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FIXED POINT THEOREMS ON CARTESIAN PRODUCT

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Abstract. In this paper we study the existence of the fixed point for operators on cartesian product $f : X \times Y \to X \times Y$, $f = (f_1, f_2)$, in terms of the operators $f_1(\cdot, y) : X \to X$ and $f_2(x, \cdot) : Y \to Y$.
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1. Introduction

In this article we study the existence of the fixed points for operators defined on cartesian product of structured sets by the following form:

$$f : X \times Y \to X \times Y$$
$$f(x, y) = (f_1(x, y), f_2(x, y))$$

The problem studied is:

Problem 1.1. *If $f : X \times Y \to X \times Y$ satisfies the following conditions:*

(H1) $f_1(\cdot, y) : X \to X$ *has a fixed point for all* $y \in Y$;

(H2) $f_2(x, \cdot) : Y \to Y$ *has a fixed point for all* $x \in X$.

In which conditions $f : X \times Y \to X \times Y$ has a fixed point.

Let $f : X \times Y \to X \times Y$, $f = (f_1, f_2)$ satisfies conditions $(H1)$, $(H2)$.

We define the following multivalued mappings:

$$P : Y \multimap X, \quad P(y) = \{x \in X : \; x = f_1(x, y)\} \tag{1}$$

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$$Q : X \multimap Y, \quad Q(x) = \{y \in Y : \ y = f_2(x, y)\} \tag{2}$$

$$H : Y \multimap Y, \quad H(y) = \{ \, f_2(x, y) : \ x \in P(y) \, \} \tag{3}$$

We have the following general principles for the existence of the fixed point for operator $f = (f_1, f_2)$.

Theorem 1.1. *(M.A. Şerban [29], [30]) Suppose that $f : X \times Y \to X \times Y$, $f = (f_1, f_2)$ satisfies conditions $(H1)$, $(H2)$. If the mapping $P \circ Q : X \multimap X$ has at least a fixed point or the mapping $Q \circ P : Y \multimap Y$ has at least a fixed point then the mapping f has at least a fixed point.*

Proof. Let $x^* \in F_{P \circ Q}$ which means that $x^* \in P \circ Q\,(x^*) = \bigcup\limits_{y \in Q(x^*)} P\,(y)$. Therefore there exists $y^* \in Q\,(x^*)$ such that $x^* \in P\,(y^*)$.

$$x^* \in P\,(y^*) \Longrightarrow x^* = f_1(x^*, y^*)$$

$$y^* \in Q\,(x^*) \Longrightarrow y^* = f_2(x^*, y^*)$$

so $(x^*, y^*) \in F_f$.

Similarly we can prove the existence of the fixed point in the case of $F_{Q \circ P} \neq \emptyset$. □

Theorem 1.2. *(I.A. Rus [15]) Suppose that $f : X \times Y \to X \times Y$, $f = (f_1, f_2)$ satisfies condition $(H1)$. If the mapping H has at least a fixed point then the mapping f has at least a fixed point.*

Proof. Let $y^* \in F_H$ therefore $y^* \in H\,(y^*)$, so there exists $x^* \in P\,(y^*)$ such that

$$y^* = f_2(x^*, y^*)$$

$$x^* \in P\,(y^*) \Longrightarrow x^* = f_1(x^*, y^*)$$

which implies that $(x^*, y^*) \in F_f$. □

Remark 1.1. *If instead of conditions $(H1)$ and $(H2)$ we use the following conditions*

(H1') $f_1\,(\cdot, y) : X \to X$ *has a unique fixed point for all* $y \in Y$*;*
(H2') $f_2\,(x, \cdot) : Y \to Y$ *has a unique fixed point for all* $x \in X$*;*

the mappings P, Q, H become singlevalued

$$P : Y \to X, \quad P(y) = x^* (y) \, , \; F_{f_1(\cdot,y)} = \{x^* (y)\} \tag{4}$$

$$Q : X \to Y, \quad Q(x) = y^* (x) \, , \; F_{f_2(x,\cdot)} = \{y^* (x)\} \tag{5}$$

$$H : Y \to Y, \quad H(y) = f_2(P (y) \, , y) \tag{6}$$

and we can formulate the following results:

(i) *If in Theorem 1.1 we suppose that the mapping $P \circ Q : X \to X$ has a unique fixed point or the mapping $Q \circ P : Y \to Y$ has a unique fixed point then the mapping f has a unique fixed point.*

(ii) *If in Theorem 1.2 we suppose that the mapping H has a unique fixed point then the mapping f has a unique fixed point.*

2. Operators on cartesian product of ordered sets

In this section we consider the case of ordered sets and we give some applications of the Theorem 1.1 and Theorem 1.2.

