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GENERAL ITERATION SCHEME WITH PERTURBED MAPPING FOR COMMON FIXED POINTS OF A FINITE FAMILY OF NONEXPANSIVE MAPPINGS

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Abstract. A new general composite implicit random iteration scheme with perturbed mapping is proposed and obtain necessary and sufficient conditions for strong convergence of proposed iteration scheme to random fixed point of a finite family of random nonexpansive mappings are obtained.

Key Words and Phrases: General iteration scheme;, nonexpansive random operator, fixed point, perturbed mapping.
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1. INTRODUCTION

Random fixed point theory has received much attention in last three decades, after the publication of the survey article by Bharucha-Reid [4] in 1976. Random fixed point theorems are stochastic versions of (classical or deterministic) fixed point theorems, and are required for the theory of random equations. A lot of efforts have been devoted to random fixed point theory and applications (see [1, 2, 3, 11, 12, 17] and references therein). On the other hand, after the existence results for fixed point of nonexpansive mappings independently given by Browder [5], Göhde [9] and Kirk [10], the metric fixed point theory has been developed rapidly in recent years [8]. One important aspect of metric fixed point theory is to study convergence of iterative schemes to fixed point of nonexpansive mappings. Convergence results for nonexpansive mappings have been established by a number of authors (e.g. [7] and references therein). More recently Xu and Ori [19], introduced implicit iteration process to approximate common fixed of a finite family of nonexpansive mappings. Since than several authors used implicit iteration to approximate common fixed point of various class of mappings (see e.g., [6, 13]). Several modifications of implicit iteration scheme have also been suggested by various authors to approximate common fixed points of various class of nonlinear mappings. Su and Qin [16], introduced a general composite implicit iteration schemes for common fixed point of a finite family of nonexpansive mappings. Motivated by hybrid steepest-descent algorithm [18, 20], Zeng and Yao [21] recently proposed a new implicit iteration scheme with perturbed mapping for approximation of common fixed point of a finite family of nonexpansive self-mappings. The purpose of this paper is to continue discussions of this line, that is, we propose a new general composite implicit iteration scheme with perturbed mapping and obtain necessary and sufficient conditions for strong convergence of proposed iteration scheme to random fixed point of a finite family of random nonexpansive mappings.

2. PRELIMINARIES

Let H be a real separable Hilbert space with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$. Let (Ω,Σ) be a measurable space (Σ – sigma algebra). A mapping $\xi\colon \Omega \to H$ is measurable if $\xi^{-1}(U) \in \Sigma$ for each open subset U of H. The mapping $T\colon \Omega \times H \to H$ is a random map if and only if for each fixed $x \in H$ the mapping $T(\cdot,x)\colon \Omega \to H$ is measurable and it is continuous if for each $\omega \in \Omega$, the mapping $T(\omega,\cdot) : H \longrightarrow H$ is continuous. A measurable mapping $\xi\colon \Omega \to E$ is the random fixed point of the random map $T\colon \Omega\times H \to H$ if and only if $T(\omega,\xi(\omega)) = \xi(\omega)$, for each $\omega \in \Omega$.

We denote the nth iterate $T(\omega, T(\omega, T(\omega, \cdots, T(\omega, x))))$ of T by $T^n(\omega, x)$. $RF(T)$ denote the set of random fixed points of T.

Definition 2.1. Let $T\colon \Omega \times H \to H$ be a random operator. Then T is said to be nonexpansive operator if for each $\omega \in \Omega$ and for all $x, y \in H$

$$\|T(\omega,x) - T(\omega,y)\| \leq \|x-y\|.$$

Let $A\colon \Omega \times H \to H$ be a random operator, then

Definition 2.2. A is said to be Lipschitzian if there exists a real valued random variable $\kappa(\omega) > 0$ such that

$$\|A(\omega, x) - A(\omega, y)\| \leq \kappa(\omega)\|x - y\|\,, \quad \forall\, x, y \in H,\ \omega \in \Omega\,.$$

Definition 2.3. A is said to be strongly monotone if there exists a real valued random variable $\theta(\omega) > 0$ such that

$$\langle A(\omega, x) - A(\omega, y), x - y\rangle \geq \theta(\omega)\|x - y\|^2\,, \quad \forall\, x, y \in H,\ \omega \in \Omega\,.$$

Definition 2.4. Let $\{T_1, T_2, \cdots, T_N\}$ be a family of random nonexpansive operators from $\Omega \times K \to K$, where K is a closed, convex subset of a separable Hilbert space H. Let $F = \bigcap_{i=1}^{N} RF(T_i) \neq \emptyset$, where $RF(T_i)$ is the set of all random fixed points of a random operator T_i for each $i \in \{1, 2, \cdots, N\}$. Let $\xi_0\colon \Omega \to K$ be any fixed measurable map and $\{\alpha_n\} \subset [0, 1]$, then the sequence of function $\{\xi_n\}$ defined by :

$$\begin{cases} \xi_1(\omega) = \alpha_1\xi_0(\omega) + (1-\alpha_1)T_1(\omega, \xi_1(\omega)), \\ \xi_2(\omega) = \alpha_2\xi_1(\omega) + (1-\alpha_2)T_2(\omega, \xi_2(\omega)), \\ \qquad \vdots \\ \xi_N(\omega) = \alpha_N\xi_{N-1}(\omega) + (1-\alpha_N)T_N(\omega, \xi_N(\omega)), \\ \xi_{N+1}(\omega) = \alpha_{N+1}\xi_N(\omega) + (1-\alpha_{N+1})T_1(\omega, \xi_{N+1}(\omega)), \\ \qquad \vdots \\ \xi_{2N}(\omega) = \alpha_{2N}\xi_{2N-1}(\omega) + (1-\alpha_{2N})T_N(\omega, \xi_{2N}(\omega)), \\ \xi_{2N+1}(\omega) = \alpha_{2N+1}\xi_{2N}(\omega) + (1-\alpha_{2N+1})T_1(\omega, \xi_{2N+1}(\omega)), \\ \qquad \vdots \end{cases} \tag{2.1}$$

is called the implicit random iteration process for a finite family of nonexpansive random operators $\{T_1, T_2, \cdots, T_N\}$.

