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FUNCTIONAL DIFFERENTIAL EQUATIONS WITH FRACTIONAL ORDER AND INFINITE DELAY

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Abstract. In this paper, we shall establish sufficient conditions for the existence of mild solutions for some densely defined semilinear functional and neutral functional differential equations with fractional order and infinite delay. Our approach is based on a nonlinear alternative of Leray-Schauder type.
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1. Introduction

It is well-established to model the evolution of some physical, biological and economic systems using functional and partial functional differential equations, in which the response of the system depends not only on the current state of the system, but also on the past history of the system. For more details on

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this topic, see for example, the books of Kolmanovskii and Myshkis [28], Hale and Verduyn Lunel [19] and Wu [42], and the references therein.

Fractional calculus is a generalization of ordinary differentiation and integration to arbitrary non-integer order. The subject is as old as the differential calculus, and goes back to time when Leibnitz and Newton invented differential calculus. The idea of fractional calculus has been a subject of interest not only among mathematicians, but also among physicists and engineers. Differential equations of fractional order have recently proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering. Indeed we can find numerous applications in viscoelasticity, electrochemistry, electromagnetism, etc. For details, see the monographs of Kilbas *et al* [26], Kiryakova [27], Miller and Ross [32], Podlubny [37] and Samko *et al* [40], and the papers of Diethelm *et al* [6, 7, 8], El-Sayed [11, 12, 13], Gaul *et al* [14], Glockle and Nonnenmacher [15], Mainardi [30], Metzler *et al* [31], Podlubny *et al* [39], and the references therein. Some classes of evolution equations have been considered by El- Borai [9, 10], Jaradat *et al* [24] studied the existence and uniqueness of mild solutions for a class of initial value problem for a semilinear integrodifferential equation involving the Caputo's fractional derivative. In a series of papers (see [1, 2, 3]) the authors considered some classes of initial value problems for functional differential equations involving the Riemann-Liouville and Caputo fractional derivatives of order $0 < \alpha \leq 1$. For more details on the geometric and physical interpretation for fractional derivatives of both the Riemann-Liouville and Caputo types see [21, 38]. The web site http://people.tuke.sk/igor.podlubny/, authored by Igor Podlubny contains more information on fractional calculus and its applications, and hence it is very useful for those that are interested in this field.

In the literature devoted to equations with finite delay, the phase space is much of time the space of all continuous functions on $[-r, 0]$, $r > 0$, endowed with the uniform norm topology. When the delay is infinite, the notion of the phase space B plays an important role in the study of both qualitative and quantitative theory, a usual choice is a seminormed space satisfying suitable axioms, which was introduced by Hale and Kato [18], see also Kappel and Schappacher [25] and Schumacher [41]. For detailed discussion on this topic, we refer the reader to the books by Hino *et al.* [23], Lakshmikantham *et al* [29] and the survey paper by Corduneanu and Lakshmikantham [4].

This paper is organized as follows: in Section 2, we will recall briefly some basic definitions and preliminary facts which will be used throughout the following sections. In Section 3, we give existence results of mild solutions defined on a compact real interval for fractional order semilinear functional differential equations (SFDEs for short) of the form:

$$D^{\alpha}y(t) = Ay(t) + f(t, y_t), \quad t \in J := [0, b] \tag{1}$$

$$y_0 = \phi \in \mathcal{B} \tag{2}$$

where D^{α} is the standard Riemann-Liouville fractional derivative of order $\alpha \in (0, 1)$, $f : J \times \mathcal{B} \to E$ is a continuous function, $\mathcal{B}$ the phase space that will specified later (see Section 2), $A : D(A) \subset E \to E$ is the infinitesimal generator of a strongly continuous semigroup $\{T(t)\}_{t \geq 0}$, $\phi : \mathcal{B} \to E$ a continuous function with $\phi(0) = 0$ and $(E, |\cdot|)$ a real Banach space. For any $t \in J$ the function $y_t \in \mathcal{B}$ is defined by

$$y_t(\theta) = y(t + \theta), \quad \theta \in (-\infty, 0].$$

Section 4 is devoted to the existence of solutions of some neutral semilinear functional differential equations (NSFDEs for short) of fractional order of the form:

$$D^{\alpha}[y(t) - h(t, y_t)] = A[y(t) - h(t, y_t)] + f(t, y_t), \quad t \in J := [0, b], \tag{3}$$

$$y_0 = \phi \in \mathcal{B}. \tag{4}$$

where f, A, ϕ are as in Problem (1)-(2) and h is a given function. These results can be considered as a contribution to this emerging field.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper.

