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AN EXISTENCE RESULT FOR A FREDHOLM-TYPE INTEGRAL INCLUSION

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Abstract. We consider a nonconvex integral inclusion of Fredholm type and we prove a Filippov type existence theorem by applying the contraction principle in the space of selections of the multifunction.
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1. Introduction

This note is concerned with the following integral inclusion

$$x(t) = \lambda(t) + \int_0^1 f(t, s, u(s))ds, \tag{1.1}$$

$$u(t) \in F(t, x(t)), \quad a.e.\ (I), \tag{1.2}$$

where $I = [0, 1]$, $\lambda(.) : I \to X$, $F(.,.) : I \times X \to \mathcal{P}(X)$, $f(.,.,.) : I \times I \times X \to X$, are given mappings and X is a separable Banach space.

The aim of this note is to obtain a version of Filippov's theorem concerning the existence of solutions for problem (1.1)-(1.2). Recall that for a differential inclusion defined by a lipschitzian set-valued map with nonconvex values, Filippov's theorem consists in proving the existence of a solution starting from a given almost solution.

Existence results and the stability of the solution set with respect to small perturbations of the free term for problem (1.1)-(1.2) are obtained via fixed point techniques in Theorem 2.8.9 in [9] (see also the references therein). Our approach is different from the one in [9] and consists in applying the contraction principle in the space of selections of the multifunctions instead of the space of solutions. The idea of applying the set-valued contraction principle due to Covitz and Nadler ([6]) in the space of "derivatives" of the solutions belongs to Kannai and Tallos ([7]) and it was already used for similar results concerning differential and integral inclusions ([2,3,4,5,7] etc.). We note that a similar result for Volterra type integral inclusions is obtained in [4], but it is well known that, in general, the results for Fredholm integral equations cannot be obtained from the corresponding results for Volterra integral equations.

The paper is organized as follows: in Section 2 we recall some preliminary facts that we need in the sequel and in Section 3 we prove our main result.

2. Preliminaries

Let $I := [0, 1]$ and denote by $\mathcal{L}(I)$ the σ-algebra of all Lebesgue measurable subsets of I. Consider X a real separable Banach space with the norm $||.||$ and denote by $\mathcal{P}(X)$ the family of all nonempty subsets of X, by $\mathcal{B}(X)$ the family of all Borel subsets of X. The unit ball in X will be denoted by B.

In what follows, as usual, we denote by $C(I, X)$ the Banach space of all continuous functions $x(.) : I \to X$ endowed with the norm $||x(.)||_C = \sup_{t\in I} ||x(t)||$ and by $L^1(I, X)$ the Banach space of all (Bochner) integrable functions $u(.) : I \to X$ endowed with the norm $||u(.)||_1 = \int_0^1 ||u(t)||dt$.

In order to study problem (1.1)-(1.2) we introduce the following assumption.

Hypothesis 2.1. Let $F(.,.) : I \times X \to \mathcal{P}(X)$ be a set-valued map with nonempty closed values that verify:

i) The set-valued map $F(.,.)$ is $\mathcal{L}(I) \otimes \mathcal{B}(X)$ measurable.

ii) There exists $L(.) \in L^1(I, \mathbf{R}_+)$ such that, for almost all $t \in I$, $F(t,.)$ is $L(t)$-Lipschitz in the sense that

$$d_H(F(t, x), F(t, y)) \le L(t)||x - y|| \quad \forall\, x, y \in X,$$

where d_H is the Pompeiu-Hausdorff generalized metric on $\mathcal{P}(X)$ defined by

$$d_H(A, B) = \max\{d^*(A, B), d^*(B, A)\}, \quad d^*(A, B) = \sup\{d(a, B); a \in A\}.$$

iii) $d(0, F(t, 0)) \le L(t)\ \forall t \in I$.

iv) The mapping $f : I \times I \times X \to X$ is continuous, and there exists $M > 0$ such that

$$||f(t, s, u_1) - f(t, s, u_2)|| \leq M||u_1 - u_2||, \quad \forall t, s \in I, \; u_1, u_2 \in X,$$

$$||f(t, s, 0)|| \leq L(s) \quad \forall t, s \in I.$$

To simplify the notations, we set

$$\Phi(u)(t) = \int_0^1 f(t, \tau, u(\tau))d\tau, \quad t \in I. \tag{2.1}$$

Then the integral inclusion system (1.1)-(1.2) becames

$$x(t) = \lambda(t) + \Phi(u)(t), \quad u(t) \in F(t, x(t)) \quad a.e.\ (I), \tag{2.2}$$

which may be written in the more "compact" form

$$u(t) \in F(t, \lambda + \Phi(u)(t)) \quad a.e.\ (I),$$

but the integral operator $\Phi(.)$ in (2.1) plays a certain role in the proof of our main result.

