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ISHIKAWA ITERATIONS FOR EQUILIBRIUM AND FIXED POINT PROBLEMS FOR NONEXPANSIVE MAPPINGS IN HILBERT SPACES

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Abstract. In this paper, we introduce an iterative scheme Ishikawa-type for finding a common element of the set $EP(G)$ of the equilibrium points of a bifunction G and the set $Fix(T)$ of fixed points of a nonexpansive mapping T in a Hilbert space H. We prove that the method converges strongly to an element $z \in Fix(T) \bigcap EP(G)$ which is the unique solution of the variational inequality $\langle (A-\gamma f)z, x-z\rangle \geq 0$ for every $x \in Fix(T) \cap EP(G)$. The results presented here are situated on the line of research of [5, 6, 7, 10, 12, 13].

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1. Introduction

Let H be a real Hilbert space. A mapping $T : H \to H$ is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\| \quad \forall\, x, y \in H\,.$$

We denote by $Fix(T)$ the set of fixed points of T; that is,

$$Fix(T) = \{x \in H : Tx = x\}.$$

An operator $A : H \to H$ is said to be strongly positive bounded linear operator on H if there exists $\bar{\gamma} > 0$ such that

$$\langle Ax, x\rangle \geq \bar{\gamma}||x||^2 \quad \forall\, x \in H\,.$$

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A mapping $f : H \to H$ is said to be a contraction if there exists a constant $\alpha \in [0, 1[$ such that

$$\|f(x) - f(y)\| \le \alpha \|x - y\| \quad \forall\, x, y \in H\,.$$

Recently, iterative methods for nonexpansive mappings are widely used to solve convex minimization problems (see [5] and references therein).

A typical problem is to minimize over the set $Fix(T)$ of fixed points of a nonexpansive mapping T, the quadratic function

$$\frac{1}{2}\langle Ax, x\rangle - \langle x, b\rangle$$

In [13], Xu proved that, if x_0 is a point chosen arbitrarily in H, the sequence $(x_n)_{n\in\mathbb{N}}$ defined by

$$x_{n+1} = (I - \alpha_n A)Tx_n + \alpha_n b \quad n \ge 0\,,$$

strongly converges to the unique solution of

$$\min_{x\in Fix(T)} \frac{1}{2}\langle Ax, x\rangle - \langle x, b\rangle\,. \tag{1.1}$$

under certain conditions on $(\alpha_n)_{n\in\mathbb{N}}$.

Moudafi in [6] introduced the viscosity approximation method for nonexpansive mappings

$$x_{n+1} = (I - \sigma_n)Tx_n + \sigma_n f(x_n) \quad n \ge 0$$

where $(\sigma_n)_{n\in\mathbb{N}}$ is a sequence in $]0, 1[$ and f is a contraction on H.

It is proved that by suitable assumptions on $(\sigma_n)_{n\in\mathbb{N}}$, the sequence $(x_n)_{n\in\mathbb{N}}$ strongly converges to the unique solution z in $Fix(T)$ of the variational inequality

$$\big\langle (I - f)z, x - z\big\rangle \ge 0 \quad \forall\, x \in Fix(T)\,.$$

Recently, Marino and Xu [5] considered the iterative method

$$x_{n+1} = (I - \alpha_n A)Tx_n + \alpha_n \gamma f(x_n) \quad n \ge 0$$

where they combined the iterative method introduced by Xu with the viscosity approximation introduced by Moudafi.

They proved that, by suitable hypotheses on $(\alpha_n)_{n\in\mathbb{N}}$, the sequence $(x_n)_{n\in\mathbb{N}}$ converges strongly to the unique solution of the variational inequality

$$\big\langle (A - \gamma f)z, x - z\big\rangle \ge 0 \quad \forall\, x \in Fix(T)$$

which is the optimality condition for the minimization problem

$$\min_{x\in Fix(T)} \frac{1}{2}\langle Ax, x\rangle - h(x) \tag{1.2}$$

where h is a potential function for γf (i.e. $h' = \gamma f$ on H).

On the other hand, let $G : H \times H \to \mathbb{R}$ be a bifunction. We say that $x \in H$ is an equilibrium point if and only if

$$G(x, y) \geq 0 \qquad \forall y \in H\,.$$

The problem of finding the equilibrium points for a bifunction G is well known as equilibrium problem.

Many problems in applied sciences reduce to finding the equilibrium points of G (see [1, 8] and the references quoted therein). In the sequel we indicate with $EP(G)$ the set of such points.