More compactly, we can write the above table in the form :

$$\xi_n(\omega) = \alpha_n\xi_{n-1}(\omega) + (1-\alpha_n)T_n(\omega, \xi_n(\omega))\,, \quad \forall\, n \geq 1\,, \tag{2.2}$$

where $N = n \mod N$, $\{\alpha_n\}$ be sequence in $[0, 1]$.

Definition 2.5. Let $\{T_1, T_2, \cdots, T_N\}$ be a family of random nonexpansive operators from $\Omega \times K \to K$, where K is a closed, convex subset of a separable Hilbert space H with $F = \bigcap_{i=1}^{N} RF(T_i) \neq \emptyset$. Let $\xi_0 \colon \Omega \to K$ be any fixed measurable map, then we consider the sequence of function $\{\xi_n\}$ defined by :

$$\begin{cases} \xi_n(\omega) = \alpha_n \xi_{n-1}(\omega) + (1-\alpha_n) T_n(\omega, \eta_n(\omega)) \,, \\ \eta_n(\omega) = r_n \xi_n(\omega) + s_n \xi_{n-1}(\omega) + t_n T_n(\omega, \xi_n(\omega)) + w_n T_n(\omega, \xi_{n-1}(\omega)) \\ \{\alpha_n\}, \{r_n\}, \{s_n\}, \{t_n\}, \{w_n\} \in [0,1], \quad r_n + s_n + t_n + w_n = 1 \end{cases} \tag{2.3}$$

where $n = n \mod N$.

The above iteration scheme is called random general implicit iteration process. Let $\lambda(\omega)$ be a real valued random variable in $[0,1)$, $\mu(\omega) > 0$ be a real valued random variable and $A \colon \Omega \times H \to H$ be a Lipschitzian strongly monotone. Associating with a nonexpansive random operator $T \colon \Omega \times H \to H$, we define a random operator $T^{\lambda} \colon \Omega \times H \to H$ by

$$T^{\lambda}(\omega, x) = T(\omega, x) - \lambda(\omega)\mu(\omega) A(\omega, T(\omega, x)) \quad \forall\, x \in H \ \ \omega \in \Omega \,.$$

We now propose random general implicit random iteration scheme with perturbed operator as below:

Definition 2.6. Let $\{T_1, T_2, \cdots, T_N\}$ be a family of random nonexpansive operators from $\Omega \times K \to K$, where K is a closed, convex subset of a separable Hilbert space H with $F = \bigcap_{i=1}^{N} RF(T_i) \neq \emptyset$. Let $\xi_0 \colon \Omega \to K$ be any fixed measurable map, then we consider the sequence of function $\{\xi_n\}$ defined by :

$$\begin{cases} \xi_n(\omega) = \alpha_n \xi_{n-1}(\omega) + (1-\alpha_n) T_n^{\lambda_n}(\omega, \eta_n(\omega)) \\ \qquad\quad = \alpha_n \xi_{n-1}(\omega) + (1-\alpha_n)[T_n \eta_n(\omega) - \lambda_n(\omega)\mu(\omega) A(T_n \eta_n(\omega))] \,, \\ \eta_n(\omega) = r_n \xi_n(\omega) + s_n \xi_{n-1}(\omega) + t_n T_n^{\lambda_n}(\omega, \xi_n(\omega)) + w_n T_n^{\lambda_n}(\omega, \xi_{n-1}(\omega)) \,; \\ \{\alpha_n\}, \{r_n\}, \{s_n\}, \{t_n\}, \{w_n\} \in [0,1], \quad r_n + s_n + t_n + w_n = 1 \end{cases} \tag{2.4}$$

$n = n \mod N$, where λ_n be a sequence of measurable mappings from Ω to $[0,1)$, and $\mu(\omega) \in (0, 2\theta(\omega)/\kappa^2(\omega))$ is a real valued random variable.

We next show that proposed iteration scheme (2.4) is well defined. First we prove following Lemma.

Lemma 2.7. *If* $\mu(\omega) < \frac{2\theta(\omega)}{\kappa^2(\omega)}$*, then* T^λ *is a random contraction, i.e. there holds*

$$\|T^\lambda(\omega, x) - T^\lambda(\omega, y)\| \le (1 - \lambda(\omega)\tau(\omega))\|x - y\| \quad \forall\, x, y \in H\,,\ \omega \in \Omega$$

where $\tau(\omega) = 1 - \sqrt{1 - \mu(\omega)(2\theta(\omega) - \mu(\omega)k^2(\omega))} \in (0, 1)$.

