

ON NONLINEAR QUASI-CONTRACTIONS

IVAN D. ARANĐELOVIĆ*, MILOJE RAJOVIĆ** AND VEDA KILIBARDA***

*Faculty of Mechanical Engineering, Kraljice Marije 16,
11000 Beograd, Serbia
E-mail: iva@alfa.mas.bg.ac.yu

**Faculty of Mechanical Engineering, Dositejeva 19,
36000 Kraljevo, Serbia

***College of Railway Engineering and Services,
Zdravka Čelara 14, 11000 Beograd, Serbia

Abstract. Lj. Ćirić [2] first introduced the notion of quasi - contractions and proved the fixed point theorem for this class of mappings. Ćirić's result was extended to nonlinear quasi - contractions by A. A. Ivanov [4]. In this paper we present new generalizations of Ivanov's fixed point theorem.

Key Words and Phrases: fixed point, nonlinear quasi-contraction.

2000 Mathematics Subject Classification: 54H25, 47H10.

1. INTRODUCTION AND PRELIMINARIES

The contraction mapping principle, formulated and proved in Ph. D. dissertation of S. Banach in 1920 which was published in 1922 is one of the most important theorems in classical functional analysis because it give:

1. the existence and uniqueness of fixed point,
2. method to obtain approximative fixed points,
3. error estimates for approximative fixed point obtained by 2.

There are many generalizations and partial generalizations (which consider only statements 1. and 2.) of the Banach principle.

Lj. Ćirić [2] first introduced the notion of quasi - contractions and proved the fixed point theorem for this class of mappings. Ćirić's result was extended to nonlinear quasi - contractions by A. A. Ivanov [4]. In this paper we present new generalizations of Ivanov's fixed point theorem.

Let X be a nonempty set and $f : X \rightarrow X$ arbitrary mapping. $x \in X$ is a fixed point for f if $x = f(x)$. If $x_0 \in X$, we say that a sequence (x_n) defined by $x_n = f^n(x_0)$ is a sequence of Picard iterates of f at point x_0 or that (x_n) is the orbit of f at point x_0 .

Function $f : \mathbb{R} \rightarrow \mathbb{R}$ is right upper semi continuous at point $r \in \mathbb{R}$ if and only if

$$\overline{\lim}_{t \rightarrow r^+} f(t) \leq f(r).$$

Let (X, d) be a metric space and $A \subseteq X$. Then $\delta(A)$ denotes the diameter of set A .

By Φ we denote the set of real functions $\varphi : [0, \infty) \rightarrow [0, \infty)$ which have the following properties:

- (a) $\varphi(0) = 0$;
- (b) $\varphi(r) < r$ for all $r > 0$;
- (c) $\lim_{x \rightarrow \infty} (x - \varphi(x)) = \infty$.

Define

$$\Phi_1 = \{\varphi \in \Phi : \varphi \text{ is monotone nondecreasing and } \overline{\lim}_{t \rightarrow r^+} \varphi(t) < r \text{ for any } r > 0\},$$

$$\Phi_2 = \{\varphi \in \Phi : \overline{\lim}_{t \rightarrow r} \varphi(t) < r \text{ for any } r > 0\}.$$

2. RESULTS

Our first result is the following generalizations of A. A. Ivanov's [4] fixed point theorem.

Theorem 1. *Let (X, d) be a metric space and $f : X \rightarrow X$. If there exists $\varphi \in \Phi_1$ such that*

$$d(f(x), f(y)) \leq \max\{\varphi(d(x, y)), \varphi(d(x, f(x))), \varphi(d(y, f(y))), \\ \varphi(d(x, f(y))), \varphi(d(f(x), y))\},$$

for any $x, y \in X$, then f has unique fixed point $y \in X$ and for each $x \in X$ sequence of Picard iterates defined by f at x converge to y .

Proof. Let $x_0 \in X$ be arbitrary. Define $\mathcal{O}_n(x_0) = \{f^k(x_0) | k = 0, 1, 2, \dots, n\}$,
 $\mathcal{O}(x_0) = \{f^k(x_0) | k = 0, 1, 2, \dots\}$,
 $d(x_0, f(x_0)) = d_0$.

