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UNIQUENESS RESULTS FOR PARTIAL FUNCTIONAL DIFFERENTIAL EQUATIONS IN FRÉCHET SPACES

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Abstract. We are giving, in this paper, the existence of the unique mild solution on a semi-infinite interval for a first order semilinear functional differential equation using a recent nonlinear Frigon-Granas alternative for contractions in Fréchet spaces, combined with semigroup theory.
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1. Introduction

In this paper, we consider the existence and uniqueness of mild solutions for a class of first order semilinear functional differential equations. In Section 3 we establish the existence of the unique mild solution, defined on a semi-infinite interval $J = [0, +\infty)$ for the following first-order semilinear evolution equation:

$$y'(t) = A(t)y(t) + f(t, y_t), \quad \text{a.e.} \quad t \in J := [0, +\infty) \tag{1}$$

$$y(t) = \varphi(t), \quad t \in H := [-r, 0], \tag{2}$$

where $0 < r < +\infty, f : J \times C([-r, 0], E) \to E$ and $\varphi \in C([-r, 0]; E)$ are given functions; $\{A(t)\}_{t\geq 0}$ is a family of linear closed (not necessarily bounded) operators from E into E that generate an evolution system of operators $\{U(t,s)\}_{(t,s)\in J\times J}$ for $0 \le s \le t < +\infty$ and E is a real Banach space with the norm $|\ .\ |$.

For any continuous function y defined on $[-r,0]$ and any $t \in [0,+\infty)$, we denote by y_t the element of $C([-r,0];E)$ defined by $y_t(\theta) = y(t+\theta)$ for $\theta \in [-r,0]$. Here $y_t(\cdot)$ represents the history of the state from time $t-r$ up to the present time t. The historical interval is $H := [-r,0]$.

Finally, an extension of these existence results is given in Section 4 for the following semilinear evolution equation with nonlocal conditions

$$y'(t) = A(t)y(t) + f(t, y_t), \quad \text{a.e.} \quad t \in J = [0, +\infty) \tag{3}$$

$$y(t) + h_t(y) = \varphi(t), \quad t \in H = [-r, 0], \tag{4}$$

where $A(\cdot)$, f and φ are as in problem $(1)-(2)$ and $h_t : C([-r,0];E) \to E$ is a given function.

Functional differential equations arise in a variety of areas of biological, physical, and engineering applications, see, for example, the books of Hale and Verduyn Lunel [17] and Kolmanovskii and Myshkis [19], and the references therein. During the last decades, existence and uniqueness of mild, strong and classical solutions of semilinear functional differential equations has been studied extensively by many authors using the semigroup theory, fixed point argument, degree theory and measures of noncompactness. We mention, for instance, the books Ahmed [1], Engel and Nagel [13], Kamenski *et al* [18], Pazy [22] and Wu [23].

Nonlinear evolution equations arise not only from many fields of mathematics, but also from other branches of science such as physics, mechanics and material sciences. For example, Navier-Stokes and Euler equations from fluid mechanics, nonlinear reaction-diffusion equations from heat transfers and biological sciences, nonlinear Klein-Gorden equations and nonlinear Schrödinger equations from quantum mechanics and Cahn-Hilliard equations from material science are some special examples of nonlinear evolution equations. Complexity of nonlinear evolution equations and challenges in their theoretical study have attracted a lot of interest from many mathematicians and scientists in nonlinear sciences.

The nonlocal Cauchy problem for a semilinear evolution equation has been studied first by Byszewski in 1991 [10] (see also [9, 11, 12]). Then, Balachandran and his collaborators have considered various classes of nonlinear integrodifferential systems ([3], see also references cited in [6]).

Benchohra and Ntouyas give existence of mild solution for many semilinear evolution equations and inclusions with nonlocal conditions in [4, 6, 7, 8] using fixed-point theorems and semigroup theory. Controllability results are given also in [4, 5, 21].