Proof. Define a mapping $F\colon \Omega \times H \to H$ given by $F(\omega, x) = \mu(\omega)A(\omega, x) - I(\omega, x)$. By Lipschitz continuity and strong monotonicity of A, we have

$$\begin{aligned}
&\|F(\omega, x) - F(\omega, y)\|^2 = \mu(\omega)^2\|A(\omega, x) - A(\omega, y)\|^2\\
&- 2\mu(\omega)\langle x - y, A(\omega, x) - A(\omega, y)\rangle + \|x - y\|^2\\
&\le \mu(\omega)^2\kappa^2(\omega)\|x - y\|^2 - 2\mu(\omega)\theta(\omega)\|x - y\|^2 + \|x - y\|^2\\
&= \left\{1 - \mu(\omega)(2\theta(\omega) - \mu(\omega)\kappa^2(\omega))\right\}\|x - y\|^2, \quad \forall\, x, y \in H\,,\ \omega \in \Omega.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\|T^\lambda(\omega, x) - T^\lambda(\omega, y)\| = \|(1 - \lambda(\omega))(T(\omega, x) - T(\omega, y))\\
&\quad - \lambda(\omega)\{F(\omega, T(\omega, x)) - F(\omega, T(\omega, y))\}\|\\
&\le (1 - \lambda(\omega))\|T(\omega, x) - T(\omega, y)\|\\
&\quad + \lambda(\omega)\|F(\omega, T(\omega, x)) - F(\omega, T(\omega, y))\|\\
&\le (1 - \lambda(\omega))\|x - y)\|\\
&\quad + \lambda(\omega)\sqrt{1 - \mu(\omega)(2\theta(\omega) - \mu(\omega)\kappa^2(\omega))}\|x - y\|\\
&= \left[1 - \lambda(\omega)\left(1 - \sqrt{1 - \mu(\omega)(2\theta(\omega) - \mu(\omega)\kappa^2(\omega))}\right)\right]\|x - y\|\\
&= (1 - \lambda(\omega)\tau(\omega))\|x - y\|
\end{aligned}$$

where $\tau(\omega) = 1 - \sqrt{1 - \mu(\omega)(2\theta(\omega) - \mu(\omega)\kappa^2(\omega))}$.

This completes the proof. ☐

Observe that for every $u \in H$, $\omega \in \Omega$ and $\alpha, r, s, t, w \in [0, 1]$ and positive integer n, the random operator $S = S_{(\alpha,r,s,t,w,n)}\colon \Omega \times H \to H$ defined by

$$S(\omega, x) = \alpha u + (1 - \alpha)T^\lambda\left(\omega, \left(rx + su + tT^\lambda(\omega, x) + wT^\lambda(\omega, u)\right)\right)$$

satisfies,

$$
\begin{aligned}
\|S(\omega,x)-S(\omega,y)\| &= (1-\alpha)\|T^{\lambda}\left(\omega,\left(rx+su+tT^{\lambda}(\omega,x)+wT^{\lambda}(\omega,u)\right)\right)\\
&\quad -T^{\lambda}\left(\omega,\left(ry+su+tT^{\lambda}(\omega,y)+wT^{\lambda}(\omega,u)\right)\right)\|\\
&\le (1-\alpha)(1-\lambda(\omega)\tau(\omega))\|\left(rx+tT^{\lambda}(\omega,x)\right)-\left(ry+tT^{\lambda}(\omega,y)\right)\|\\
&\le (1-\alpha)(1-\lambda(\omega)\tau(\omega))(r+t(1-\lambda(\omega)\tau(\omega)))\|x-y\|\\
&\le (1-\alpha)(1-\lambda(\omega)\tau(\omega))(r+t)\|x-y\|
\end{aligned}
$$

So, if $(1-\alpha)(1-\lambda(\omega)\tau(\omega))(r+t)<1$, then S is a random contraction. Thus by [3, Theorem 2.1], S has a unique random fixed point. This shows that, the random iteration scheme (2.4) with perturbed operator A is well-defined and can be employed for the approximation of random common fixed point of a finite family of nonexpansive random operators.

We will now give some definitions and lemmas which will be used in the rest of this paper.

Lemma 2.8. [14] *Let* $\{a_n\}$, $\{b_n\}$ *and* $\{c_n\}$ *are sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1}\le(1+\delta_n)a_n+b_n \quad n\ge 1$$

If $\sum_{n=1}^{\infty}\delta_n<\infty$ *and* $\sum_{n=1}^{\infty}b_n<\infty$, *then* $\lim_{n\to\infty}a_n$ *exists. If in addition* $\{a_n\}$ *has a subsequence which converges strongly to zero, then* $\lim_{n\to\infty}a_n=0$.

Lemma 2.9. [15] *Let* E *be a uniformly convex Banach space,* b,c *be two constants with with* $0<b<c<1$. *Suppose that* $\{t_n\}$ *be a real sequence in* $[b,c]$ *and* $\{x_n\}$, $\{y_n\}$ *are two sequences in* E *such that :*

$$
\begin{cases}
\limsup\limits_{n\to\infty}\|x_n\|\le a; \quad \limsup\limits_{n\to\infty}\|y_n\|\le a;\\
\lim\limits_{n\to\infty}\|t_nx_n+(1-t_n)y_n\|=a,
\end{cases}
$$

then $\lim_{n\to\infty}\|x_n-y_n\|=0$, *where* $a\ge 0$ *is some constant.*

3. Main Results

We now present main results of this paper.