Theorem 2.1. *(M.A. Şerban* [29]*) Let $(X, \leq_1)$, $(Y, \leq_2)$ be two complete lattices and $f : X \times Y \to X \times Y$, $f = (f_1, f_2)$, such that:*

(i) *the mapping $f_1(\cdot, y)$ is monotone increasing for any $y \in Y$;*

(ii) *the mapping $f_2(x, \cdot)$ is monotone increasing for any $x \in X$;*

(iii) *for every $x_1, x_2 \in X$ such that $x_1 \leq_1 x_2$ and $y_1 = f_2(x_1, y_1)$, $y_2 = f_2(x_2, y_2)$ we have $y_1 \leq_2 y_2$;*

(iv) *for every $y_1, y_2 \in Y$ such that $y_1 \leq_2 y_2$ and $x_1 = f_1(x_1, y_1)$, $x_2 = f_1(x_2, y_2)$ we have $x_1 \leq_1 x_2$.*

In these conditions f has at least a fixed point.

Proof. The conditions (i) and (ii) show us that $f_1(\cdot, y)$ and $f_2(x, \cdot)$ satisfy the Knaster-Tarski Fixed Point Theorem for any $y \in Y$, respectively for any $x \in X$.

The conditions (iii) and (iv) can be write in the terms of mappings P and Q as follow:

(iii) for every $x_1, x_2 \in X$ such that $x_1 \leq_1 x_2$ and $y_1 \in Q(x_1)$, $y_2 \in Q(x_2)$ we have $y_1 \leq_2 y_2$;

(iv) for every $y_1, y_2 \in Y$ such that $y_1 \leq_2 y_2$ and $x_1 \in P(y_1)$, $x_2 \in P(y_2)$ we have $x_1 \leq_1 x_2$.

Let $y_1 \leq_2 y_2$, $x_1 \in P(y_1)$, $x_2 \in P(y_2)$ then $x_1 \leq_1 x_2$. For $x_1 \leq_1 x_2$, $y_1' \in Q(x_1)$, $y_2' \in Q(x_2)$ then $y_1' \leq_2 y_2'$. So, any selection $g : Y \to Y$ of multivalued mapping $Q \circ P$ is monotone increasing and thus g is in the conditions of Knaster-Tarski Fixed Point Theorem, therefore $F_{Q\circ P} \neq \emptyset$. Applying Theorem 1.1 we obtain the conclusion. □

Theorem 2.2. *(M.A. Şerban [29]) Let $(X, \leq_1)$, $(Y, \leq_2)$ be two right inductively ordered sets and $f : X \times Y \to X \times Y$, $f = (f_1, f_2)$, such that:*

(i) *for fixed $y \in Y$ we have $x \leq_1 f_1(x, y)$, $\forall x \in X$;*

(ii) *for fixed $x \in X$ we have $y \leq_2 f_2(x, y)$, $\forall y \in Y$;*

(iii) *for every $x \in X$ and $y \in F_{f_2(x,\cdot)}$ there exist $x' \in F_{f_1(\cdot,y)}$ such that $x \leq_1 x'$;*

or the condition holds:

(iii') *for every $y \in Y$ and $x \in F_{f_1(\cdot,y)}$ there exist $y' \in F_{f_2(x,\cdot)}$ such that $y \leq_2 y'$;*

In these conditions f has at least a fixed point.

Proof. From Bourbaki-Birkhoff Fixed Point Theorem, conditions (i) and (ii) imply the conditions $(H1)$ and $(H2)$ of Theorem 1.1. Condition (iii) can be formulate as:

(iii) for every $x \in X$ and $y \in Q(x)$ there exist $x' \in P(y)$ such that $x \leq_1 x'$,

this means that there is a selection h of multivalued mapping $P \circ Q$ such that:

$$h : X \to X, \quad x \longmapsto x'$$

Using condition (iii) we deduce that h satisfies: $x \leq_1 h(x)$, $\forall x \in X$, which means that h satisfies the Bourbaki-Birkhoff Fixed Point Theorem, therefore $F_{P\circ Q} \neq \emptyset$.