By means of a recent Frigon-Granas nonlinear alternative for contractions maps in Fréchet spaces [16] combined with the semigroup theory controllability results for semilinear functional differential equations in Fréchet spaces were given by Arara, Benchohra and Ouahab [2]. Applications of such alternative to differential and integral equations were given by Frigon in [15].

Here we establish sufficient conditions to get the existence of the unique mild solution for semilinear evolution equations $(1)-(2)$ and nonlocal problem $(3)-(4)$. The method we are going to use is to reduce the existence of the unique mild solution to the search for the existence of the unique fixed-point of an appropriate operator by applying a nonlinear alternative of Leray-Schauder type for contraction maps in Fréchet spaces due to Frigon-Granas [16].

2. Preliminaries

We introduce notations, definitions and theorems which are used throughout this paper.

Let $C([-r,0];E)$ be the Banach space of continuous functions with the norm

$$\|y\| = \sup \{ \ |y(t)| : t \in [-r,0] \ \}.$$

Let $B(E)$ be the space of all bounded linear operators from E into E, with the norm

$$\|N\|_{B(E)} = \sup \{ \ |N(y)| \ : \ |y| = 1 \ \}.$$

A measurable function $y : [0,+\infty) \to E$ is Bochner integrable if and only if $|y|$ is Lebesgue integrable. (For properties of the Bochner integral, see for instance, Yosida [24]).

Let $L^1([0,+\infty),E)$ denotes the Banach space of measurable functions $y:[0,+\infty)\to E$ which are Bochner integrable normed by

$$\|y\|_{L^1}=\int_0^{+\infty}|y(t)|\,dt.$$

Definition 2.1. *A function $f:J\times E\to E$ is said to be an L^1 -Carathéodory function if it satisfy :*

(i) *for each $t\in J$ the function $f(t,.):E\to E$ is continuous ;*
(ii) *for each $y\in E$ the function $f(.,y):J\to E$ is measurable ;*
(iii) *for every positive integer k there exists $h_k\in L^1(J;\mathbb{R}_+)$ such that*

$$|f(t,y)|\le h_k(t)\quad for\ all\ |y|\le k\quad and\ almost\ each\ t\in J.$$

In what follows, for the family $\{A(t),\ t\in J\}$ of closed densely defined linear unbounded operators on the Banach space E we assume that it satisfies the following assumptions (see [1], p. 158).

(P1) The domain $D(A(t))$ is independent of t and is dense in E.
(P2) For $t\ge 0$, the resolvent $R(\lambda,A(t))=(\lambda I-A(t))^{-1}$ exists for all λ with $Re\lambda\le 0$, and there is a constant M independent of λ and t such that

$$\|R(t,A(t))\|\le M(1+|\lambda|)^{-1},\ \text{for } Re\lambda\le 0.$$

(P3) There exist constants $L>0$ and $0<\alpha\le 1$ such that

$$\|(A(t)-A(\theta))A^{-1}(\tau)\|\le L|t-\tau|^{\alpha},\ \text{for } t,\theta,\tau\in J.$$

Lemma 2.2. *([1], p. 159). Under assumptions (P1)-(P3), the Cauchy problem*

$$y'(t)-A(t)y(t)=0,\ t\in J,\ y(0)=y_0,$$

has a unique evolution system $U(t,s)$, $(t,s)\in\Delta:=\{(t,s)\in J\times J:0\le s\le t<+\infty\}$ satisfying the following properties:

(1) *$U(t,t)=I$ where I is the identity operator in E,*
(2) *$U(t,s)\ U(s,\tau)=U(t,\tau)$ for $0\le\tau\le s\le t<+\infty$,*
(3) *$U(t,s)\in B(E)$ the space of bounded linear operators on E, where for every $(t,s)\in\Delta$ and for each $y\in E$, the mapping $(t,s)\to U(t,s)\ y$ is continuous.*

More details on evolution systems and their properties from semigroup theory could be found on the books of Ahmed [1], Engel and Nagel [13] and Pazy [22].

