$ is a solution of the variational inequality (3) in F. Consequently, any subsequence of $\{x_n\}$ has a strongly convergent subsequence with limit $\hat{x}$, and hence $\{x_n\}$ converges strongly to $\hat{x}$. □

By Theorem 3.2, for each $m \in \mathbf{N}$, the sequences $\{x_n^{(m)}\}_{n=1}^\infty$ and $\{z_n^{(m)}\}_{n=1}^\infty$ defined by (16) and (17) both converge strongly to a fixed point $\hat{x}_m$ of T_m which is the unique solution of the variational inequality (3) in $\mathrm{Fix}(T_m)$ if and only if for all $m \in \mathbf{N}$ we have $\lim_{n\to\infty} t_n^{(m)} = 1$, $\sum_{n=1}^\infty [\,1 - (t_n^{(m)}/k_n^{(m)})\,] = \infty$, $\lim_{n\to\infty} \|x_n^{(m)} - T_m x_n^{(m)}\| = 0$ and $\lim_{n\to\infty} \|z_n^{(m)} - T_m z_n^{(m)}\| = 0$. Hence, the following result is a direct consequence of Theorem 3.2 and Theorem 4.1.

Theorem 4.2. *Let X be a uniformly convex Banach space with uniformly Gâteaux differentiable norm satisfying Opial's condition, C a nonempty bounded closed convex subset of X, $f : C \to C$ a contraction, $\{T_m\}_{m=1}^\infty$ a sequence of asymptotically nonexpansive self-mappings of C with sequences $\{k_n^{(m)}\}_{n=1}^\infty \subset [1,\infty)$ ($m \in \mathbf{N}$) such that $F = \bigcap_{m=1}^\infty \mathrm{Fix}(T_m) \neq \emptyset$. For $m \in \mathbf{N}$, let $\{t_n^{(m)}\}_{n=1}^\infty$ be a sequence in $(0,1)$ such that $\lim_{n\to\infty} t_n^{(m)} = 1$ and $\lim_{n\to\infty}(k_n^{(m)} - 1)/(k_n^{(m)} - t_n^{(m)}) = 0$. For each $m \in \mathbf{N}$, the sequences $\{x_n^{(m)}\}_{n=1}^\infty$ and $\{z_n^{(m)}\}_{n=1}^\infty$ defined by (16) and (17) respectively converge strongly to a point $\hat{x}_m$ of $\mathrm{Fix}(T_m)$ such that the sequence $\{\hat{x}_m\}_{m=1}^\infty$ converges strongly to a point of F which is the unique solution of the variational inequality (3) in F if and only if the following conditions hold:*

(i) $\lim\limits_{n\to\infty} t_n^{(m)} = 1$, *for* $m \in \mathbf{N}$;

(ii) $\sum\limits_{n=1}^{\infty} [\,1 - (t_n^{(m)}/k_n^{(m)})\,] = \infty$, *for* $m \in \mathbf{N}$;

(iii) $\lim\limits_{n\to\infty} \|x_n^{(m)} - T_m x_n^{(m)}\| = 0$ *and* $\lim\limits_{n\to\infty} \|z_n^{(m)} - T_m z_n^{(m)}\| = 0$, *for* $m \in \mathbf{N}$.

(iv) $\lim\limits_{m\to\infty} \|\hat{x}_m - T_i \hat{x}_m\| = 0$, *for* $i \in \mathbf{N}$.

Proof. The result follows directly from Theorem 3.2 and Theorem 4.1. □

Remark. Using the same iterations and arguments as Theorems 4.1 and 4.2 in the case when C is a closed, not necessarily bounded, convex set, since the sequence $\{\hat{x}_m\}_{m=1}^\infty$ is bounded, cf. [17, proof of Theorem 3.1], it is seen that the results in [17] can be extended to an infinite sequence of asymptotically nonexpansive mappings. Accordingly, the boundedness on C in Theorems 4.1 and 4.2 can be dropped.

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