By $C(J, E)$ we denote the Banach space of all continuous functions from J into E with the norm

$$\|y\|_{\infty} =: \sup\{|y(t)| : t \in J\},$$

and $B(E)$ denotes the Banach space of bounded linear operators from E into E, with norm

$$\|N\|_{B(E)} = \sup\{|N(y)| : |y| = 1\}.$$

Definition 2.1. *([37, 40]). The Riemann-Liouville fractional primitive of order α of a function $h:(0,b]\to E$ of order $\alpha\in\mathbb{R}^+$ is defined by*

$$I_0^\alpha h(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)ds,$$

provided the right side is pointwise defined on $(0,b]$, and where Γ is the gamma function.

Definition 2.2. *([37, 40]). The Riemann-Liouville fractional derivative of order $\alpha>0$ of a continuous function $h:(0,b]\to E$ is defined by*

$$\begin{aligned}\frac{d^\alpha h(t)}{dt^\alpha} &= \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t (t-s)^{-\alpha}h(s)ds\\ &= \frac{d}{dt}I_0^{1-\alpha}h(t).\end{aligned}$$

Consider the following space

$$\mathcal{B}_b=\{y:(-\infty,b]\to E: y|_J\in C(J,E),\quad y_0\in\mathcal{B}\},$$

where $y|_J$ is the restriction of y to J. Let $\|\cdot\|_b$ be the seminorm in $\mathcal{B}_b$ defined by:

$$\|y\|_b=\|y_0\|_{\mathcal{B}}+sup\{|y(s)|:0\le s\le b\},y\in\mathcal{B}_b.$$

In all this paper, we assume that the phase space $(\mathcal{B},|\cdot|)$ is a seminormed linear space of functions mapping $(-\infty,0]$ into E, and satisfying the following axioms introduced at first by Hale and Kato in [18]:

(A1) There exist a positive constant H and functions $K(\cdot), M(\cdot):\mathbb{R}^+\to\mathbb{R}^+$ with K continuous and M locally bounded, such that for any $b>0$, if $x:(-\infty,b]\to E$, $x\in\mathcal{B}$, and $x(\cdot)$ is continuous on $[0,b]$, then for every $t\in[0,b]$ the following conditions hold:

(i) x_t is in $\mathcal{B}$;

(ii) $|x(t)|\le H\|x_t\|_{\mathcal{B}}$;

(iii) $\|x_t\|_{\mathcal{B}}\le K(t)\sup\{|x(s)|:\ 0\le s\le t\}+M(t)\|x_0\|_{\mathcal{B}}$, and H,K and M are independent of $x(\cdot)$.

Denote

$$K_b=\sup\{K(t):\quad t\in J\}\quad\text{and}\quad M_b=\sup\{M(t):\quad t\in J\}.$$

(A_2) For the function $x(.)$ in (A_1), x_t is a $\mathcal{B}-$valued continuous function on $[0,b]$.

(A_3) The space $\mathcal{B}$ is complete.

Hereafter are some examples of phase spaces. For other details we refer, for instance to the book by Hino *et al* [23].

Example 2.1. *The spaces* BC, BUC, C^∞ *and* C^0. *Let:*

BC *the space of bounded continuous functions defined from* $(-\infty, 0]$ *to* E;

BUC *the space of bounded uniformly continuous functions defined from* $(-\infty, 0]$ *to* E;

$$C^\infty := \left\{\phi \in BC : \lim_{\theta\to-\infty} \phi(\theta) \text{ exist in } E\right\};$$

$$C^0 := \left\{\phi \in BC : \lim_{\theta\to-\infty} \phi(\theta) = 0\right\}, \text{ endowed with the uniform norm}$$

$$\|\phi\| = \sup\{|\phi(\theta)| : \ \theta \le 0\}.$$

We have that the spaces BUC, C^∞ *and* C^0 *satisfy conditions* $(A_1) - (A_3)$. BC *satisfies* $(A_2), (A_3)$ *but* (A_1) *is not satisfied.*

Example 2.2. *The spaces* C_g, UC_g, C_g^∞ *and* C_g^0.