Lemma 3.1. *Let H be a real separable Hilbert space and $A\colon \Omega\times H\to H$ be a Lipschitzian and strongly monotone random operator. Let $\{T_i\}_{i=1}^N$ be N nonexpansive random operators from $\Omega\times H$ to H such that $F=\bigcap_{n=1}^N RF(T_i)\neq\emptyset$. Let $\{\lambda_n\}$ be a sequence of measurable mapping from $\Omega\to[0,1)$ such that $\sum_{n=1}^\infty\lambda_n(\omega)<\infty$ and $\{\alpha_n\}$, $\{r_n\}$, $\{s_n\}$, $\{t_n\}$, $\{w_n\}$ be five real sequences in $[0,1]$ satisfying $0<a\leq\alpha_n\leq b<1$, $t_n+w_n\leq b<1$, where a,b are some constants. The sequence $\{\xi_n(\omega)\}_{i=1}^\infty$ is defined by (2.4), then for $\mu(\omega)\in(0,2\theta(\omega)/\kappa^2(\omega))$*

- (i) $\lim_{n\to\infty}\|\xi_n(\omega)-\xi(\omega)\|$ *exists for each $\xi(\omega)\in F$,*
- (ii) $\lim_{n\to\infty}d(\xi_n(\omega),F)$ *exists, where $d(\xi_n(\omega),F)=\inf_{\xi(\omega)\in F}\|\xi_n(\omega)-\xi(\omega)\|$.*

Proof. Let $\xi(\omega)$ be an arbitrary element of F, then

$$\begin{aligned}\|\xi_n(\omega)-\xi(\omega)\|&=\|\alpha_n\xi_{n-1}(\omega)+(1-\alpha_n)T_n^{\lambda_n}(\omega,\xi_n(\omega))-\xi(\omega)\|\\&\leq\alpha_n\|\xi_{n-1}(\omega)-\xi(\omega)\|+(1-\alpha_n)\|T_n^{\lambda_n}(\omega,\eta_n(\omega))-\xi(\omega)\|\end{aligned}\tag{3.1}$$

Now, using Lemma 2.7, we have

$$\begin{aligned}&\|T_n^{\lambda_n}(\omega,\eta_n(\omega))-\xi(\omega)\|\\&=\|T_n^{\lambda_n}(\omega,\eta_n(\omega))-T_n^{\lambda_n}(\omega,\xi(\omega))+T_n^{\lambda_n}(\omega,\xi(\omega))-\xi(\omega)\|\\&\leq\|T_n^{\lambda_n}(\omega,\eta_n(\omega))-T_n^{\lambda_n}(\omega,\xi(\omega))\|+\|T_n^{\lambda_n}(\omega,\xi(\omega))-\xi(\omega)\|\\&\leq(1-\lambda(\omega)\tau(\omega))\|\eta_n(\omega)-\xi(\omega)\|+\lambda_n(\omega)\mu(\omega)\|A(\omega,\xi(\omega))\|\end{aligned}\tag{3.2}$$

Now,

$$\begin{gathered}\|\eta_n(\omega)-\xi(\omega)\|\leq r_n\|\xi_n(\omega)-\xi(\omega)\|\\+s_n\|\xi_{n-1}(\omega)-\xi(\omega)\|+t_n\|T_n^{\lambda_n}(\omega,\xi_n(\omega))-\xi(\omega)\|\\+w_n\|T_n^{\lambda_n}(\omega,\xi_{n-1}(\omega))-\xi(\omega)\|\\=r_n\|\xi_n(\omega)-\xi(\omega)\|+s_n\|\xi_{n-1}(\omega)-\xi(\omega)\|\\+(1-\lambda_n(\omega)\tau(\omega))\left[t_n\|\xi_n(\omega)-\xi(\omega)\|+w_n\|\xi_{n-1}(\omega)-\xi(\omega)\|\right]\\+(t_n+w_n)\lambda_n(\omega)\mu(\omega)\|A(\omega,\xi(\omega))\|\\=\left(r_n+t_n(1-\lambda_n(\omega)\tau(\omega))\right)\|\xi_n(\omega)-\xi(\omega)\|\\+\left(s_n+w_n(1-\lambda_n(\omega)\tau(\omega))\right)\|\xi_{n-1}(\omega)-\xi(\omega)\|\\+(t_n+w_n)\lambda_n(\omega)\mu(\omega)\|A(\omega,\xi(\omega))\|\end{gathered}$$

$$\le (r_n+t_n)\,\|\xi_n(\omega)-\xi(\omega)\|+(s_n+w_n)\,\|\xi_{n-1}(\omega)-\xi(\omega)\| \\ +\lambda_n(\omega)\mu(\omega)\|A(\omega,\xi(\omega))\| \tag{3.3}$$

By (3.3) and (3.2), we have

$$\begin{aligned}\|T_n^{\lambda_n}(\omega,\eta_n(\omega))-\xi(\omega)\| \le{}& (1-\lambda_n(\omega)\tau(\omega))\,[\,(r_n+t_n)\,\|\xi_n(\omega)-\xi(\omega)\| \\ &+(s_n+w_n)\,\|\xi_{n-1}(\omega)-\xi(\omega)\| \\ &+\lambda_n(\omega)\mu(\omega)\|A(\omega,\xi(\omega))\|\ ] \\ &+\lambda_n(\omega)\mu(\omega)\|A(\omega,\xi(\omega))\| \end{aligned}\tag{3.4}$$