In 2007, in [12] S. Takahashi and W. Takahashi introduced the following iterative method for finding an equilibrium point of G which is also a fixed point of a nonexpansive mapping T

$$\begin{cases} G(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, & \forall y \in H\,, \\ \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Tu_n, & \forall n \geq 0 \end{cases} \tag{1.3}$$

More recently, in [10] S. Plubtieng and R. Punpaeng generalized the above scheme introducing the following iterative method, in which appears the operator A, in order to find an equilibrium point of G which is also a fixed point of a mapping $T : H \to H$

$$\begin{cases} G(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, & \forall y \in H \\ \\ x_{n+1} = \alpha_n f(x_n) + (I - \alpha_n\, A)Tu_n, & \forall n \geq 0 \end{cases} \tag{1.4}$$

They proved that, in appropriate conditions on $(\alpha_n)_{n\in\mathbb{N}}$ and $(r_n)_{n\in\mathbb{N}}$, the sequences $(x_n)_{n\in\mathbb{N}}$ and $(u_n)_{n\in\mathbb{N}}$ are strongly convergent to the unique solution $z \in Fix(T) \bigcap EP(G)$ of the variational inequality

$$\langle (A - \gamma f)z, x - z\rangle \geq 0\,, \quad \forall x \in Fix(T) \cap EP(G) \tag{1.5}$$

which is the optimality condition for the minimization problem

$$\min_{x\in Fix(T)\cap EP(G)} \frac{1}{2}\left\langle Ax,x\right\rangle - h(x)\,.$$

In this paper, motivated from [10], we introduce the following Ishikawa iterations

$$\begin{cases} x_0 \in H \\ G(u_n,y)+\frac{1}{r_n}\left\langle y-u_n,u_n-x_n\right\rangle \geq 0 \quad \forall\, y\in H \\ y_n=\beta_n x_n+(1-\beta_n)Tu_n \\ x_{n+1}=\alpha_n\gamma f(x_n)+(I-\alpha_n A)y_n \end{cases}$$

which summarizes equilibrium problem and fixed points problems for nonexpansive mappings. Our aim is to find an element $z\in Fix(T)\cap EP(G)$ that is the unique point in $Fix(T)\cap EP(G)$ solving the variational inequality

$$\left\langle (A-\gamma f)z, x-z\right\rangle \geq 0 \quad \forall\, x\in Fix(T)\cap EP(G)$$

under weaker hypotheses on the coefficients α_n than one in [10] and [12].

Recall that the uniqueness is a consequence of strong monotonicity of $(A-\gamma f)$ and was proved in [5].

2. Preliminar results

We recall some basic definitions and results of Hilbert spaces which we use in the next section.

Lemma 2.1. *For all $x,y\in H$, there holds the inequality*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle.$$

If K is closed convex subset of a real Hilbert space H, the metric projection $P_K : H\to K$ is the mapping defined as follows: for each $x\in H$, $P_K x$ is the only point in K with the property

$$||x-P_K x|| = \inf_{y\in K} ||x-y||\,.$$

Lemma 2.2. *Let K be a nonempty closed convex subset of a real Hilbert space H and let P_K be the metric projection from H onto K. Given $x\in H$ and $z\in K$, $z=P_K x$ if and only if*

$$\langle x-z, y-z\rangle \leq 0 \quad \forall\, y\in K\,.$$

Lemma 2.3. [5] *Let H be a Hilbert space and let $A : H \to H$ be a strongly positive linear bounded operator with coefficient $\bar{\gamma} > 0$. If $0 < \rho \leq ||A||^{-1}$, then $||I - \rho A|| \leq (1 - \rho\bar{\gamma})$.*

In order to solve the equilibrium problem for a function $G : H \times H \to \mathbb{R}$, we assume that

(E1) $G(x, x) = 0$ for all $x \in H$;

(E2) $G(x, y) + G(y, x) \leq 0$ for all $(x, y) \in H \times H$ (i.e. G is monotone);

(E3) for each $x, y, z \in H$

$$\limsup_{t\to 0} G(tz + (1-t)x, y) \leq G(x, y)\,;$$

(E4) the function $y \to G(x, y)$ is convex and lower semicontinuous for each $x \in H$.

Lemma 2.4. [1] *Let C be a nonempty closed convex subset of H and let G be satisfying (E1)-(E4). Let $r > 0$ and $x \in H$. Then there exists $z \in C$ such that*

$$G(z, y) + \frac{1}{r}\,\langle y - z, z - x\rangle \,\geq 0 \quad \forall\, y \in C\,.$$

Lemma 2.5. [3] *Let C be a nonempty closed convex subset of H and let G be satisfying (E1)-(E4). For $r > 0$ and $x \in H$, the mapping $S_r : H \to C$ defined by*

$$S_r(x) = \Big\{z \in C : \quad G(z, y) + \frac{1}{r}\,\langle y - z, z - x\rangle \,\geq 0 \quad \forall\, y \in C\Big\}$$

(1) *S_r is single-valued;*

(2) *S_r is firmly nonexpansive, i.e.*

$$||S_r(x) - S_r(y)||^2 \leq \langle S_r(x) - S_r(y), x - y\rangle \quad \forall\, x, y \in H\,;$$