Let g *be a positive continuous function on* $(-\infty, 0]$. *We define:*

$$C_g := \left\{\phi \in C((-\infty, 0], E) : \frac{\phi(\theta)}{g(\theta)} \text{ is bounded on } (-\infty, 0]\right\};$$

$$C_g^0 := \left\{\phi \in C_g : \lim_{\theta\to-\infty} \frac{\phi(\theta)}{g(\theta)} = 0\right\}, \text{ endowed with the uniform norm}$$

$$\|\phi\| = \sup\left\{\frac{|\phi(\theta)|}{g(\theta)} : \ \theta \le 0\right\}.$$

We consider the following condition on the function g.

(g_1) *For all* $a > 0$, $\displaystyle\sup_{0\le t\le a} \sup\left\{\frac{g(t+\theta)}{g(\theta)} : -\infty < \theta \le -t\right\} < \infty.$

Then we have that the spaces C_g *and* C_g^0 *satisfy conditions* (A_3). *They satisfy conditions* (A_1) *and* (A_3) *if* g_1 *holds.*

Example 2.3. *The space* C_γ.

For any real constant γ, *we define the functional space* C_γ *by*

$$C_\gamma := \left\{\phi \in C((-\infty, 0], E) : \lim_{\theta\to-\infty} e^{\gamma\theta}\phi(\theta) \text{ exist in } E\right\}$$

endowed with the following norm

$$\|\phi\| = \sup\{e^{\gamma\theta}|\phi(\theta)| : \ \theta \le 0\}.$$

Then in the space C_γ the axioms $(A_1) - (A_3)$ are satisfied.

3. Functional differential equations

In this section we give our main existence results for problem (1)-(2). Before stating and proving these results, we give the definition of the mild solution.

Definition 3.1. *We say that a function $y \in \mathcal{B}_b$ is a mild solution of problem (1)-(2) if $y_0 = \phi$ and*

$$y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} T(t-s) f(s, y_s) ds, \quad t \in J. \tag{5}$$

Our first existence result for problem (1)-(2) is based on the Banach's contraction principle.

Theorem 3.1. *Let $f : J \times \mathcal{B} \to E$. Assume:*

(H) *There exists a nonnegative constant k such that*

$$|f(t,u) - f(t,v)| \leq k\|u - v\|_{\mathcal{B}}, \text{ for } t \in J \text{ and every } u, v \in \mathcal{B}$$

If

$$\frac{MkK_b b^\alpha}{\Gamma(\alpha+1)} < 1, \tag{6}$$

where

$$M = \sup\{\|T(t)\|_{B(E)} : t \in J\}.$$

Then there exists a unique mild solution of problem (1)-(2) on $(-\infty, b]$.

Proof. Transform the IVP (1)-(2) into a fixed point problem. Consider the operator:

$$\mathcal{N} : \mathcal{B}_b \to \mathcal{B}_b$$

defined by

$$\mathcal{N}(y)(t) = \begin{cases} \phi(t), & t \in (-\infty, 0], \\ \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1} T(t-s) f(s, y_s) ds, & t \in J. \end{cases}$$

For $\phi \in \mathcal{B}$, we define the function:

$$x(t) = \begin{cases} \phi(t), & t \in (-\infty, 0], \\ 0, & t \in J. \end{cases}$$

Then $x \in \mathcal{B}_b$. Set

$$y(t) = z(t) + x(t).$$

It is obvious that y satisfies (5) if and only if z satisfies $z_0 = 0$ and

$$z(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} T(t-s) f(s, z_s + x_s) ds, \quad t \in J.$$

Let

$$\mathcal{B}_b^0 = \{z \in \mathcal{B}_b : z_0 = 0\}.$$

For any $z \in \mathcal{B}_b^0$, we have

$$\|z\|_b = \|z_0\|_B + sup\{|z(s)| : 0 \le s \le b\} = sup\{|z(s)| : 0 \le s \le b\}.$$

Thus $(\mathcal{B}_b^0, \|\cdot\|_b)$ is a Banach space. Let the operator $\mathcal{P} : \mathcal{B}_b^0 \to \mathcal{B}_b^0$ defined by

$$\mathcal{P}(z)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} T(t-s) f(s, z_s + x_s) ds,$$