Let X be a Fréchet space with a family of semi-norms $\{\|\cdot\|_n\}_{n\in\mathbb{N}}$. Let $Y \subset X$, we say that F is bounded if for every $n \in \mathbb{N}$, there exists $M_n > 0$ such that

$$\|y\|_n \leq M_n \quad \textit{for all } y \in Y.$$

To X we associate a sequence of Banach spaces $\{(X^n, \|\cdot\|_n)\}$ as follows : For every $n \in \mathbb{N}$, we consider the equivalence relation $\sim_n$defined by : $x \sim_n y$ if and only if $\|x-y\|_n = 0$ for all $x, y \in X$. We denote $X^n = (X|_{\sim_n}, \|\cdot\|_n)$ the quotient space, the completion of X^n with respect to $\|\cdot\|_n$. To every $Y \subset X$, we associate a sequence the $\{Y^n\}$ of subsets $Y^n \subset X^n$ as follows : For every $x \in X$, we denote $[x]_n$ the equivalence class of x of subset X^n and we defined $Y^n = \{[x]_n : x \in Y\}$. We denote $\overline{Y^n}$, $int_n(Y^n)$ and $\partial_n Y^n$, respectively, the closure, the interior and the boundary of Y^n with respect to $\|\cdot\|$ in X^n. We assume that the family of semi-norms $\{\|\cdot\|_n\}$ verifies :

$$\|x\|_1 \leq \|x\|_2 \leq \ \|x\|_3 \leq ... \quad \textit{for every } x \in X.$$

Definition 2.3. [16] *A function $f : X \to X$ is said to be a contraction if for each $n \in \mathbb{N}$ there exists $k_n \in (0,1)$ such that :*

$$\|f(x) - f(y)\|_n \leq k_n \ \|x - y\|_n \quad \textit{for all } x, y \in X.$$

Theorem 2.4. *(Nonlinear Alternative,* [16]*). Let X be a Fréchet space and $Y \subset X$ a closed subset in Y and let $N : Y \to X$ be a contraction such that $N(Y)$ is bounded. Then one of the following statements holds :*

$(C1)$ N has a unique fixed point ;

$(C2)$ There exists $\lambda \in [0,1)$, $n \in \mathbb{N}$ and $x \in \partial_n Y^n$ such that $\|x - \lambda\ N\ (x)\|_n = 0$.

3. Semilinear Evolution Equations

Before stating and proving the main result, we give first the definition of mild solution of the semilinear evolution problem $(1) - (2)$.

Definition 3.1. *We say that the function* $y(\cdot) : [-r, +\infty) \to E$ *is a mild solution of* $(1) - (2)$ *if* $y(t) = \varphi(t)$ *for all* $t \in [-r, 0]$ *and* y *satisfies the following integral equation*

$$y(t) = U(t, 0)\ \varphi(0) + \int_0^t U(t, s)\ f(s, y_s)\ ds \qquad for\ each\ t \in [0, +\infty). \qquad (5)$$

We will need the following assumptions :

$(H1)$ There exists a constant $M \geq 1$ such that

$$\|U(t, s)\|_{B(E)} \leq M \qquad for\ every\ (t, s) \in \Delta;$$

$(H2)$ There exists a continuous nondecreasing function $\psi : [0, +\infty) \to (0, +\infty)$ and $p \in L^1_{loc}([0, +\infty); \mathbb{R}_+)$ such that :

$$|f(t, u)| \leq p(t)\ \psi(\|u\|)$$

for a.e. $t \in [0, +\infty)$ and each $u \in C([-r, 0]; E)$ with :

$$\int_c^{+\infty} \frac{ds}{\psi(s)} > M \int_0^a p(s)\ ds \qquad for\ each\ a > 0$$

where $c = M\ \|\varphi\|$.