(3.1) and (3.4) gives

$$\begin{aligned}\|\xi_n(\omega)-\xi(\omega)\| &\le \alpha_n\|\xi_{n-1}(\omega)-\xi(\omega)\| \\ &\quad+(1-\alpha_n)[(1-\lambda_n(\omega)\tau(\omega))[(r_n+t_n)\|\xi_n(\omega)-\xi(\omega)\| \\ &\quad+(s_n+w_n)\,\|\xi_{n-1}(\omega)-\xi(\omega)\|+\lambda_n(\omega)\mu(\omega)\|A(\omega,\xi(\omega))\|\,] \\ &\quad+(1-\alpha_n)\lambda_n(\omega)\mu(\omega)\|A(\omega,\xi(\omega))\| \\ &\le[\alpha_n+(1-\alpha_n)\,(s_n+w_n)]\,\|\xi_{n-1}(\omega)-\xi(\omega)\| \\ &\quad+(1-\alpha_n)\,(r_n+t_n)\,\|\xi_n(\omega)-\xi(\omega)\| \\ &\quad+(1-\alpha_n)(2-\lambda_n(\omega)\tau(\omega))\lambda_n(\omega)\mu(\omega)\|A(\omega,\xi(\omega))\| \\ &\le[\alpha_n+(1-\alpha_n)\,(1-(r_n+t_n))]\,\|\xi_{n-1}(\omega)-\xi(\omega)\| \\ &\quad+(1-\alpha_n)\,(r_n+t_n)\,\|\xi_n(\omega)-\xi(\omega)\| \\ &\quad+(1-\alpha_n)(2-\lambda_n(\omega)\tau(\omega))\lambda_n(\omega)\mu(\omega)\|A(\omega,\xi(\omega))\| \\ &=[1-(1-\alpha_n)\,(r_n+t_n)]\,\|\xi_{n-1}(\omega)-\xi(\omega)\| \\ &\quad+(1-\alpha_n)\,(r_n+t_n)\,\|\xi_n(\omega)-\xi(\omega)\| \\ &\quad+(1-\alpha_n)(2-\lambda_n(\omega)\tau(\omega))\lambda_n(\omega)\mu(\omega)\|A(\omega,\xi(\omega))\| \\ &\le\|\xi_{n-1}(\omega)-\xi(\omega)\|+\frac{(1-\alpha_n)(2-\lambda_n(\omega)\tau(\omega))\lambda_n(\omega)\mu(\omega)\|A(\omega,\xi(\omega))\|}{1-(1-\alpha)\,(r_n+t_n)}\end{aligned}\tag{3.5}$$

For each $\xi(\omega)\in F$, we have

$$\begin{aligned}\|A(\omega,\xi(\omega))\| &\le \|A(\omega,\xi(\omega))-A(\xi_{n-1}(\omega))\|+\|F(\omega,\xi_{n-1}(\omega))\| \\ &\le \kappa\|\xi_{n-1}(\omega)-\xi(\omega)\|+\|F(\omega,\xi_{n-1}(\omega))\|\end{aligned}\tag{3.6}$$

From (3.5) and (3.6), we have

$$\|\xi_n(\omega)-\xi(\omega)\| \le (1+\sigma_n)\|\xi_{n-1}(\omega)-\xi(\omega)\|+\sigma_n \tag{3.7}$$

where,

$$\sigma_n=\lambda_n(\omega)\cdot\left(\frac{(1-\alpha_n)(2-\lambda_n(\omega)\tau(\omega))\mu(\omega)}{1-(1-\alpha)\,(r_n+t_n)}\right)\cdot\max\{\kappa(\omega),\|F(\omega,\xi_{n-1}(\omega))\|\}\ .$$

Taking infimum over all $\xi(\omega)\in F$, we have

$$d(\xi_n(\omega),F)\le(1+\sigma_n)d(\xi_{n-1}(\omega),F)+\sigma_n \tag{3.8}$$

Since $\sum_{n=1}^{\infty}\lambda_n<\infty$ and $\{F(\omega,\xi_{n-1}(\omega))\}$ is bounded, so $\sum_{n=1}^{\infty}\sigma_n<\infty$ converges, thus by Lemma 2.8, we get that

$$\lim_{n\to\infty}\|\xi_n(\omega)-\xi(\omega)\| \quad \text{exists,} \qquad \lim_{n\to\infty}d(\xi_n(\omega),F) \quad \text{exists.}$$

□

Lemma 3.2. *Let H be a real separable Hilbert space and $A\colon\Omega\times H\to H$ be a Lipschitzian and strongly monotone random operator. Let $\{T_i\}_{i=1}^{N}$ be N nonexpansive random operators from $\Omega\times H$ to H such that $F=\bigcap_{n=1}^{N}RF(T_i)\ne\emptyset$. Let $\{\lambda_n\}$ be a sequence of measurable mappings from $\Omega\to[0,1)$ such that $\sum_{n=1}^{\infty}\lambda_n(\omega)<\infty$ and $\{\alpha_n\}$, $\{r_n\}$, $\{s_n\}$, $\{t_n\}$, $\{w_n\}$ be five real sequences in $[0,1]$ satisfying $0<a\le\alpha_n\le b<1$, $t_n+w_n\le b<1$, where a,b are some constants. The sequence $\{\xi_n(\omega)\}_{i=1}^{\infty}$ is defined by (2.4), then for $\mu(\omega)\in(0,2\theta(\omega)/\kappa^2(\omega))$*

$$\liminf_{n\to\infty}\|\xi_n(\omega)-T_l(\omega,\xi_n(\omega))\|=0 \qquad 1\le l\le N.$$

Proof. Without loss of generality, we can assume that

$$\lim_{n\to\infty}\|\xi_n(\omega)-\xi(\omega)\|=d\,, \tag{3.9}$$

where $d\ge0$ is some number.