It is obvious that $\mathcal{N}$ has a fixed point is equivalent to $\mathcal{P}$ has a fixed point, and so we turn to proving that $\mathcal{P}$ has a fixed point. We shall show that $\mathcal{P}$ is a contraction. Indeed, consider $z, \overline{z} \in \mathcal{B}_b^0$. Then for each $t \in J$,

$$\begin{aligned} |\mathcal{P}(z)(t) - \mathcal{P}(\overline{z})(t)| &\le \frac{M}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s, z_s + x_s) - f(s, \overline{z}_s + x_s)| ds \\ &\le \frac{Mk}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|z_s - \overline{z}_s\|_B ds \\ &\le \frac{Mk}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} K_b \sup_{s \in [0,t]} |z(s) - \overline{z}(s)| ds. \end{aligned}$$

Taking the supremum over t we get

$$\|\mathcal{P}(z) - \mathcal{P}(\overline{z})\|_b \le \frac{MkK_b b^\alpha}{\Gamma(\alpha+1)} \|z - \overline{z}\|_b,$$

which implies by (6) that $\mathcal{P}$ is a contraction. Therefore, by the Banach's contraction principle $\mathcal{P}$ has a unique fixed point z^*. Then $y^*(t) = z^*(t) + x(t), \quad t \in (-\infty, b]$ is a fixed point of the operator $\mathcal{N}$, which gives rise to a unique mild solution of the problem (1)-(2). □

Next we give an existence result based upon the following nonlinear alternative of Leray-Schauder applied to completely continuous operators [16].

Theorem 3.2. *Let X a Banach space, and $C \subset X$ convex with $0 \in C$. Let $F : C \to C$ be a completely continuous operator. Then either*

(a) *F has a fixed point, or*
(b) *the set $\mathcal{E} = \{x \in C : x = \lambda F(x), \quad 0 < \lambda < 1\}$ is unbounded.*

Essential for the main results of this section, we state a generalization of Gronwall's lemma for singular kernels ([20], Lemma 7.1.1).

Lemma 3.1. *Let $v, w : [0, b] \to [0, \infty)$ be continuous functions. If $w(\cdot)$ is nondecreasing and there are constants $a > 0$ and $0 < \alpha < 1$ such that*

$$v(t) \le w(t) + a \int_0^t \frac{v(s)}{(t-s)^\alpha} ds,$$

then there exists a constant $k = k(\alpha)$ such that

$$v(t) \le \omega(t) + ka \int_0^t \frac{\omega(s)}{(t-s)^\alpha} ds,$$

for every $t \in [0, b]$.

Our main result reads

Theorem 3.3. *Assume that the following hypotheses hold:*

(H1) *The semigroup $\{T(t)\}_{t \in J}$ is compact for $t > 0$.*
(H2) *There exist functions $p, q \in C(J, \mathbb{R}_+)$ such that*

$$|f(t, u)| \le p(t) + q(t)\|u\|_{\mathcal{B}}, \quad \text{for a.e. } t \in J, \text{ and each } u \in \mathcal{B}.$$

Then the problem (1)-(2) has at least one mild solution on $(-\infty, b]$.

Proof. Transform the IVP (1)-(2) into a fixed point problem. Consider the operator $\mathcal{P}$ defined in the proof of Theorem 3.1. We shall show that the operator $\mathcal{P}$ is continuous and completely continuous.

Step 1: $\mathcal{P}$ is continuous.

Let $\{z_n\}$ be a sequence such that $z_n \to z$ in $\mathcal{B}_b^0$. Then

$$\begin{aligned} &|\mathcal{P}(z_n)(t) - \mathcal{P}(z)(t)| \\ &\le \left| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} T(t-s)[f(s, z_{n_s} + x_s) - f(s, z_s + x_s)] ds \right| \\ &\le \frac{Mb^\alpha}{\alpha\Gamma(\alpha)} \|f(., z_{n.} + x_.) - f(., z_. + x_.)\|_\infty. \end{aligned}$$

Since f is a continuous function, then we have

$$\|\mathcal{P}(z_n) - \mathcal{P}(z)\|_b \le \frac{Mb^\alpha}{\Gamma(\alpha+1)} \|f(., z_{n.} + x_.) - f(., z_. + x_.)\|_\infty \to 0 \quad as \quad n \mapsto \infty.$$

Thus $\mathcal{P}$ is continuous.