Let r_0 be arbitrary real number such that:

$$r_0 - \varphi(r_0) \leq d_0 \quad \text{and} \quad r - \varphi(r) > d_0 \quad \text{for} \quad r > r_0. \quad (1)$$

Such r_0 exists because from $\lim_{r \rightarrow \infty} r - \varphi(r) = \infty$ follows that $\{r | r - \varphi(r) > d_0\}$ is nonempty open and lower bounded. So r_0 can be its greatest lower bound.

Now we prove that for each $n \geq 1$ there exists k , $0 < k \leq n$, such that $\delta(\mathcal{O}_n(x_0)) = d(x_0, f^k(x_0))$. Let $1 \leq i, j \leq n$. Then:

$$\begin{aligned} d(f^i(x_0), f^j(x_0)) &\leq \max\{\varphi(d(f^{i-1}(x_0), f^{j-1}(x_0))), \varphi(d(f^{i-1}(x_0), f^i(x_0))), \\ &\quad \varphi(d(f^{j-1}(x_0), f^j(x_0))), \varphi(d(f^{i-1}(x_0), f^j(x_0))), \varphi(d(f^{j-1}(x_0), f^i(x_0)))\} \\ &\leq \varphi(\delta(\mathcal{O}_n(x_0))), \end{aligned}$$

because φ is monotone nondecreasing. So for any $1 \leq i, j \leq n$ we have

$$d(f^i(x_0), f^j(x_0)) \leq \varphi(\delta(\mathcal{O}_n(x_0))).$$

This implies that $i = 0$ or $j = 0$ because

$$\delta(\mathcal{O}_n(x_0)) = d(f^i(x_0), f^j(x_0)) \quad \text{for some} \quad 0 \leq i, j \leq n.$$

So we obtain that for each $n \geq 1$:

$$\delta(\mathcal{O}_n) = d(x_0, f^k(x_0)) \leq d(x_0, f(x_0)) + d(f(x_0), f^k(x_0)) \leq d_0 + \varphi(\delta(\mathcal{O}_n)).$$

Hence we get

$$\delta(\mathcal{O}_n) - \varphi(\delta(\mathcal{O}_n)) \leq d_0$$

which implies $\delta(\mathcal{O}_n) \leq r_0$. Hence all sequences of Picard iterates defined by f are bounded.

So $\delta(\mathcal{O}(f(x_0))) \leq \varphi(\delta(\mathcal{O}(x_0)))$ which implies

$$\delta(\mathcal{O}(f^n(x_0))) \leq \varphi^n(\delta(\mathcal{O}(x_0))).$$

Hence $\mathcal{O}(x_0) = \{f^k(x_0) | k = 0, 1, 2, \dots\}$ is Cauchy sequence. Let $y \in X$ be its limit. Now:

$$\begin{aligned} d(f^n(y), f(y)) &\leq \max\{\varphi(d(f^{n-1}(y), y)), \varphi(d(f^{n-1}(y), f^n(y))), \\ &\quad \varphi(d(y, f(y))), \varphi(d(f^n(y), y)), \varphi(d(f^{n-1}(y), f(y)))\}. \end{aligned}$$

When $n \rightarrow \infty$ we get:

$$d(y, f(y)) \leq \varphi(d(y, f(y))),$$

which implies $y = f(y)$.

Now we show that this fixed point is unique. If there exists $y_1, y_2 \in X$ such that: $y_1 = f(y_1)$ and $y_2 = f(y_2)$, then

$$d(y_1, y_2) \leq \max\{\varphi(d(y_1, y_2)), \varphi(d(y_1, y_1)), \\ \varphi(d(y_2, y_2)), \varphi(f(y_1), y_2), \varphi(y_1), f(y_2))\} = \varphi(d(y_1, y_2)).$$

So $d(y_1, y_2) = 0$, which implies $y_1 = y_2$. □

If in our Theorem φ is right upper semi continuous then Theorem 1 reduces to the theorem of A. A. Ivanov [4] which generalize the famous theorem of Ćirić [2]. We can see that hypotheses for the gauge function φ in Theorem 1 are stronger of assumptions of Boyd - Wong's [1] fixed point theorem.

Now we prove that for Theorem 1 assumptions

$$\lim_{t \rightarrow \infty} (t - \varphi(t)) = \infty$$

and φ is monotone nondecreasing, were needed.