From (3.9), we have

$$\begin{aligned}\lim_{n\to\infty}&\|\xi_n(\omega)-\xi(\omega)\|\\ &=\lim_{n\to\infty}\|\alpha_n(\xi_{n-1}(\omega)-\xi(\omega))+(1-\alpha_n)(T_n^{\lambda_n}(\omega,\eta_n(\omega))-\xi(\omega))\|=d\end{aligned} \tag{3.10}$$

Now, by (3.4) we have

$$\begin{aligned}\|T_n^{\lambda_n}(\omega,\eta_n(\omega))-\xi(\omega)\| \le{}& (r_n+t_n)\,\|\xi_n(\omega)-\xi(\omega)\|\\ &+(s_n+w_n)\,\|\xi_{n-1}(\omega)-\xi(\omega)\|\\ &+2\lambda_n\mu(\omega)\|A(\omega,\xi(\omega))\|\end{aligned}$$

this gives that,

$$\limsup_{n\to\infty}\|T_n^{\lambda_n}(\omega,\eta_n(\omega))-\xi(\omega)\|\le d\,. \tag{3.11}$$

From Lemma 2.9, (3.9),(3.10) and (3.11) we have,

$$\lim_{n\to\infty}\|T_n^{\lambda_n}(\omega,\eta_n(\omega))-\xi_{n-1}(\omega)\|=0\,. \tag{3.12}$$

Again, from (2.4) and (3.12), we have

$$\lim_{n\to\infty}\|\xi_n(\omega)-\xi_{n-1}(\omega)\|\le\lim_{n\to\infty}(1-\alpha_n)\|T_n^{\lambda_n}(\omega,\eta_n(\omega))-\xi_{n-1}(\omega)\|=0\,, \tag{3.13}$$

so for any $j=1,2,\cdots,N$,

$$\lim_{n\to\infty}\|\xi_n(\omega)-\xi_{n+j}(\omega)\|=0\,. \tag{3.14}$$

On the otherhand, from (3.12), (3.14), we have

$$\begin{aligned}&\lim_{n\to\infty}\|\xi_n(\omega)-T_n^{\lambda_n}(\omega,\eta_n(\omega))\|\\ &\le\lim_{n\to\infty}\Big\{\|\xi_n(\omega)-\xi_{n-1}(\omega)\|+\|\xi_{n-1}(\omega)-T_n^{\lambda_n}(\omega,\eta_n(\omega))\|\Big\}\\ &=0\,.\end{aligned} \tag{3.15}$$

Now, since

$$\begin{aligned}\|T_n^{\lambda_n}(\omega,\xi_n(\omega))-\xi_n(\omega)\|&\le\|\xi_n(\omega)-\xi_{n-1}(\omega)\|+\|T_n^{\lambda_n}(\omega,\eta_n(\omega))-\xi_{n-1}(\omega)\|\\ &\quad+\|T_n^{\lambda_n}(\omega,\eta_n(\omega))-T_n^{\lambda_n}(\omega,\xi_n(\omega))\|\\ &\le\|\xi_n(\omega)-\xi_{n-1}(\omega)\|+\|T_n^{\lambda_n}(\omega,\eta_n(\omega))-\xi_{n-1}(\omega)\|\\ &\quad+(1-\lambda_n(\omega)\tau(\omega))\|\eta_n(\omega)-\xi_n(\omega)\|\\ &\le\|\xi_n(\omega)-\xi_{n-1}(\omega)\|+\|T_n^{\lambda_n}(\omega,\eta_n(\omega))-\xi_{n-1}(\omega)\|\\ &\quad+(1-\lambda_n(\omega)\tau(\omega))(\|\eta_n(\omega)-\xi_{n-1}(\omega)\|+\|\xi_n(\omega)-\xi_{n-1}(\omega)\|)\\ &=(2-\lambda_n(\omega)\tau(\omega))\|\xi_n(\omega)-\xi_{n-1}(\omega)\|\\ &\quad+\|T_n^{\lambda_n}(\omega,\eta_n(\omega))-\xi_{n-1}(\omega)\|\\ &\quad+(1-\lambda_n(\omega)\tau(\omega))\|\eta_n(\omega)-\xi_{n-1}(\omega)\|\end{aligned} \tag{3.16}$$

Now,

$$\begin{aligned}
&\|\eta_n(\omega)-\xi_{n-1}(\omega)\| \le \|r_n\xi_n(\omega)+s_n\xi_{n-1}(\omega)+t_nT_n^{\lambda_n}(\omega,\xi_n(\omega))\\
&\quad +w_nT_n^{\lambda_n}(\omega,\xi_{n-1}(\omega))-\xi_{n-1}(\omega)\|\\
&=\|r_n\xi_n(\omega)+t_nT_n^{\lambda_n}(\omega,\xi_n(\omega))+w_nT_n^{\lambda_n}(\omega,\xi_{n-1}(\omega))\\
&\quad -(1-s_n)\xi_{n-1}(\omega)\|\\
&=\|r_n\xi_n(\omega)+t_nT_n^{\lambda_n}\xi_n(\omega)+w_nT_n^{\lambda_n}\xi_{n-1}(\omega)-(r_n+t_n+w_n)\xi_{n-1}(\omega)\|\\
&=\|t_n(T_n^{\lambda_n}(\omega,\xi_n(\omega))-\xi_n(\omega))+w_n(T_n^{\lambda_n}(\omega,\xi_{n-1}(\omega))-\xi_n(\omega))\\
&\quad +(r_n+t_n+w_n)(\xi_n(\omega)-\xi_{n-1}(\omega))\|\\
&\le t_n\|T_n^{\lambda_n}(\omega,\xi_n(\omega))-\xi_n(\omega)\|+w_n\|T_n^{\lambda_n}(\omega,\xi_{n-1}(\omega))-\xi_n(\omega)\|\\
&\quad +(r_n+t_n+w_n)\|\xi_n(\omega)-\xi_{n-1}(\omega)\|\\
&\le t_n\|T_n^{\lambda_n}(\omega,\xi_n(\omega))-\xi_n(\omega)\|+w_n[\|T_n^{\lambda_n}(\omega,\xi_{n-1}(\omega))-T_n^{\lambda_n}(\omega,\xi_n(\omega))\|\\
&\quad +\|T_n^{\lambda_n}(\omega,\xi_n(\omega))-\xi_n(\omega)\|]+(r_n+t_n+w_n)\|\xi_n(\omega)-\xi_{n-1}(\omega)\|\\
&\le (t_n+w_n)\|T_n^{\lambda_n}(\omega,\xi_n(\omega))-\xi_n(\omega)\|\\
&\quad +w_n(1-\lambda_n(\omega)\tau(\omega))\|\xi_{n-1}(\omega)-\xi_n(\omega)\|\\
&\quad +(r_n+t_n+w_n)\|\xi_n(\omega)-\xi_{n-1}(\omega)\|\\
&\le (t_n+w_n)\|T_n^{\lambda_n}(\omega,\xi_n(\omega))-\xi_n(\omega)\|\\
&\quad +(r_n+t_n+w_n+w_n(1-\lambda_n(\omega)\tau(\omega)))\|\xi_n(\omega)-\xi_{n-1}(\omega)\|\\
&\le (t_n+w_n)\|T_n^{\lambda_n}(\omega,\xi_n(\omega))-\xi_n(\omega)\|\\
&\quad +(1-s_n+w_n(1-\lambda_n(\omega)\tau(\omega)))\|\xi_n(\omega)-\xi_{n-1}(\omega)\| \qquad (3.17)
\end{aligned}$$