(3) *$Fix(S_r) = EP(G)$;*

(4) *$EP(G)$ is closed and convex.*

Lemma 2.6. *Let us suppose (E1)-(E4) hold for the equilibrium function $G : H \to H$. Let $x, y \in H$, $r_1, r_2 > 0$. Then:*

$$\|S_{r_2}(y) - S_{r_1}(x)\| \leq \|y - x\| + \left|\frac{r_2 - r_1}{r_2}\right| \, \|S_{r_2}(y) - y\| \tag{2.1}$$

Proof. Calling $u_1 := S_{r_1}(x)$ and $u_2 := S_{r_2}(y)$, we know that:

$$G(u_2, z) + \frac{1}{r_2}\langle z - u_2, u_2 - y\rangle \geq 0 \;\; \forall z \in H$$

and

$$G(u_1, z) + \frac{1}{r_1}\langle z - u_1, u_1 - x\rangle \geq 0 \quad \forall z \in H.$$

In particular,

$$G(u_2, u_1) + \frac{1}{r_2}\langle u_1 - u_2, u_2 - y\rangle \geq 0$$

and

$$G(u_1, u_2) + \frac{1}{r_1}\langle u_2 - u_1, u_1 - x\rangle \geq 0.$$

Hence, summing up these two inequalities and using $(E2)$,

$$\frac{1}{r_1}\langle u_2 - u_1, u_1 - x\rangle + \frac{1}{r_2}\langle u_1 - u_2, u_2 - y\rangle \geq 0$$

so it follows that

$$\left\langle u_2 - u_1, \frac{u_1 - x}{r_1} - \frac{u_2 - y}{r_2}\right\rangle \geq 0. \tag{2.2}$$

We derive from (2.2) that

$$\begin{aligned}
&\left\langle u_2 - u_1, u_1 - x - \frac{r_1}{r_2}(u_2 - y)\right\rangle \geq 0 \Rightarrow \\
&\left\langle u_2 - u_1, u_1 - u_2 - x + u_2 - \frac{r_1}{r_2}(u_2 - y)\right\rangle \geq 0 \Rightarrow \\
&-\|u_2 - u_1\|^2 + \langle u_2 - u_1, (u_2 - y)\left(1 - \frac{r_1}{r_2}\right) + (y - x)\rangle \geq 0
\end{aligned}$$

Then:

$$\begin{aligned}
\|u_2 - u_1\|^2 &\leq \langle u_2 - u_1, (u_2 - y)\left(1 - \frac{r_1}{r_2}\right) + (y - x)\rangle \\
&\leq \|u_2 - u_1\| \left(\|y - x\| + \left|1 - \frac{r_1}{r_2}\right| \|u_2 - y\|\right)
\end{aligned}$$

and so,

$$\|u_2 - u_1\| \leq \|y - x\| + \left|1 - \frac{r_1}{r_2}\right| \|u_2 - y\|$$

□

The following Suzuki's result is a fundamental tool in order to prove the asymptotic regularity of $(x_n)_{n\in\mathbb{N}}$ under minimal conditions on $(\beta_n)_{n\in\mathbb{N}}$.

Lemma 2.7. *Let* $(x_n)_{n\in\mathbb{N}}$ *and* $(z_n)_{n\in\mathbb{N}}$ *be bounded sequences in a Banach space* X *and let* $(\beta_n)_{n\in\mathbb{N}}$ *be a sequence in* $[0,1]$ *with* $0 < \liminf\limits_{n\to\infty} \beta_n \le \limsup\limits_{n\to\infty} \beta_n < 1$. *Suppose*

$$x_{n+1} = \beta_n x_n + (1-\beta_n) z_n$$

for all integers $n \ge 0$ *and*

$$\limsup_{n\to\infty} (\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \le 0.$$

Then $\lim_{n\to\infty} \|x_n - z_n\| = 0$.

Lemma 2.8. [9] *Let* X *be a Banach space with weakly sequentially continuous duality mapping* J, *and suppose* $(x_n)_{n\in\mathbb{N}}$ *converges weakly to* $x_0 \in X$. *Then for any* $x \in X$,

$$\liminf_n \|x_n - x_0\| \le \liminf_n \|x_n - x\|$$

Moreover if X *is uniformly convex, equality holds if and only if* $x_0 = x$.

Finally we conclude this section with a result on real sequences which play an important role in many papers regarding explicit iteration schemes.