Step 2: $\mathcal{P}$ maps bounded sets into bounded sets in $\mathcal{B}_b^0$.

It is enough to show that for any $\rho > 0$ there exists a positive constant δ such that for each $z \in B_\rho = \{z \in \mathcal{B}_b^0 : \|z\|_b \le \rho\}$ we have $\mathcal{P}(z) \in B_\delta$. Let $z \in B_\rho$, then

$$\begin{aligned}\|z_s + x_s\|_{\mathcal{B}} &\le \|z_s\|_{\mathcal{B}} + \|x_s\|_{\mathcal{B}}\\ &\le K_b\rho + M_b\|\phi\|_{\mathcal{B}} := \rho^*.\end{aligned}$$

Then we have for each $t \in J$

$$\begin{aligned}|\mathcal{P}(z)(t)| &= \left|\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}T(t-s)f(s,y_s)ds\right|\\ &\le \frac{M}{\Gamma(\alpha)}\int_0^t \sup_{s\in[0,t]}|p(s)|(t-s)^{\alpha-1}ds\\ &+ \frac{M\rho^*}{\Gamma(\alpha)}\int_0^t \sup_{s\in[0,t]}|q(s)|(t-s)^{\alpha-1}ds.\end{aligned}$$

Taking the supremum over t we have

$$\|\mathcal{P}(z)\|_b \le \frac{Mb^\alpha}{\Gamma(\alpha+1)}(\|p\|_\infty + \rho^*\|q\|_\infty) =: \delta.$$

Step 3: $\mathcal{P}$ maps bounded sets into equicontinuous sets of $\mathcal{B}_b^0$.

We consider B_ρ as in Step 2. Let $\tau_1, \tau_2 \in J$, $\tau_2 > \tau_1$. thus if $\epsilon > 0$ and $\epsilon \le \tau_1 \le \tau_2$ we have

$$\begin{aligned}|\mathcal{P}(z)(\tau_2) - \mathcal{P}(z)(\tau_1)| &\le \frac{1}{\Gamma(\alpha)}\left|\int_0^{\tau_1-\epsilon}\big[(\tau_2-s)^{\alpha-1}T(\tau_2-s)\right.\\ &\quad \left. - (\tau_1-s)^{\alpha-1}T(\tau_1-s)\big]f(s,z_s+x_s)ds\right|\\ &\quad + \frac{1}{\Gamma(\alpha)}\left|\int_{\tau_1-\epsilon}^{\tau_1}\big[(\tau_2-s)^{\alpha-1}T(\tau_2-s)\right.\\ &\quad \left. - (\tau_1-s)^{\alpha-1}T(\tau_1-s)\big]f(s,z_s+x_s)ds\right|\\ &\quad + \frac{1}{\Gamma(\alpha)}\left|\int_{\tau_1}^{\tau_2}(\tau_2-s)^{\alpha-1}T(\tau_2-s)f(s,z_s+x_s)ds\right|\\ &\le M\frac{\|p\|_\infty + \rho^*\|q\|_\infty}{\Gamma(\alpha)}\left(\left|\int_0^{\tau_1-\epsilon}\big[(\tau_2-s)^{\alpha-1}\right.\right.\\ &\quad \left. - (\tau_1-s)^{\alpha-1}\big]T(\tau_1-s)ds\right|\\ &\quad + \left|\int_0^{\tau_1-\epsilon}(\tau_2-s)^{\alpha-1}T(\tau_1-\epsilon-s)\left(T(\tau_2-\tau_1-\epsilon)-T(\epsilon)\right)ds\right|\end{aligned}$$

$$+\int_{\tau_1-\epsilon}^{\tau_1}\left((\tau_2-s)^{\alpha-1}-(\tau_1-s)^{\alpha-1}\right)ds$$

$$+\int_{\tau_1}^{\tau_2}(\tau_2-s)^{\alpha-1}ds\Bigg)$$

$$\leq M\frac{\|p\|_\infty+\rho^*\|q\|_\infty}{\Gamma(\alpha)}\left(\int_0^{\tau_1-\epsilon}\left[(\tau_2-s)^{\alpha-1}-(\tau_1-s)^{\alpha-1}\right]ds\right.$$