By substituting (3.17) into (3.16), we get

$$\begin{aligned}
&\|T_n^{\lambda_n}(\omega,\xi_n(\omega))-\xi_n(\omega)\| \le (2-\lambda_n(\omega)\tau(\omega))\|\xi_n(\omega)-\xi_{n-1}(\omega)\|\\
&+\|T_n^{\lambda_n}(\omega,\eta_n(\omega))-\xi_{n-1}(\omega)\|\\
&+(1-\lambda_n(\omega)\tau(\omega))\Big[(t_n+w_n)\|T_n^{\lambda_n}(\omega,\xi_n(\omega))-\xi_n(\omega)\|\\
&+(1-s_n+w_n(1-\lambda_n(\omega)\tau(\omega)))\|\xi_n(\omega)-\xi_{n-1}(\omega)\|\,\Big]\\
&\le (3+w_n)\|\xi_n(\omega)-\xi_{n-1}(\omega)\|+\|T_n^{\lambda_n}(\omega,\eta_n(\omega))-\xi_{n-1}(\omega)\|\\
&+(1-\lambda_n(\omega)\tau(\omega))(t_n+w_n)\|T_n^{\lambda_n}(\omega,\xi_n(\omega))-\xi_n(\omega)\|\,,
\end{aligned}$$

since $t_n + w_n \le b < 1$, so above inequality gives,

$$[1-(1-\lambda_n(\omega)\tau(\omega))b]\,\|T_n^{\lambda_n}(\omega,\xi_n(\omega))-\xi_n(\omega)\| \le (3+w_n)\|\xi_n(\omega)-\xi_{n-1}(\omega)\| \\ +\|T_n^{\lambda_n}(\omega,\eta_n(\omega))-\xi_{n-1}(\omega)\|$$

From (3.12), (3.13) and above inequality, we have

$$\lim_{n\to\infty}\|T_n^{\lambda_n}(\omega,\xi_n(\omega))-\xi_n(\omega)\| = 0\,. \tag{3.18}$$

Using (3.18), we have

$$\begin{aligned}\|\xi_n(\omega)-T_n(\omega,\xi_n(\omega))\| &\le \|\xi_n(\omega)-T_n^{\lambda_n}(\omega,\xi_n(\omega))\| \\ &\quad +\|T_n^{\lambda_n}(\omega,\xi_n(\omega))-T_n(\omega,\xi_n(\omega))\| \\ \le \|\xi_n(\omega)-T_n^{\lambda_n}(\omega,\xi_n(\omega))\| &+ \lambda_n(\omega)\mu(\omega)\|A\,(\omega,T_n(\omega,\xi_n(\omega)))\,\| \to \ \text{as } n\to\infty\end{aligned} \tag{3.19}$$

Consequently, we have for each $j = 1,2,\cdots,N$

$$\begin{aligned}\|\xi_n(\omega)-T_{n+j}(\omega,\xi_n(\omega))\| &\le \|\xi_n(\omega)-\xi_{n+j}(\omega)\| + \|\xi_{n+j}(\omega)-T_{n+j}(\omega,\xi_n(\omega))\| \\ &\quad + \|T_{n+j}(\omega,\xi_{n+j}(\omega))-T_{n+j}(\omega,\xi_n(\omega))\| \\ &\le 2\|\xi_n(\omega)-\xi_{n+j}(\omega)\| + \|x_{n+j}-T_{n+j}(\omega,\xi_{n+j}(\omega))\| \\ &\to \ 0 \quad \text{as} \quad n\to\infty\,,\end{aligned} \tag{3.20}$$

and so $\lim_{n\to\infty}\|\xi_n(\omega)-T_{n+j}(\omega,\xi_n(\omega))\| = 0$ for each $j = 1,2,\cdots,N$. This implies that

$$\lim_{n\to\infty}\|\xi_n(\omega)-T_l(\omega,\xi_n(\omega))\| = 0 \quad \text{for each} \ \ l = 1,2,\cdots,N\,. \tag{3.21}$$

□

Next, we prove necessary and sufficient condition for the strong convergence of the random iteration scheme (2.4) to random common fixed point of finite family of random nonexpansive mappings :