$(H3)$ For all $R > 0$, there exists $l_R \in L^1_{loc}([-r, +\infty); \mathbb{R}_+)$ such that :

$$|f(t, u) - f(t, v)| \leq l_R(t)\ \|u - v\|$$

for all $u, v \in C([-r, 0]; E)$ with $\|u\| \leq R$ and $\|v\| \leq R$.

Theorem 3.2. *Suppose that hypotheses* $(H1) - (H3)$ *are satisfied. Then the problem* $(1) - (2)$ *has a unique mild solution.*

Proof. For every $n \in \mathbb{N}$, we define in $C([-r, +\infty); E)$ the semi-norms by :

$$\|y\|_n := \sup\ \{\ e^{-\tau\ L_n^*(t)}\ |y(t)| : t \in [0, n]\ \}$$

where : $L_n^*(t) = \int_0^t \bar{l}_n(s)\ ds$, $\bar{l}_n(t) = M\ l_n(t)$ and l_n is the function from $(H3)$.

In what follows we will choose $\tau > 1$.

Then $C([-r, +\infty); E)$ is a Fréchet space with the family of semi-norms $\{\|\cdot\|_n\}_{n \in \mathbb{N}}$.

Transform the problem $(1)-(2)$ into a fixed-point problem. Consider the operator $N : C([-r,+\infty); E) \to C([-r,+\infty); E)$ defined by :

$$N(y)(t) = \begin{cases} \varphi(t), & \text{if } t \in [-r,0]; \\ U(t,0)\ \varphi(0) + \int_0^t U(t,s)\ f(s,y_s)\ ds, & \text{if } t \in [0,+\infty). \end{cases} \tag{6}$$

Clearly, the fixed points of the operator N are mild solutions of the problem $(1)-(2)$.

Let y be a possible solution of the problem $(1)-(2)$. Given $n \in \mathbb{N}$ and $t \leq n$, then from (5), $(H1)$ and $(H2)$ we have :

$$\begin{aligned} |y(t)| &\leq |U(t,0)|\ |\varphi(0)| + \int_0^t \|U(t,s)\|_{B(E)}\ |f(s,y_s)| ds \\ &\leq M\ |\varphi(0)| + M \int_0^t p(s)\ \psi(\|y_s\|)\ ds \\ &\leq M\ \|\varphi\| + M \int_0^t p(s)\ \psi(\|y_s\|)\ ds. \end{aligned}$$

We consider the function μ defined by

$$\mu(t) := \sup\{\ |y(s)|\ :\ 0 \leq s \leq t\ \}, \quad 0 \leq t < +\infty.$$

Let $t^* \in [-r,t]$ be such that $\mu(t) = |y(t^*)|$.

If $t^* \in [0,n]$, by the previous inequality we have

$$\mu(t) \leq M\ \|\varphi\| + M \int_0^t p(s)\ \psi(\mu(s))\ ds, \quad t \in [0,n].$$

If $t^* \in [-r,0]$, then $\mu(t) = \|\varphi\|$ and the previous inequality holds.

Let us take the right-hand side of the above inequality as $v(t)$. Then we have

$$\mu(t) \leq v(t) \ \text{ for all } t \in [0,n].$$

From the definition of v, we have

$$c := v(0) = M\|\varphi\|$$

and

$$v'(t) = M\ p(t)\ \psi(\mu(t)) \quad \text{a.e. } \ t \in [0,n].$$

Using the nondecreasing character of ψ, we get

$$v'(t) \leq M\ p(t)\ \psi(v(t)) \quad \text{a.e. } t \in [0,n].$$

This implies that for each $t \in [0,n]$ and using $(H2)$ we get

$$\int_c^{v(t)} \frac{ds}{\psi(s)} \leq M \int_0^t p(s)\ ds \leq M \int_0^n p(s)\ ds < \int_c^{+\infty} \frac{ds}{\psi(s)}.$$