$$+\|T(\tau_2-\tau_1-\epsilon)-T(\epsilon)\|_{B(E)}\int_0^{\tau_1-\epsilon}(\tau_2-s)^{\alpha-1}ds$$

$$+\int_{\tau_1-\epsilon}^{\tau_1}\left((\tau_2-s)^{\alpha-1}-(\tau_1-s)^{\alpha-1}\right)ds$$

$$\left.+\int_{\tau_1}^{\tau_2}(\tau_2-s)^{\alpha-1}ds\right).$$

As $\tau_1 \to \tau_2$ and ϵ sufficiently small, the right-hand side of the above inequality tends to zero, since $T(t)$ is a strongly continuous operator and the compactness of $T(t)$ for $t > 0$ implies the continuity in the uniform operator topology (see [36]). As a consequence of steps 1 to 3 together with Arzelá-Ascoli theorem it suffices to show that $\mathcal{P}$ maps B_ρ into a precompact set in E.

Let $0 < t < b$ be fixed and let ϵ be a real number satisfying $0 < \epsilon < t$. For $z \in B_\rho$ we define

$$\mathcal{P}_\epsilon(z)(t)=\frac{T(\epsilon)}{\Gamma(\alpha)}\int_0^{t-\epsilon}(t-s-\epsilon)^{\alpha-1}T(t-s-\epsilon)f(s,z_s+x_s)ds.$$

Since $T(t)$ is a compact operator for $t > 0$, the set

$$Z_\epsilon(t)=\{\mathcal{P}_\epsilon(z)(t):\quad z\in B_\rho\}$$

is precompact in E for every ϵ, $\;0 < \epsilon < t$. Moreover

$$|\mathcal{P}(z)(t)-\mathcal{P}_\epsilon(z)(t)|$$

$$\leq M\frac{\|p\|_\infty+\rho^*\|q\|_\infty}{\Gamma(\alpha)}\left(\int_0^{t-\epsilon}\left[(t-s)^{\alpha-1}-(t-s-\epsilon)^{\alpha-1}\right]ds\right.$$

$$\left.+\int_{t-\epsilon}^{t}(t-s)^{\alpha-1}ds.\right)$$

$$\leq M\frac{\|p\|_\infty+\rho^*\|q\|_\infty}{\Gamma(\alpha)}\left(t^\alpha-(t-\epsilon)^\alpha\right).$$

Therefore, the set $Z(t)=\{\mathcal{P}(z)(t): z\in B_\rho\}$ is precompact in E. Hence the operator $\mathcal{P}$ is completely continuous.

Step 4: A priori bounds on solutions.

Now, it remains to show that the set

$$\mathcal{E} = \left\{ z \in \mathcal{B}_b^0 : z = \lambda \mathcal{P}(z) \text{ for some } 0 < \lambda < 1 \right\}$$

is bounded.

Let $z \in \mathcal{E}$ be any element. Then, for each $t \in J$,

$$z(t) = \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} T(t-s) f(s, z_s + x_s) ds.$$

Then

$$|z(t)| \leq \frac{Mb^\alpha \|p\|_\infty}{\Gamma(\alpha+1)} + \frac{M\|q\|_\infty}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|z_s + x_s\|_B ds. \tag{7}$$

But

$$\begin{aligned} \|z_t + x_t\|_B &\leq K(t) \sup\{|z(s)| : 0 \leq s \leq t\} + M(t)\|z_0\|_B \\ &+ K(t) \sup\{|x(s)| : 0 \leq s \leq t\} + M(t)\|x_0\|_B \\ &\leq K_b \sup\{|z(s)| : 0 \leq s \leq t\} + M_b\|\phi\|_B. \end{aligned}$$

Take the right hand side of the above inequality as $v(t)$, then by (7) we have

$$|z(t)| \leq \frac{Mb^\alpha \|p\|_\infty}{\Gamma(\alpha+1)} + \frac{M\|q\|_\infty}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v(s) ds.$$

Using the above inequality and the definition of v we have

$$v(t) \leq \frac{MK_b b^\alpha \|p\|_\infty}{\Gamma(\alpha+1)} + \frac{MK_b\|q\|_\infty}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v(s) ds + M_b\|\phi\|_B.$$