If we are using condition (iii)' instead of (iii) we deduce the existence of selection $g : Y \to Y$ of multivalued mapping $Q \circ P$ such that $y \leq_2 g(y)$, $\forall y \in Y$, thus $F_{Q\circ P} \neq \emptyset$. □

From the Theorem 1.2 point of view we get the following results:

Theorem 2.3. *(I.A. Rus [15]). Let $(X, \leq_1)$, $(Y, \leq_2)$ be two complete lattices and $f : X \times Y \to X \times Y$, $f = (f_1, f_2)$, such that:*

(i) *the mapping $f_1(\cdot, y) : X \to X$ is monotone increasing for any $y \in Y$;*

(ii) *the mapping $f_2(x, \cdot) : Y \to Y$ is monotone increasing for any $x \in X$;*

(iii) *the mapping* $f_2(\cdot,y): X \to Y$ *is monotone increasing for any* $x \in X$*;*
(iv) *for every* $y_1, y_2 \in Y$ *such that* $y_1 \leq_2 y_2$ *and* $x_1 = f_1(x_1,y_1)$*,* $x_2 = f_1(x_2,y_2)$ *we have* $x_1 \leq_1 x_2$*.*

In these conditions f has at least a fixed point.

Proof. We show that multivalued mapping H, defined by (3), has a fixed point. For $y_1 \leq_2 y_2$ and $x_1 \in P(y_1)$, $x_2 \in P(y_2)$ we have $x_1 \leq_1 x_2$, therefore:

$$f_2(x_1,y_1) \leq_2 f_2(x_2,y_1) \leq_2 f_2(x_2,y_2).$$

Thus, any selection $s: Y \to Y$ of multivalued mapping H is monotone increasing, so s is in the conditions of Knaster-Tarski Fixed Point Theorem, which implies that $F_H \neq \emptyset$. □

Theorem 2.4. *(M.A. Şerban* [29]*) Let* $(X,\leq_1)$*,* $(Y,\leq_2)$ *be two right inductively ordered sets and* $f: X\times Y \to X \times Y$*,* $f=(f_1,f_2)$*, such that:*

(i) *for fixed* $y \in Y$ *we have* $x \leq_1 f_1(x,y)$*,* $\forall x \in X$*;*
(ii) *for every* $y \in Y$ *and* $x \in F_{f_1(\cdot,y)}$ *there exist* $y' \in F_{f_2(x,\cdot)}$ *such that* $y \leq_2 y'$*.*

In these conditions f has at least a fixed point.

Proof. Condition (ii) ensure the existence of selection $s: Y \to Y$ of multivalued mapping H, defined by (3), such that

$$y \leq_2 s(y), \ \forall y \in Y$$

which implies that $F_H \neq \emptyset$. □

3. Operators on cartesian product of metric spaces

In this section we present some applications of the Theorem 1.1 and Theorem 1.2 in the case of cartesian product of metric spaces.

3.1. **Equivalent conditions.** Let (X,d) and (Y,ρ) two complete metric spaces. We have:

Theorem 3.1.1. *(M.A. Şerban* [29], [30]*)* $f: X\times Y \to X\times Y$, $f=(f_1,f_2)$*, such that:*

(i) $f_1(\cdot,y): X \to X$ *is* a_1*-contraction* $\forall y \in Y$;
(ii) $f_2(x,\cdot): Y \to Y$ *is* a_2*-contraction* $\forall x \in X$;

(iii) $f_1(x,\cdot) : Y \to X$ *is* L_1*-lipschitz* $\forall x \in X$;
(iv) $f_2(\cdot,y) : X \to Y$ *is* L_2*-lipschitz* $\forall y \in Y$;
(v) $\dfrac{L_1 L_2}{(1-a_1)(1-a_2)} < 1$.

Then f has a unique fixed point.

Proof. Since (X,d) and (Y,ρ) are two complete metric spaces and from (i) and (ii) we have that $f_1(\cdot,y) : X \to X$ satisfies condition $(H1')$ and $f_2(x,\cdot) : Y \to Y$ satisfies condition $(H2')$ therefore P and Q are singlevalued operators. Using (i) and (iii) we get that operator P is lipschitz:

$$d\left(P\left(y_1\right),P\left(y_2\right)\right) = d\left(f_1\left(P\left(y_1\right),y_1\right),f_1\left(P\left(y_2\right),y_2\right)\right) \leq$$
$$\leq d\left(f_1\left(P\left(y_1\right),y_1\right),f_1\left(P\left(y_2\right),y_1\right)\right) + d\left(f_1\left(P\left(y_2\right),y_1\right),f_1\left(P\left(y_2\right),y_2\right)\right) \leq$$
$$\leq a_1 \cdot d\left(P\left(y_1\right),P\left(y_2\right)\right) + L_1 \cdot \rho\left(y_1,y_2\right)$$

thus

$$d(P(y_1),P(y_2)) \leq \frac{L_1}{1-a_1}\rho(y_1,y_2), \qquad \forall y_1,y_2 \in Y.$$