Thus there exists a constant K_n such that $v(t) \leq K_n,\ t \in [0,n]$ and hence $\mu(t) \leq K_n,\ t \in [0,n]$. Since for every $t \in [0,n]$, $\|y_t\| \leq \mu(t)$, we have

$$\|y\|_n \leq \max\{\ \|\varphi\|\ ,\ K_n\ \} := M_n.$$

Set

$$Y = \{\ y \in C([-r,+\infty);E) : \sup\{|y(t)|\ : 0 \leq t \leq n\} \leq M_n{+}1 \ \text{ for all } n \in \mathbb{N}\ \}.$$

Clearly, Y is a closed subset of $C([-r,+\infty);E)$.

We shall show that $N : Y \to C([-r,+\infty);E)$ is a contraction operator.

Indeed, consider $y, \overline{y} \in C([-r,+\infty);E)$, thus using $(H1)$ and $(H3)$ for each $t \in [0,n]$ and $n \in \mathbb{N}$

$$\begin{aligned}
|N(y)(t) - N(\overline{y})(t)| &= \left| \int_0^t U(t,s)\ [f(s,y_s) - f(s,\overline{y}_s)]\ ds \right| \\
&\leq \int_0^t \|U(t,s)\|_{B(E)}\ |f(s,y_s) - f(s,\overline{y}_s)|\ ds \\
&\leq \int_0^t M\ |f(s,y_s) - f(s,\overline{y}_s)|\ ds \\
&\leq \int_0^t M\ l_n(s)\ \|y_s - \overline{y}_s\|\ ds \\
&\leq \int_0^t [\bar{l}_n(s)\ e^{\tau\ L_n^*(s)}\ ]\ [e^{-\tau\ L_n^*(s)}\ \|y_s - \overline{y}_s\|]\ ds \\
&\leq \int_0^t [\bar{l}_n(s)\ e^{\tau\ L_n^*(s)}]\ ds\ \|y - \overline{y}\|_n \\
&\leq \int_0^t \frac{1}{\tau}[e^{\tau\ L_n^*(s)}]'\ ds\ \|y - \overline{y}\|_n \\
&\leq \frac{1}{\tau}\ e^{\tau\ L_n^*(t)}\ \|y - \overline{y}\|_n.
\end{aligned}$$

Therefore,

$$\|N(y) - N(\overline{y})\|_n \leq \frac{1}{\tau} \, \|y - \overline{y}\|_n.$$

So, for $\tau > 1$, the operator N is a contraction for all $n \in \mathbb{N}$. From the choice of Y there is no $y \in \partial Y^n$ such that $y = \lambda \, N(y)$ for some $\lambda \in (0,1)$. Then the statement $(C2)$ in Theorem 2.4 does not hold. A consequence of the nonlinear alternative of Frigon and Granas that $(C1)$ holds, we deduce that the operator N has a unique fixed-point which is the unique mild solution of the problem $(1) - (2)$.

An Example. As an application of our results we consider the following partial functional differential equation

$$\begin{cases} \dfrac{\partial z}{\partial t}(t,x) = a(t,x)\dfrac{\partial^2 z}{\partial x^2}(t,x) + Q(t, z(t-r,x)) & t \in [0,+\infty), \quad x \in [0,\pi] \\ z(t,0) = z(t,\pi) = 0 & t \in [0,+\infty) \\ z(t,x) = \Phi(t,x) & t \in [-r,0], \quad x \in [0,\pi], \end{cases} \tag{7}$$

where $r > 0$, $a(t,x) : [0,\infty) \times [0,\pi] \to \mathbb{R}$ is a continuous function and is uniformly Hölder continuous in t, $Q : [0,+\infty) \times \mathbb{R} \to \mathbb{R}$ and $\Phi : [-r,0] \times [0,\pi] \to \mathbb{R}$ are continuous functions.