Lemma 2.9. [13] *Let* $(\alpha_n)_{n\in\mathbb{N}}$ *be a sequence of nonnegative real numbers satisfying the following inequality*

$$\alpha_{n+1} \le (1-\gamma_n)\alpha_n + \gamma_n \sigma_n \,, \quad \forall\, n \ge 0 \,,$$

where $(\gamma_n)_{n\in\mathbb{N}}$ *is a sequence in* $]0,1[$ *and* $(\sigma_n)_{n\in\mathbb{N}}$ *is a sequence of real numbers such that*

(1) $\lim\limits_{n\to+\infty} \gamma_n = 0 \quad$ *and* $\quad \sum\limits_{n\ge 0} \gamma_n = +\infty \,.$

(2) $\limsup\limits_{n\to+\infty} \sigma_n \le 0 \quad$ *or* $\quad \sum\limits_{n\ge 0} |\gamma_n \sigma_n| < +\infty \,.$

Then $(\alpha_n)_{n\in\mathbb{N}}$ *converges to zero.*

3. MAIN RESULT.

In this section we consider the following scheme:

$$\begin{cases} x_0 \in H \\ G(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \ge 0 \quad \forall\, y \in H \\ y_n = \beta_n x_n + (1-\beta_n) T u_n \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A) y_n \end{cases} \tag{3.1}$$

and we suppose that the following hypotheses hold.

Let $G : H \times H \to \mathbb{R}$ be a bifunction satisfying (E1)-(E4) and $T : H \to H$ a nonexpansive mapping. Suppose that $EP(G) \cap Fix(T) \neq \emptyset$.

Let $(r_n)_{n\in\mathbb{N}}$ be a sequence of positive real number such that:

(R1) $\liminf\limits_{n\to+\infty} r_n = r > 0$;

(R2) $\limsup\limits_{n} |r_n - r_{n-1}| = 0$.

Let $(\alpha_n)_{n\in\mathbb{N}}$, $(\beta_n)_{n\in\mathbb{N}}$ be sequences in $]0, 1[$ such that:

(A1) $\alpha_n \to 0$ and $\sum\limits_{n\in\mathbb{N}} \alpha_n = +\infty$;

(B1) $0 < \liminf\limits_{n\to+\infty} \beta_n \leq \limsup\limits_{n\to+\infty} \beta_n < 1$.

Let suppose that $f : H \to H$ be a $\alpha-$contraction and A be a strongly positive linear bounded self-adjoint operator of H into itself with coefficient $\bar{\gamma} > 0$. Let γ be a real number such that $0< \gamma < \bar{\gamma}/\alpha$.

We notice that by Lemma 2.5 $u_n = S_{r_n} x_n$ and for all $p \in EP(G)$ and $n \in \mathbb{N}$, it results

$$\|u_n - p\| \leq \|x_n - p\| \tag{3.2}$$

First, we prove some properties of sequences $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$, $(u_n)_{n\in\mathbb{N}}$ which will be used in the proof of our main result.

We recall that a sequence $(x_n)_{n\in\mathbb{N}}$ is said to be asymptotically regular if

$$||x_{n+1} - x_n|| \to 0, \qquad n \to \infty.$$

In the sequel we denote by $\omega_w(x_n)$ the set of weak cluster points of $(x_n)_{n\in\mathbb{N}}$ and with $\omega_s(x_n)$ the set of strong cluster points of $(x_n)_{n\in\mathbb{N}}$.

Lemma 3.1. *Let $G : H\times H \to \mathbb{R}$ be a bifunction such that (E1)-(E4) hold and $T : H \to H$ a nonexpansive mapping. Let suppose that $EP(G) \cap Fix(T) \neq \emptyset$.*

Let $(r_n)_{n\in\mathbb{N}}$ be a sequence of positive real number such that (R1) and (R2) hold.

Let $(\alpha_n)_{n\in\mathbb{N}}$, $(\beta_n)_{n\in\mathbb{N}}$ be two sequences in $]0, 1[$ such that (A1) and (B1) hold.

Let suppose that $f : H \to H$ be a $\alpha-$contraction and A be a strongly positive linear bounded self-adjoint operator on H with coefficient $\bar{\gamma} > 0$. Let γ be a real number such that $0< \gamma < \bar{\gamma}/\alpha$.

Then

(i) $(x_n)_{n\in\mathbb{N}}$ $(y_n)_{n\in\mathbb{N}}$, $(u_n)_{n\in\mathbb{N}}$ *defined in (3.1) are bounded;*

(ii) *the sequences* $(x_n)_{n\in\mathbb{N}}$*,* $(u_n)_{n\in\mathbb{N}}$ *and* $(y_n)_n$ *are asymptotically regulars;*

(iii) $(x_n - y_n)_{n\in\mathbb{N}}$*,* $(x_n - u_n)_{n\in\mathbb{N}}$*,* $(x_n - Tu_n)_{n\in\mathbb{N}}$ *are null sequences;*

(iv) *one of* $(x_n)_{n\in\mathbb{N}}$*,* $(y_n)_{n\in\mathbb{N}}$*,* $(u_n)_{n\in\mathbb{N}}$ *converges if and only if also the other two converge to the same limit;*

(v) $\omega_s(x_n) = \omega_s(y_n) = \omega_s(u_n)$ *and* $\omega_w(x_n) = \omega_w(y_n) = \omega_w(u_n)$*.*

Proof. Since $\lim\limits_{n\to+\infty} \alpha_n = 0$, we have $\alpha_n < ||A||^{-1}$ for big $n \in \mathbb{N}$.