By the Lemma 3.1, there exists a constant $K = K(\alpha)$ such that we have

$$v(t) \leq \left[\frac{MK_b b^\alpha \|p\|_\infty}{\Gamma(\alpha+1)} + M_b\|\phi\|_B \right] \left[1 + \frac{MKK_b b^\alpha \|q\|_\infty}{\Gamma(\alpha+1)} \right] := \Lambda.$$

Then there exists a constant $d = d(\Lambda)$ such that $\|z\|_b \leq d$. This shows that the set $\mathcal{E}$ is bounded. As a consequence of the Theorem 3.2, we deduce that the operator $\mathcal{P}$ has a fixed point which gives rise to a mild solution of the problem (1)-(2). □

4. Neutral functional differential equations

In this section we give existence results for the problem (3)-(4). Before stating and proving these results, we give the definition of the mild solution.

Definition 4.1. *We say that a function $y \in \mathcal{B}_b$ is a mild solution of problem (3)-(4) if $y_0 = \phi$ and*

$$y(t) = h(t, y_t) - T(t)h(0, \phi) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} T(t-s) f(s, y_s) ds, \quad t \in J. \tag{8}$$

Our first existence result for problem (3)-(4) is based on the Banach's contraction principle.

Theorem 4.1. *Assume that (H) holds and moreover*

(C1) *There exists a nonnegative constant l such that*

$$|h(t,u) - h(t,v)| \le l \|u - v\|_{\mathcal{B}}, \ \textit{for } t \in J \quad \textit{and every } u, v \in \mathcal{B}.$$

If

$$K_b \left(l + \frac{Mkb^\alpha}{\Gamma(\alpha+1)} \right) < 1. \tag{9}$$

Then there exists a unique mild solution of problem (3)-(4) on $(-\infty, b]$.

Proof. Consider the operator $\widetilde{N} : \mathcal{B}_b \to \mathcal{B}_b$ defined by

$$\widetilde{N}(y)(t) = \begin{cases} \phi(t), & t \in (-\infty, 0], \\ h(t, y_t) - T(t)h(0, \phi) & \\ + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} T(t-s) f(s, y_s) ds, & t \in J. \end{cases}$$

In analogy to Theorem 3.1, we consider the operator $\widetilde{\mathcal{P}} : \mathcal{B}_b^0 \to \mathcal{B}_b^0$ defined by

$$\widetilde{\mathcal{P}}(z)(t) = h(t, z_t + x_t) - T(t)h(0, \phi) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} T(t-s) f(s, z_s + x_s) ds.$$

We shall show that the operator $\widetilde{\mathcal{P}}$ is a contraction. Consider $z, \overline{z} \in \mathcal{B}_b^0$, then for each $t \in J$,

$$\begin{aligned} |\widetilde{\mathcal{P}}(z)(t) - \widetilde{\mathcal{P}}(\overline{z})(t)| &\leq |h(t, z_t + x_t) - h(t, \overline{z}_t + x_t)| \\ &+ \frac{M}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s, z_s + x_s) - f(s, \overline{z}_s + x_s)| ds \\ &\leq l\|z_t - \overline{z}_t\|_B + \frac{Mk}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|z_s - \overline{z}_s\|_B ds \\ &\leq lK_b \sup_{s\in[0,t]} |z(s) - \overline{z}(s)| \\ &+ \frac{Mk}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} K_b \sup_{s\in[0,t]} |z(s) - \overline{z}(s)| ds. \end{aligned}$$

Taking the supremum over t we get

$$\|\widetilde{\mathcal{P}}(z) - \widetilde{\mathcal{P}}(\overline{z})\|_b \leq K_b \left(l + \frac{Mkb^\alpha}{\Gamma(\alpha+1)} \right) \|z - \overline{z}\|_b,$$

which implies by (9) that $\widetilde{\mathcal{P}}$ is a contraction. Therefore, by the Banach's contraction principle $\widetilde{\mathcal{P}}$ has a unique fixed point $\widetilde{z}^*$. Then $\widetilde{y}^*(t) = \widetilde{z}^*(t) + x(t), \quad t \in (-\infty, b]$ is a fixed point of the operator $\widetilde{\mathcal{N}}$, which gives rise to a unique mild solution of the problem (1)-(2). □

Next we give an existence result based upon the nonlinear alternative of Leray-Schauder.