Let

$$f(t, w_t)(x) = Q(t, w(t-x)), \; t \in [0,+\infty), \; x \in [0,\pi].$$

Consider $E = L^2([0,\pi], \mathbb{R})$ and define $A(t)$ by

$$A(t)w = a(t,x)w''$$

with domain

$$D(A) = \{w \in E : w, \; w' \text{ are absolutely continuous}, w'' \in E, \; w(0) = w(\pi) = 0\}$$

Then $A(t)$ generates an evolution system $U(t,s)$ satisfying assumption $(H1)$ (see [14, 20]).

Thus, under the above definitions of f and $A(\cdot)$, the system (7) can be represented by the abstract evolution problem $(1) - (2)$. Furthermore, more appropriate conditions on Q ensure the existence of unique mild solution for (7) by Theorems 3.2 and 2.4.

4. Nonlocal Semilinear Evolution Equations

In this section we give the existence results for the semilinear evolution equation with nonlocal conditions $(3)-(4)$.

The nonlocal condition can be applied in physics with better effect than the classical initial condition $y(0)=y_0$. For example, $h_t(y)$ may be given by

$$h_t(y)=\sum_{i=1}^{p} c_i\ y(t_i+t), \qquad t\in[-r,0]$$

where $c_i,\ i=1,...,p$ are given constants and $0<t_1<\ ...\ <t_p<+\infty$.

At time $t=0$, we have

$$h_0(y)=\sum_{i=1}^{p} c_i\ y(t_i).$$

Nonlocal conditions were initiated by Byszewski [10] to which we refer for motivation and other references.

Before giving the main result, we give first the definition of mild solution of the nonlocal semilinear evolution problem $(3)-(4)$.

Definition 4.1. *A function $y\in C([-r,+\infty);E)$ is said to be a mild solution of $(3)-(4)$ if $y(t)=\varphi(t)-h_t(y)$ for all $t\in[-r,0]$ and y satisfies the following integral equation*

$$y(t)=U(t,0)\ [\varphi(0)-h_0(y)]+\int_0^t U(t,s)\ f(s,y_s)\ ds \qquad for\ each\ t\in[0,+\infty). \tag{8}$$

Let us consider the following conditions needed in the proof of the main result of this section.

$(H4)$ For all $k\geq 0$, there exists a constant $C=C(k)>0$ such that :

$$|h_t(u)-h_t(v)|\leq C\ \|u-v\|$$

for all $u,v\in C([-r,0];E)$ with $\|u\|\leq k$ and $\|v\|\leq k$;

$(H5)$ there exists $D>0$ such that

$$|h_t(u)|\leq D \text{ for each } u\in C([-r,0];E), \text{ and } t\in J.$$

Theorem 4.2. *Assume that the hypotheses $(H1)-(H5)$ hold (the constant c in $(H2)$ becomes here $\widetilde{c}:=M\ [\|\varphi\|+D]$). Then the nonlocal semilinear evolution problem $(3)-(4)$ has a unique mild solution.*

Sketch of the proof. Transform the problem $(3)-(4)$ into a fixed-point problem. Consider the operator $\widetilde{N}: C([-r,+\infty);E) \to C([-r,+\infty);E)$ defined by :

$$\widetilde{N}(y)(t) = \begin{cases} \varphi(t) - h_t(y), & \text{if } t \in [-r,0]; \\ U(t,0)\ [\varphi(0) - h_0(y)] + \displaystyle\int_0^t U(t,s)\ f(s,y_s)\ ds, & \text{if } t \in [0,+\infty). \end{cases}$$

Clearly, the fixed points of the operator $\widetilde{N}$ are mild solutions of the problem $(3)-(4)$.

Then, by parallel steps of the Theorem 3.2's proof, we can easily show that the operator $\widetilde{N}$ is a contraction which have a unique fixed-point by statement $(C1)$ in Theorem 2.4. The details are left to the reader.

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