The proof of (i) follows by the fact that $Fix(T)$ and $EP(G)$ have a common element. Indeed if $p \in Fix(T) \cap EP(G)$, we have:

$$\begin{aligned} \|x_{n+1} - p\| &= \|\alpha_n(\gamma f(x_n) - Ap) + (I - \alpha_n A)(y_n - p)\| \\ &= \|\alpha_n\gamma(f(x_n) - f(p)) + \alpha_n(\gamma f(p) - Ap) + (I - \alpha_n A)(y_n - p)\| \\ &\leq \alpha_n\gamma\alpha\|x_n - p\| + \alpha_n\|\gamma f(p) - Ap\| + (1 - \alpha_n\bar{\gamma})\|y_n - p\| \end{aligned}$$

It is enough to observe that:

$$\begin{aligned} \|y_n - p\| &= \|\beta_n(x_n - p) + (1 - \beta_n)(Tu_n - p)\| \\ &\leq \beta_n\|x_n - p\| + (1 - \beta_n)\|Tu_n - Tp\| \\ &\leq \beta_n\|x_n - p\| + (1 - \beta_n)\|u_n - p\| \qquad (3.3) \\ &\leq \|x_n - p\| \qquad (3.4) \end{aligned}$$

to obtain that:

$$\begin{aligned} \|x_{n+1} - p\| &\leq \alpha_n\gamma\alpha\|x_n - p\| + \alpha_n\|\gamma f(p) - Ap\| + (1 - \alpha_n\bar{\gamma})\|x_n - p\| \\ &= [1 - \alpha_n(\bar{\gamma} - \alpha\gamma)]\|x_n - p\| + \alpha_n\|\gamma f(p) - Ap\| \end{aligned}$$

By induction we have:

$$\|x_n - p\| \leq \max\left\{\|x_1 - p\|, \frac{\|\gamma f(p) - Ap\|}{\bar{\gamma} - \alpha\gamma}\right\}$$

i.e. $(x_n)_{n\in\mathbb{N}}$ is bounded. From (3.2) and (3.3) it follows that also $(u_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$ are bounded.

(ii) In order to use Lemma 2.7, we set $x_{n+1} := \beta_n x_n + (1 - \beta_n)z_n$.

Then we have

$$z_n = \frac{\alpha_n\gamma f(x_n) + (I - \alpha_n A)y_n - \beta_n x_n}{1 - \beta_n} = \frac{\alpha_n}{1 - \beta_n}(\gamma f(x_n) - Ay_n) + Tu_n$$

and

$$\begin{aligned}\|z_{n+1}-z_n\| &= \left\|\frac{\alpha_{n+1}}{1-\beta_{n+1}}(\gamma f(x_{n+1})-Ay_{n+1})-\frac{\alpha_n}{1-\beta_n}(\gamma f(x_n)-Ay_n)\right.\\ &\quad\left.+Tu_{n+1}-Tu_n\right\|\\ &\leq \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|\gamma f(x_{n+1})-Ay_{n+1}\|+\frac{\alpha_n}{1-\beta_n}\|\gamma f(x_n)-Ay_n\|\\ &\quad+\|Tu_{n+1}-Tu_n\|\\ &\leq \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|\gamma f(x_{n+1})-Ay_{n+1}\|+\frac{\alpha_n}{1-\beta_n}\|\gamma f(x_n)-Ay_n\|\\ &\quad+\|u_{n+1}-u_n\|\end{aligned}$$

By Lemma 2.6, (choosing $x=x_n, y=x_{n+1}$ and $r_1=r_n, r_2=r_{n+1}$)

$$||u_{n+1}-u_n||\leq ||x_{n+1}-x_n||+\frac{|r_{n+1}-r_n|}{r_{n+1}}||u_{n+1}-x_{n+1}|| \tag{3.5}$$

so, if $b>0$ is a definitive minorant for $(r_n)_{n\in\mathbb{N}}$:

$$||u_{n+1}-u_n||\leq ||x_{n+1}-x_n||+\frac{|r_{n+1}-r_n|}{b}||u_{n+1}-x_{n+1}||\,. \tag{3.6}$$

Hence we have:

$$\begin{aligned}\|z_{n+1}-z_n\| &\leq \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|\gamma f(x_{n+1})-Ay_{n+1}\|\\ &\quad+\frac{\alpha_n}{1-\beta_n}\|\gamma f(x_n)-Ay_n\|+||x_{n+1}-x_n||\\ &\quad+\frac{|r_{n+1}-r_n|}{b}||u_{n+1}-x_{n+1}||\Rightarrow\\ \|z_{n+1}-z_n\|-||x_{n+1}-x_n|| &\leq \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|\gamma f(x_{n+1})-Ay_{n+1}\|\\ &\quad+\frac{\alpha_n}{1-\beta_n}\|\gamma f(x_n)-Ay_n\|\\ &\quad+\frac{|r_{n+1}-r_n|}{b}||u_{n+1}-x_{n+1}||\end{aligned}$$

By (i), (A1) and (R2) we obtain that $\limsup_n \|z_{n+1}-z_n\|-||x_{n+1}-x_n||\leq 0$.