Theorem 4.2. *Assume that* $(H1), (H2)$ *holds and moreover*

(C2) *The function h is continuous and completely continuous, and for every bounded set $B \in \mathcal{B}_b^0$, the set $\{t \mapsto h(t, y_t), \quad y \in B\}$ is equicontinuous in E.*

(C3) *There exists constants:* $0 \leq c_1 < \dfrac{1}{K_b}$, *and* $c_2 \geq 0$ *such that*

$$|h(t,u)| \leq c_1\|u\|_{\mathcal{B}} + c_2, \text{ for } t \in J \text{ and } u \in \mathcal{B}.$$

Then the problem (3)-(4) has at least one mild solution on $(-\infty, b]$.

Proof. Let $\widetilde{\mathcal{P}} : \mathcal{B}_b^0 \to \mathcal{B}_b^0$ defined as in Theorem 4.1. We can easily show that the operator $\widetilde{\mathcal{P}}$ is continuous and completely continuous. Using $(C2)$ it suffices to show that the operator $\mathcal{P} : \mathcal{B}_b^0 \to \mathcal{B}_b^0$ defined by

$$\mathcal{P}(z)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} T(t-s) f(s, z_s + x_s) ds,$$

is continuous and completely continuous. Now, it remains to show that the set

$$\widetilde{\mathcal{E}} = \left\{ z \in \mathcal{B}_b^0 : z = \lambda \widetilde{\mathcal{P}}(z) \ \text{ for some } \ 0 < \lambda < 1 \right\}$$

is bounded.

Let $z \in \widetilde{\mathcal{E}}$ be any element. Then, for each $t \in J$,

$$\begin{aligned} |z(t)| &\leq |h(t, z_t + x_t)| + M|h(0,\phi)| + \frac{M}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s, z_s + x_s)| ds \\ &\leq c_1 \|z_t + x_t\|_B + c_2 + M\|\phi\|_B + Mc_2 + \frac{Mb^\alpha \|p\|_\infty}{\Gamma(\alpha+1)} \\ &+ \frac{M\|q\|_\infty}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|z_s + x_s\|_B ds. \end{aligned}$$

Denote $v(t)$ as in Theorem 4.1. Then

$$\|z_t + x_t\|_B \leq v(t),$$

and

$$\begin{aligned} v(t) &\leq c_1 K_b v(t) + K_b \left[(M + \frac{M_b}{K_b}) \|\phi\|_B + (1+M)c_2 + \frac{Mb^\alpha \|p\|_\infty}{\Gamma(\alpha+1)} \right] \\ &+ \frac{K_b M \|q\|_\infty}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v(s) ds. \end{aligned}$$

Then

$$\begin{aligned} v(t) &\leq \frac{K_b}{1 - c_1 K_b} \left[(M + \frac{M_b}{K_b}) \|\phi\|_B + (1+M)c_2 + \frac{Mb^\alpha \|p\|_\infty}{\Gamma(\alpha+1)} \right] \\ &+ \frac{K_b}{1 - c_1 K_b} \frac{M\|q\|_\infty}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v(s) ds. \end{aligned}$$

By the Lemma 3.1 there exists a constant $K = K(\alpha)$ such that:

$$v(t) \leq \frac{d_1 K_b}{1 - c_1 K_b} \left(1 + \frac{K K_b}{1 - c_1 K_b} \frac{Mb^\alpha \|q\|_\infty}{\Gamma(\alpha+1)} \right) := \widetilde{\Lambda}$$

where

$$d_1 = (M + \frac{M_b}{K_b}) \|\phi\|_B + (1+M)c_2 + \frac{Mb^\alpha \|p\|_\infty}{\Gamma(\alpha+1)}.$$

Then there exists a constant $\widetilde{d} = \widetilde{d}(\widetilde{\Lambda})$ such that $\|z\|_b \leq \widetilde{d}$. This shows that the set $\widetilde{\mathcal{E}}$ is bounded. As a consequence of the Theorem 3.2, we deduce that the operator $\widetilde{\mathcal{P}}$ has a fixed point which gives rise to a mild solution of the problem (3)-(4). □

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