Theorem 3.3. *Let H be a real separable Hilbert space and $A\colon \Omega\times H\to H$ be a Lipschitzian and strongly monotone random operator. Let $\{T_i\}_{i=1}^N$ be N nonexpansive random operators from $\Omega\times H$ to H such that $F=\bigcap_{n=1}^N RF(T_i)\ne\emptyset$. Let $\{\lambda_n\}$ be a sequence of measurable mappings from $\Omega\to[0,1)$ such that $\sum_{n=1}^\infty \lambda_n(\omega)<\infty$ and $\{\alpha_n\}$, $\{r_n\}$, $\{s_n\}$, $\{t_n\}$, $\{w_n\}$ be five real sequences in $[0,1]$ satisfying $0<a\le\alpha_n\le b<1$, $t_n+w_n\le b<1$, where a,b are some constants. The sequence $\{\xi_n(\omega)\}_{i=1}^\infty$ is defined by (2.4), converges to a* *common fixed point of a random operator* $\{T_i\colon i \in 1, 2, \cdots, N\}$ *if and only if* $\liminf_{n\to\infty} d(\xi_n(\omega), F) = 0$, *where* $\mu(\omega) \in (0, 2\theta(\omega)/\kappa^2(\omega))$.

Proof. From (3.8), we have

$$d(\xi_n(\omega), F) \le (1 + \sigma_n) d(\xi_{n-1}(\omega), F) + \sigma_n$$

for all $n \ge 1$, where $\sum_{n=1}^{\infty} \sigma_n < \infty$.

If $\{\xi_n(\omega)\}$ converges strongly to a common fixed point $\xi(\omega)$ of the family $\{T_i\}_{i=1}^N$, then $\lim_{n\to\infty} \|\xi_n(\omega) - \xi(\omega)\| = 0$. Since

$$0 \le d(\xi_n(\omega), F) \le \|\xi_n(\omega) - \xi(\omega)\|,$$

we have $\liminf_{n\to\infty} d(\xi_n(\omega), F) = 0$.

Conversely suppose $\liminf_{n\to\infty} d(\xi_n(\omega), F) = 0$, then our Lemma 3.1 implies that $\lim_{n\to\infty} d(\xi_n(\omega), F) = 0$. Thus for arbitrary $\varepsilon > 0$, there exists a positive integer N_0 such that

$$d(\xi_n(\omega), F) < \frac{\varepsilon}{4M}, \quad \forall\, n \ge N_0 .$$

Also since $\sum_{n=1}^{\infty} \sigma_n < \infty$, there exists a positive integer N_1 such that,

$$\sum_{j=n}^{\infty} \sigma_n < \frac{\varepsilon}{4M}, \quad \forall\, n \ge N_1 .$$

From (3.7), we have

$$\begin{aligned}
\|\xi_{n+m}(\omega) - \xi(\omega)\| &\le (1 + \sigma_{n+m-1})\|\xi_{n+m-1}(\omega) - \xi(\omega)\| + \sigma_{n+m-1} \\
&\le (1 + \sigma_{n+m-1})(1 + \sigma_{n+m-2})\|\xi_{n+m-2}(\omega) - \xi(\omega)\| \\
&\quad + (1 + \sigma_{n+m-1})\sigma_{n+m-2} + \sigma_{n+m-1} \\
&\le \prod_{i=n}^{n+m-1} (1 + \sigma_i)\|\xi_n(\omega) - \xi(\omega)\| \\
&\quad + \sum_{j=n+1}^{n+m-1} \left[\left\{ \prod_{i=j}^{n+m-1} (1 + \sigma_i) \right\} \sigma_{j-1} \right] + \sigma_{n+m-1} \\
&\le M\|\xi_n(\omega) - \xi(\omega)\| + M \left[\left\{ \sum_{j=n+1}^{n+m-1} \sigma_{j-1} \right\} + \sigma_{n+m-1} \right] \\
&\le M \left[\|\xi_n(\omega) - \xi(\omega)\| + \sum_{j=n}^{n+m-1} \sigma_j \right]
\end{aligned}$$

Choose $N = \max\{N_0, N_1\}$. Then for all $n, m \geq N$ and for all $\xi(\omega) \in F$, we have

$$\|\xi_{n+m}(\omega) - \xi_n(\omega)\| \leq \|\xi_{n+m}(\omega) - \xi(\omega)\| + \|\xi_n(\omega) - \xi(\omega)\|$$

$$\leq M \left[\|\xi_N(\omega) - \xi(\omega)\| + \sum_{j=N}^{n+m-1} \sigma_j \right] + M \left[\|\xi_N(\omega) - \xi(\omega)\| + \sum_{j=N}^{n+m-1} \sigma_j \right]$$

$$\leq 2M \left[\|\xi_N(\omega) - \xi(\omega)\| + \sum_{j=N}^{\infty} \sigma_j \right]$$

Taking infimum over all $\xi(\omega) \in F$, we obtain

$$\|\xi_{n+m}(\omega) - \xi_n(\omega)\| \leq 2M \left(d(\xi_N(\omega), F) + \frac{\varepsilon}{4M} \right) \leq 2M \left(\frac{\varepsilon}{4M} + \frac{\varepsilon}{4M} \right) = \varepsilon$$

Thus $\{\xi_n(\omega)\}_{n=1}^{\infty}$ is Cauchy sequence for each $\omega \in \Omega$. Therefore, $\xi_n(\omega) \to p(\omega)$ for each $\omega \in \Omega$, $p\colon \Omega \to H$, being the limit of the sequence of measurable function, is also measurable. Since $\lim_{n\to\infty} d(\xi_n(\omega), F) = 0$ for each $\omega \in \Omega$, and each member of the family if continuous, using the similar arguments as in [1, Theorem 3.1], we have $p \in F$.

This completes the proof. □

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