By Suzuki's Lemma 2.7 we conclude that $\|z_n-x_n\|\to 0$. As rule:

$$\|x_{n+1}-x_n\|=(1-\beta_n)\|x_n-z_n\|\to 0$$

as $n\to\infty$.

Let us observe that by (3.6) and (R2), $\|u_{n+1}-u_n\| \to 0$, i.e. $(u_n)_n$ is asymptotically regular too.

In order to prove that the asymptotical regularity of $(y_n)_{n\in\mathbb{N}}$ we need of (iii)

$$\|x_n - y_n\| \to 0\,. \tag{3.7}$$

Since

$$\begin{aligned} \|x_n - y_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| \\ &= \|x_n - x_{n+1}\| + \alpha_n \|\gamma f(x_n) - Ay_n\| \end{aligned}$$

by (i) and (ii) follows (3.7).

If we observe that

$$\|y_{n+1} - y_n\| \leq \|y_{n+1} - x_{n+1}\| + \|x_{n+1} - x_n\| + \|x_n - y_n\|$$

by (ii) and (3.7), $(y_n)_{n\in\mathbb{N}}$ is asymptotically regular.

Now we prove that

$$\|x_n - u_n\| \to 0\,. \tag{3.8}$$

By the firm nonexpansivity of S_{r_n}, if $p \in EP(G)$ we have

$$\begin{aligned} \|u_n - p\|^2 &= \langle u_n - p, S_{r_n}x_n - S_{r_n}p\rangle \leq \langle u_n - p, x_n - p\rangle \\ &= \frac{1}{2}(\|u_n - p\|^2 + \|x_n - p\|^2 - \|x_n - u_n\|^2) \end{aligned}$$

from which:

$$\|u_n - p\|^2 \leq \|x_n - p\|^2 - \|x_n - u_n\|^2 \tag{3.9}$$

In particular, if $p \in EP(G) \cap Fix(T)$ we have:

$$\begin{aligned} \|y_{n+1} - p\|^2 &\leq \beta_n \|x_n - p\|^2 + (1-\beta_n)\|Tu_n - p\|^2 \\ &\leq \beta_n \|x_n - p\|^2 + (1-\beta_n)\|u_n - p\|^2 \\ &\leq \beta_n \|x_n - p\|^2 + (1-\beta_n)\|x_n - p\|^2 - (1-\beta_n)\|x_n - u_n\|^2 \\ &= \|x_n - p\|^2 - (1-\beta_n)\|x_n - u_n\|^2 \end{aligned}$$

Hence

$$(1-\beta_n)\|x_n - u_n\|^2 \leq \|x_n - p\|^2 - \|y_{n+1} - p\|^2$$

i.e.

$$\begin{aligned}\|x_n-u_n\|^2 &\le \frac{1}{1-\beta_n}\|x_n-y_{n+1}\|\big[\|x_n-p\|+\|y_{n+1}-p\|\big]\\ &\le \frac{1}{1-\beta_n}\|x_n-y_{n+1}\|M\,,\quad M:=\sup_n\big[\|x_n-p\|+\|y_{n+1}-p\|\big]\\ &\le \frac{M}{1-\beta_n}\big[\|x_n-x_{n+1}\|+\|x_{n+1}-y_{n+1}\|\big]\,. \qquad (3.10)\end{aligned}$$

From (B1), (ii), and (3.7), (3.8) follows.

In order to prove

$$\|x_n-Tu_n\|\to 0 \qquad (3.11)$$

we observe that:

$$\begin{aligned}\|x_n-Tu_n\| &\le \|x_n-y_n\|+\|y_n-Tu_n\|\\ &\le \|x_n-y_n\|+\beta_n\|x_n-Tu_n\|\end{aligned}$$

i.e.

$$(1-\beta_n)\|x_n-Tu_n\|\le\|x_n-y_n\|$$

and by (B1) and (3.7) follows (3.11).

Finally, (iv) and (v) follow directly by the property (iii), indipendently as the sequnces are defined, and this is true in the broader setting of Banach spaces. ☐

Corollary 3.2. *We suppose that the hypotheses of Lemma 3.1 are satisfied. Then the weak cluster points of* $(x_n)_{n\in\mathbb{N}}$ *are equilibrium points of* G *and fixed points of* T, *i.e.* $\omega_w(x_n)\subseteq Fix(T)\bigcap EP(G)$.