Denote $L^* := \int_0^1 L(s)ds$ and denote by $S(\lambda)$ the solution set of (1.1)-(1.2).

Finally we recall some basic results concerning set valued contractions that we shall use in the sequel.

Let (Z, d) be a metric space and consider a set valued map T on Z with nonempty closed values in Z. T is said to be a l-contraction if there exists $0 < l < 1$ such that:

$$d_H(T(x), T(y)) \leq l d(x, y) \quad \forall x, y \in Z.$$

If Z is complete, then every set valued contraction has a fixed point, i.e. a point $z \in Z$ such that $z \in T(z)$ ([6]).

We denote by $Fix(T)$ the set of all fixed point of the multifunction T. Obviously, $Fix(T)$ is closed.

Proposition 2.2. ([8]) *Let Z be a complete metric space and suppose that T_1, T_2 are l-contractions with closed values in Z. Then*

$$d_H(Fix(T_1), Fix(T_2)) \leq \frac{1}{1-l} \sup_{z \in Z} d_H(T_1(z), T_2(z)).$$

3. The main result

We are able now to prove our main result.

Theorem 3.1. *Let Hypothesis 2.1 be satisfied and assume that*

$$M\int_0^1 L(s)ds < 1.$$

Consider $\lambda(.), \mu(.) \in C(I,X)$ *and let* $v(.) \in L^1(I,X)$ *be such that*

$$d(v(t), F(t, y(t))) \le p(t) \quad a.e.\ (I),$$

where $p(.) \in L^1(I, \mathbf{R}_+)$ *and* $y(t) = \mu(t) + \Phi(v)(t)$, $\forall t \in I$.

Then for every $\varepsilon > 0$ *there exists* $x(.) \in S(\lambda)$ *such that for every* $t \in I$

$$||x(t) - y(t)|| \le \frac{1}{1-ML^*}||\lambda - \mu||_C + \frac{M}{1-ML^*}\int_0^1 p(t)dt + \varepsilon.$$

Proof. For $\lambda \in C(I,X)$ and $u \in L^1(I,X)$ define

$$x_{u,\lambda}(t) = \lambda(t) + \int_0^1 f(t,s,u(s)ds.$$

Consider $\lambda \in C(I,X), \sigma \in L^1(I,X)$ and define the set valued maps:

$$M_{\lambda,\sigma}(t) := F(t, x_{\sigma,\lambda}(t)), \quad t \in I, \tag{3.1}$$

$$T_\lambda(\sigma) := \{\psi(.) \in L^1(I,X); \quad \psi(t) \in M_{\lambda,\sigma}(t) \quad a.e.\ (I)\}. \tag{3.2}$$

We shall prove first that $T_\lambda(\sigma)$ is nonempty and closed for every $\sigma \in L^1(I,X)$.

The fact that that the set valued map $M_{\lambda,\sigma}(.)$ is measurable is well known. For example the map $t \to F(t, x_{\sigma,\lambda}(t))$ can be approximated by step functions and we can apply Theorem III. 40 in [1]. Since the values of F are closed, with the measurable selection theorem (e.g. Theorem III.6 in [1]) we infer that $M_{\lambda,\sigma}(.)$ admits a measurable selection $\varphi(.)$. One has

$$||\varphi(t)|| \le d(0, F(t,0)) + d_H(F(t,0), F(t, \lambda(t) + \int_0^1 f(t,s,\sigma(s))ds) \le L(t)(1+$$

$$||\lambda(t)||+\int_0^1 ||f(t,s,\sigma(s))||ds) \le L(t)(1+||\lambda(t)||+\int_0^1 L(s)ds+\int_0^1 M||\sigma(s)||ds)$$

which shows that $\phi \in L^1(I,X)$ and $T_\lambda(\sigma)$ is nonempty.

The set $T_\lambda(\sigma)$ is closed. Indeed, if $\psi_n \in T_\lambda(\sigma)$ and $||\psi_n - \psi||_1 \to 0$, then we can pass to a subsequence ψ_{n_k} such that $\psi_{n_k}(t) \to \psi(t)$ for a.e. $t \in I$ and we find that $\psi \in T_\lambda(\sigma)$.

The next step of the proof will show that $T_\lambda(.)$ is a contraction on $L^1(I, X)$.

Let $\sigma_1, \sigma_2 \in L^1(I, X)$ be given, $\psi_1 \in T_\lambda(\sigma_1)$ and let $\delta > 0$. Consider the following set valued map

$$G(t) := M_{\lambda,\sigma_2}(t) \cap \{z \in X; ||\psi_1(t) - z|| \leq ML(t)\int_0^1 ||\sigma_1(s) - \sigma_2(s)||ds + \delta\}.$$

Since

$$\begin{aligned} d(\psi_1(t), M_{\lambda,\sigma_2}(t)) &\leq d_H(F(t, x_{\sigma_1,\lambda}(t)), F(t, x_{\sigma_2,\lambda}(t))) \\ &\leq L(t)||x_{\sigma_1,\lambda}(t) - x_{\sigma_2,\lambda}(t)|| \end{aligned}$$

$$\leq L(t)\int_0^1 ||f(t, s, \sigma_1(s)) - f(t, s, \sigma_2(s))||ds \leq ML(t)\int_0^1 ||\sigma_1(s) - \sigma_2(s)||ds$$

we deduce that $G(.)$ has nonempty closed values.