Example 1. - Ćirić [3]. Let $X = (0, +\infty)$ with usual metric, $\varphi : [0, +\infty) \rightarrow [0, +\infty)$ mapping defined by:

$$\varphi(t) = \begin{cases} \frac{t}{2}, t \in [0, 1); \\ t - \frac{1}{2}, t \in [1, +\infty) \end{cases}$$

and $f(x) = x + 1$.

If $x < y$ then:

$$d(f(x), f(y)) = d(x, y) < (d(x, y) + 1) - \frac{1}{2} = \varphi(d(x, f(y))).$$

φ is continuous, monotone nondecreasing and $\varphi(t) < t$, but

$$\lim_{t \rightarrow \infty} (t - \varphi(t)) \neq \infty$$

and f is fixed point free.

Example 2. - Ćirić [3]. Let $X = (0, +\infty)$,

$$d(x, y) = \frac{|x - y|}{1 + |x - y|},$$

$\varphi : [0, +\infty) \rightarrow [0, +\infty)$ mapping defined by:

$$\varphi(t) = \begin{cases} \frac{t}{2}, t \in [0, \frac{1}{2}) \cup [1, +\infty); \\ t - (1-t)^2, t \in [\frac{1}{2}, 1), \end{cases}$$

and $f(x) = x + 1$.

Then φ is not monotone nondecreasing and all other conditions of Theorem 1 are satisfied. f is also fixed point free.

Now we need the following Lemma.

Lemma 1. *Let $\varphi \in \Phi_2$. Then there exists $\psi \in \Phi_1$ such that*

$$\varphi(x) \leq \psi(x) < x,$$

for each $x > 0$.

Proof. Let real function $\psi : [0, \infty) \rightarrow [0, \infty)$ be defined by

$$\psi(x) = \sup_{t \in [0, x]} \varphi(t).$$

Then ψ is monotone nondecreasing, $\psi(0) = 0$,

$\overline{\lim}_{t \rightarrow x^+} \psi(t) < x$ and $\varphi(x) \leq \psi(x) < x$ for any $x > 0$.

If φ is bounded then ψ is bounded, which implies that

$$\lim_{x \rightarrow \infty} (x - \psi(x)) = \infty.$$

Suppose that

$$\lim_{x \rightarrow \infty} \psi(x) = \infty.$$

This implies that

$$\overline{\lim}_{x \rightarrow \infty} \varphi(x) = \infty.$$

Then for arbitrary $\varepsilon > 0$ there exists real function $g : [0, \infty) \rightarrow [0, \infty)$ such that:

$$g(x) \in \{t \in [0, x] : \psi(x) - \varphi(t) < \varepsilon\}.$$

Now, for any $R > 0$ there exists $x > 0$ such that

$$x > \varphi(x) > R + \varepsilon,$$

which implies that $g(x) > R$. So

$$\lim_{x \rightarrow \infty} g(x) = \infty.$$

Then

$$\begin{aligned} \lim_{x \rightarrow \infty} (x - \psi(x)) &= \lim_{x \rightarrow \infty} ((x - g(x)) + (g(x) - \varphi(g(x))) + (\varphi(g(x)) - \psi(x))) \\ &\geq \lim_{x \rightarrow \infty} (x - g(x)) + \lim_{x \rightarrow \infty} (g(x) - \varphi(g(x))) + \lim_{x \rightarrow \infty} (\varphi(g(x)) - \psi(x)) = \infty, \end{aligned}$$

because

$$\lim_{x \rightarrow \infty} (g(x) - \varphi(g(x))) = \infty.$$

So

$$\lim_{x \rightarrow \infty} (x - \psi(x)) = \infty$$

which implies that $\psi \in \Phi_1$. □

Now we shall prove our next result.

Theorem 2. *Let (X, d) be a metric space and $f : X \rightarrow X$. If there exists $\varphi \in \Phi_2$ such that*

$$\begin{aligned} d(f(x), f(y)) \leq \max\{\varphi(d(x, y)), \varphi(d(x, f(x))), \varphi(d(y, f(y))), \\ \varphi(d(x, f(y))), \varphi(d(f(x), y))\}, \end{aligned}$$

for any $x, y \in X$, then f has unique fixed point $y \in X$ and for each $x \in X$ sequence of Picard iterates defined by f at x converge to y .