Proof. Let q a weak cluster point of $(x_n)_{n\in\mathbb{N}}$ and let $(x_{n_m})_{m\in\mathbb{N}}$ be a subsequence of $(x_n)_{n\in\mathbb{N}}$ weakly converging to q. We show that $q\in EP(G)$.

At first, note that by $(E2)$ we have

$$\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle\ge G(y,u_n).$$

In particular,

$$\left\langle y-u_{n_m},\frac{u_{n_m}-x_{n_m}}{r_{n_m}}\right\rangle\ge G(y,u_{n_m}). \qquad (3.12)$$

By condition $(E4)$, for $x\in H$ fixed, the function $G(x,\cdot)$ is lower semicontinuos and convex, and thus weakly lower semicontinuous [2].

Since $\|x_n - u_n\| \to 0$, as $n \to \infty$ and by (R1) we obtain $(u_{n_m} - x_{n_m})/r_{n_m} \to 0$. Therefore, letting $m \to \infty$ in (3.12) yields

$$G(y, q) \leq \lim_{m\to\infty} G(y, u_m) \leq 0, \quad y \in H.$$

Replacing y with $y_\tau := \tau y + (1-\tau)q$ with $\tau \in [0,1]$ and using $(E1)$ and $(E4)$, we obtain

$$0 = G(y_\tau, y_\tau) \leq \tau G(y_\tau, y) + (1-\tau)G(y_\tau, q) \leq \tau G(y_\tau, y).$$

Hence:

$$G(\tau y + (1-\tau)q, y) \geq 0, \quad \tau \in (0,1], \ y \in H.$$

Letting $\tau \to 0^+$ and using assumption $(E3)$, we conclude

$$G(q, y) \geq 0, \quad y \in H\,,$$

therefore, $q \in EP(G)$.

Show now that $q \in Fix(T)$. By Opial's Lemma 2.8 we have

$$\begin{aligned}
\liminf_{m\to+\infty} \|x_{n_m} - q\| &\leq \liminf_{m\to+\infty} \|x_{n_m} - Tq\| \\
&\leq \liminf_{m\to+\infty} \big[\|x_{n_m} - Tu_{n_m}\| + \|Tu_{n_m} - Tq\|\big] \\
&\leq \liminf_{m\to+\infty} \big[\|x_{n_m} - Tu_{n_m}\| + \|u_{n_m} - q\|\big] \\
\text{(by (3.11))} &= \liminf_{m\to+\infty} \|u_{n_m} - q\| \\
\text{(by } q \in EP(G) \text{ and Lemma 2.5)} &= \liminf_{m\to+\infty} \|S_{r_{n_m}} x_{n_m} - S_{r_{n_m}} q\| \\
&\leq \liminf_{m\to+\infty} \|x_{n_m} - q\|
\end{aligned}$$

i.e. $\liminf_{m\to+\infty} \|x_{n_m} - q\| = \liminf_{m\to+\infty} \|x_{n_m} - Tq\|$. Still by Opial's Lemma 2.8, we obtain $q = Tq$ since H is uniformly convex. □

Theorem 3.3. *Let $G : H \times H \to \mathbb{R}$ be a bifunction such that (E1)-(E4) hold and $T : H \to H$ a nonexpansive mapping. Let suppose that $EP(G) \cap Fix(T) \neq \emptyset$.*

Let $(r_n)_{n\in\mathbb{N}}$ be a sequence of positive real number such that (R1) and (R2) hold.

Let $(\alpha_n)_{n\in\mathbb{N}}$, $(\beta_n)_{n\in\mathbb{N}}$ be two sequences in $]0,1[$ such that (A1) and(B1) hold.

Let suppose $f : H \to H$ be a $\alpha-$contraction and A be a strongly positive linear bounded self-adjoint operator on H into itself with coefficient $\bar{\gamma} > 0$. Let γ be a real number such that $0 < \gamma < \bar{\gamma}/\alpha$.

Then $(x_n)_{n\in\mathbb{N}}$ *and* $(u_n)_{n\in\mathbb{N}}$ *defined by*

$$\begin{cases} x_0 \in H \\ G(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0 \quad \forall\, y \in H \\ y_n = \beta_n x_n + (1-\beta_n)Tu_n \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)y_n \end{cases}$$

are strongly convergent to z*, where* z *is the unique solution in* $Fix(T)\cap EP(G)$ *of the variational inequality*

$$\langle(\gamma f - A)z, p - z)\rangle \leq 0 \quad \forall\, p \in Fix(T)\cap EP(G)\,. \tag{3.13}$$

Proof. From Lemma 2.1 it follows

$$\begin{aligned} ||x_{n+1} - z||^2 &= ||(I-\alpha_n A)(y_n - z) + \alpha_n(\gamma f(x_n) - Az)||^2 \\ &\leq ||(I-\alpha_n A)(y_n - z)||^2 + 2\alpha_n\langle \gamma f(x_n) - Az, x_{n+1} - z\rangle\,. \end{aligned}$$