Moreover, according to Proposition III.4 in [1], $G(.)$ is measurable.

Let $\psi_2(.)$ be a measurable selection of $G(.)$. It follows that $\psi_2 \in T_\lambda(\sigma_2)$ and

$$\begin{aligned} ||\psi_1 - \psi_2||_1 = \int_0^1 ||\psi_1(t) - \psi_2(t)||dt &\leq \int_0^1 ML(t)(\int_0^1 ||\sigma_1(s) - \sigma_2(s)||ds)dt + \delta \\ &= ML^*||\sigma_1 - \sigma_2||_1 + \delta. \end{aligned}$$

Since δ is arbitrarly, we deduce that

$$d(\psi_1, T_\lambda(\sigma_2)) \leq ML^*||\sigma_1 - \sigma_2||_1.$$

Replacing $\sigma_1(.)$ with $\sigma_2(.)$, we obtain

$$d_H(T_\lambda(\sigma_1), T_\lambda(\sigma_2)) \leq ML^*||\sigma_1 - \sigma_2||_1.$$

Hence $T_\lambda(.)$ is a contraction on $L^1(I, X)$.

We consider the following set-valued maps

$$\tilde{F}(t, x) := F(t, x) + p(t)B,$$

$$\tilde{M}_{\lambda,\sigma}(t) = \tilde{F}(t, x_{\sigma,\lambda}(t)),$$

$$\tilde{T}_\mu(\sigma) = \{\psi \in L^1(I, X); \quad \psi(t) \in \tilde{M}_{\mu,\sigma}(t) \quad a.e.\ (I)\}$$

Obviously, $\tilde{F}(., .)$ satisfies Hypothesis 2.1.

Repeating the previous step of the proof we obtain that $\tilde{T}_\mu$ is also a ML^*-contraction on $L^1(I, X)$ with closed nonempty values.

We prove next the following estimate

$$d_H(T_\lambda(\sigma), \tilde{T}_\mu(\sigma)) \leq L^*||\lambda - \mu||_C + \int_0^1 p(t)dt. \tag{3.3}$$

Let $\phi \in T_\lambda(\sigma), \delta > 0$ and define

$$G_1(t) = \tilde{M}_{\lambda,\sigma}(t) \cap \{z \in X;\ ||\phi(t) - z|| \leq L(t)||\lambda - \mu||_C + p(t) + \delta\}.$$

With the same arguments used for the set valued map $G(.)$, we deduce that $G_1(.)$ is measurable with nonempty closed values. Let $\psi(.) \in \tilde{T}_\mu(\sigma)$. One has:

$$||\phi - \psi||_1 = \int_0^1 ||\phi(t) - \psi(t)||dt \leq \int_0^1 [L(t)||\lambda - \mu||_C + p(t) + \delta]dt =$$

$$= L^*||\lambda - \mu||_C + \int_0^1 p(t)dt + \delta.$$

Since δ is arbitrarly, as above we obtain (3.3).

Applying Proposition 2.2 we obtain

$$d_H(Fix(T_\lambda), Fix(\tilde{T}_\mu)) \leq \frac{L^*}{1 - ML^*}||\lambda - \mu||_C + \frac{1}{1 - ML^*}\int_0^1 p(t)dt.$$

Since $v(.) \in Fix(\tilde{T}_\mu)$, it follows that there exists $u(.) \in Fix(T_\lambda)$ such that

$$||v - u||_1 \leq \frac{L^*}{1 - ML^*}||\lambda - \mu||_C + \frac{1}{1 - ML^*}\int_0^1 p(t)dt + \frac{\varepsilon}{M}. \tag{3.4}$$

We define

$$x(t) = \lambda(t) + \int_0^1 f(t, s, u(s))ds.$$

One has

$$||x(t) - y(t)|| \leq ||\lambda(t) - \mu(t)|| + M\int_0^1 ||u(s) - v(s)||ds \leq$$

$$\leq ||\lambda - \mu||_C + \frac{ML^*}{1 - ML^*}||\lambda - \mu||_C + \frac{M}{1 - ML^*}\int_0^1 p(t)dt + \varepsilon =$$

$$= \frac{1}{1 - ML^*}||\lambda - \mu||_C + \frac{M}{1 - ML^*}\int_0^1 p(t)dt + \varepsilon$$

and the proof is complete.

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