Proof. From Lemma 1 follows that there exists $\psi \in \Phi_1$ such that:

$$\begin{aligned} d(f(x), f(y)) \leq \max\{\psi(d(x, y)), \psi(d(x, f(x))), \psi(d(y, f(y))), \\ \psi(d(x, f(y))), \psi(d(f(x), y))\}. \end{aligned}$$

Therefore, the hypotheses of Theorem 1 are satisfied. □

Lemma 2. *Let $\varphi_1, \dots, \varphi_n \in \Phi_1$. Then there exists $\psi \in \Phi_1$ such that*

$$\varphi_k(x) \leq \psi(x) < x,$$

for each $1 \leq k \leq n$ and $x > 0$.

Proof. Let

$$\psi(x) = \max\{\varphi_1, \dots, \varphi_n\}.$$

Then it is easy to show that: $\psi(0) = 0$, ψ is monotone nondecreasing, $\varphi_k(t) \leq \psi(t) < t$ ($1 \leq k \leq n$) for all $t > 0$ and

$$\overline{\lim}_{t \rightarrow x^+} \psi(t) < x.$$

Now we shall prove that $\lim_{x \rightarrow \infty} (x - \psi(x)) = \infty$.

Let $R > 0$ be arbitrary. Then for any $1 \leq k \leq n$ there exists $S_k > 0$ such that $t - \varphi_k(t) > R$ for all $t > S_k$. Then for all

$$t > \max_{1 \leq k \leq n} S_k$$

we have

$$t - \psi(t) = t - \max_{1 \leq k \leq n} \varphi_k(t) = \min_{1 \leq k \leq n} (t - \varphi_k(t)) > R.$$

So $\lim_{x \rightarrow \infty} (x - \psi(x)) = \infty$. □

Now we shall prove our main result, which is a common generalization of Theorem 1 and Theorem 2.

Theorem 3. *Let (X, d) be a metric space and $f : X \rightarrow X$. If there exists $\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5 \in \Phi_1 \cup \Phi_2$ such that*

$$d(f(x), f(y)) \leq \max\{\varphi_1(d(x, y)), \varphi_2(d(x, f(x))), \varphi_3(d(y, f(y))), \\ \varphi_4(d(x, f(y))), \varphi_5(d(f(x), y))\},$$

for any $x, y \in X$, then f has unique fixed point $y \in X$ and for each $x \in X$ sequence of Picard iterates defined by f at x converge to y .

Proof. Let

$$\varphi_k^*(t) = \begin{cases} \varphi_k(t), \varphi_k \in \Phi_1 \setminus \Phi_2; \\ \sup_{s \in [0, t]} \varphi_k(s), \varphi_k \in \Phi_2 \end{cases} \quad (1 \leq k \leq 5).$$

From Lemma 1 it follows that $\varphi_k^* \in \Phi_1$ ($1 \leq k \leq 5$).

From Lemma 2 it follows that there exists real function $\varphi \in \Phi_1$ such that:

$$\varphi_k^*(x) \leq \varphi(x) < x, (1 \leq k \leq n) \text{ for each } x > 0, \text{ which implies}$$

$$d(f(x), f(y)) \leq \max\{\varphi(d(x, y)), \varphi(d(x, f(x))), \varphi(d(y, f(y))), \\ \varphi(d(x, f(y))), \varphi(d(f(x), y))\}.$$

Therefore, the hypotheses of Theorem 1 are satisfied. \square

REFERENCES

- [1] D. W. Boyd, J. S. W. Wong, *On nonlinear contractions*, Proc. Amer. Math. Soc., **20** (1969), 458-464.
- [2] Lj. B. Ćirić, *A generalization of Banach's contraction principle*, Proc. Amer. Math. Soc., **45** (1974), 267-273.
- [3] Lj. B. Ćirić, *On Ivanov open problem related to the nonlinear quasi-contraction mappings*, (in Russian), Publ. Inst. Math. (Beograd), **41(55)** (1987), 149-150.
- [4] A. A. Ivanov, *Fixed points of mappings on metric spaces*, (in Russian), Studies in topology II - Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov (LOMI), **66** (1976), 5-102.

Received: January 31, 2008; Accepted: April 21, 2008.