From $\|y_n - z\| \leq \|x_n - z\|$ we obtain

$$\begin{aligned} ||x_{n+1} - z||^2 &\leq (1-\alpha_n\bar{\gamma})^2\,||x_n - z||^2 \qquad\qquad (3.14) \\ &\quad +2\alpha_n\langle \gamma f(x_n) - Az, x_{n+1} - z\rangle \\ &= (1-\alpha_n\bar{\gamma})^2\,||x_n - z||^2 + 2\alpha_n\gamma\langle f(x_n) - f(z), x_{n+1} - z\rangle \\ &\quad +2\alpha_n\langle \gamma f(z) - Az, x_{n+1} - z\rangle \\ &\leq (1-\alpha_n\bar{\gamma})^2\,||x_n - z||^2 + 2\alpha_n\gamma\alpha||x_n - z||\,||x_{n+1} - z|| \\ &\quad +2\alpha_n\langle \gamma f(z) - Az, x_{n+1} - z\rangle \\ &\leq (1-\alpha_n\bar{\gamma})^2\,||x_n - z||^2 + \alpha_n\gamma\alpha\big(||x_n - z||^2 + ||x_{n+1} - z||^2\big) \\ &\quad +2\alpha_n\langle \gamma f(z) - Az, x_{n+1} - z\rangle \\ &\leq \big((1-\alpha_n\bar{\gamma})^2 + \alpha_n\gamma\alpha\big)\,||x_n - z||^2 + \alpha_n\gamma\alpha||x_{n+1} - z||^2 \\ &\quad +2\alpha_n\langle \gamma f(z) - Az, x_{n+1} - z\rangle\,. \end{aligned}$$

This implies that

$$\begin{aligned}
||x_{n+1}-z||^2 &\leq \frac{1-2\alpha_n\bar{\gamma}+(\alpha_n\bar{\gamma})^2+\alpha_n\gamma\alpha}{1-\alpha_n\gamma\alpha}\,||x_n-z||^2 \\
&\quad +\frac{2\alpha_n}{1-\alpha_n\gamma\alpha}\,\langle\gamma f(z)-Az, x_{n+1}-z\rangle \\
&= \left(1-\frac{2(\bar{\gamma}-\gamma\alpha)\alpha_n}{1-\alpha_n\gamma\alpha}\right)||x_n-z||^2+\frac{(\alpha_n\bar{\gamma})^2}{1-\alpha_n\gamma\alpha}\,||x_n-z||^2 \\
&\quad +\frac{2\alpha_n}{1-\alpha_n\gamma\alpha}\,\langle\gamma f(z)-Az, x_{n+1}-z\rangle
\end{aligned}$$

Set $M:=\sup\limits_{n\in\mathbb{N}}||x_n-z||$, we obtain

$$\begin{aligned}
||x_{n+1}-z||^2 \leq \left(1-\frac{2(\bar{\gamma}-\gamma\alpha)\alpha_n}{1-\alpha_n\gamma\alpha}\right)||x_n-z||^2+\frac{2(\bar{\gamma}-\gamma\alpha)\alpha_n}{1-\alpha_n\gamma\alpha}\left(\frac{\alpha_n\bar{\gamma}^2M}{2(\bar{\gamma}-\gamma\alpha)}\right. \\
\left.+\frac{1}{(\bar{\gamma}-\gamma\alpha)}\,\langle\gamma f(z)-Az, x_{n+1}-z\rangle\right).
\end{aligned}$$

If we prove that

$$\limsup_{n\to+\infty}\langle\gamma f(z)-Az, x_n-z\rangle\leq 0\,,$$

from Lemma 2.9 we obtain $x_n\to z$ and consequently $u_n\to z$.

Let $(x_{n_k})_{k\in\mathbb{N}}$ be a subsequence of $(x_n)_{n\in\mathbb{N}}$ such that

$$\limsup_{n\to+\infty}\langle\gamma f(z)-Az, x_n-z\rangle=\lim_{k\to+\infty}\langle\gamma f(z)-Az, x_{n_k}-z\rangle \tag{3.15}$$

Without loss of generality, we can assume that $(x_{n_k})_{k\in\mathbb{N}}$ weakly converges to p. By Corollary 3.2, $p\in Fix(T)\cap EP(G)$; hence

$$\lim_{k\to+\infty}\langle\gamma f(z)-Az, x_{n_k}-z\rangle=\langle\gamma f(z)-Az, p-z\rangle\leq 0$$

□

Remark 3.4. The assumptions on the coefficients α_n are weaker than the corresponding assumptions in [10] and [12] where the convergence of the iteration schemes Mann-type (1.3) and (1.4) was proved.

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