Analogue, using (ii) and (iii), we obtain that operator Q is lipschitz:

$$\rho(Q(x_1),Q(x_2)) \leq \frac{L_2}{1-a_2} d\left(x_1,x_2\right) \qquad \forall x_1,x_2 \in X.$$

The conclusion is obtained from Theorem 1.1 and Remark 1.1 since the operator $P \circ Q : X \to X$ is contraction. □

Theorem 3.1.2. *(I.A. Rus* [16]*)* $f : X \times Y \to X \times Y$, $f = (f_1, f_2)$ *satisfies conditions (i) - (iv) from Theorem 3.1.1 and*

(v') $a_2 + \dfrac{L_1 L_2}{(1-a_1)} < 1$.

Then f has a unique fixed point.

Proof. We consider the operator $H : Y \to Y$ defined by (6) which is a contraction because of condition (v'):

$$\rho(H(y_1),H(y_2)) = \rho(f_2\left(P(y_1),y_1\right),f_2\left(P\left(y_2\right),y_2\right)) \leq$$
$$\leq \rho(f_2\left(P(y_1),y_1\right),f_2\left(P\left(y_1\right),y_2\right)) + \rho(f_2\left(P(y_1),y_2\right),f_2\left(P\left(y_2\right),y_2\right)) \leq$$
$$\leq a_2 \cdot \rho(y_1,y_2) + L_2 \cdot d(P(y_1),P(y_2)) \leq \left(a_2 + \frac{L_1 L_2}{1-a_1}\right) \cdot \rho(y_1,y_2)$$

Applying the Theorem 1.2 and Remark 1.1 we get the conclusion. □

Remark 3.1.1. $\dfrac{L_1 L_2}{(1-a_1)(1-a_2)} < 1 \Longleftrightarrow a_2 + \dfrac{L_1 L_2}{(1-a_1)} < 1.$

Theorem 3.1.3. *(I.A. Rus [16]) If $f : X \times Y \to X \times Y$, $f = (f_1, f_2)$ satisfies:*

(i) *There exist $a_1 \in [0; 1[$ and $L_1 > 0$ such that:*

$$d(f_1(x_1, y_1), f_1(x_2, y_2)) \leq a_1 d(x_1, x_2) + L_1 \rho(y_1, y_2) \tag{7}$$

for all (x_1, y_1), $(x_2, y_2) \in X \times Y$;

(ii) *There exist $a_2 \in [0; 1[$ and $L_2 > 0$ such that:*

$$\rho(f_2(x_1, y_1), f_2(x_2, y_2)) \leq L_2 d(x_1, x_2) + a_2 \rho(y_1, y_2) \tag{8}$$

for all (x_1, y_1), $(x_2, y_2) \in X \times Y$;

(iii) *Condition (v) from Theorem 3.1.1 or condition (v') from Theorem 3.1.2 hold.*

Then f has a unique fixed point.

Proof. The proof of this theorem is similar with the proof of Theorem 3.1.1 because conditins (i)-(ii) are equivalent with conditions (i)-(iv) from Theorem 3.1.1. □

Theorem 3.1.4. *(Perov Theorem) If $f : X \times Y \to X \times Y$, $f = (f_1, f_2)$ such that:*

(i) *there exist $a_1, a_2, L_1, L_2 \in \mathbb{R}_+$ such that (7) and (8) hold;*

(ii) *the matrix*

$$A = \begin{pmatrix} a_1 & L_1 \\ L_2 & a_2 \end{pmatrix}$$

has the property that $A^n \to 0$.

Then f has a unique fixed point.

The Perov Theorem is obtained using the vectorial metric $\delta : (X \times Y)^2 \to \mathbb{R}_+^2$:

$$\delta\left((x_1, y_1), (x_2, y_2)\right) = \begin{pmatrix} d\left(x_1, x_2\right) \\ \rho\left(y_1, y_2\right) \end{pmatrix}$$

and conditions (i) can be written in the following form:

$$\delta\left(f\left(x_1, y_1\right), f\left(x_2, y_2\right)\right) \leq A \cdot \delta\left(\left(x_1, y_1\right), \left(x_2, y_2\right)\right).$$

Remark 3.1.2. *The conditions of the Perov Theorem are equivalent with the conditions of Theorem 3.1.1 and Theorem 3.1.2 because:*

$A^n \to 0 \Longleftrightarrow$ *the matrix* A *has eigenvalues with*

$$|\lambda| < 1 \Longleftrightarrow \frac{L_1 L_2}{(1-a_1)(1-a_2)} < 1.$$

Theorem 3.1.5. *(St. Czerwik* [9], *J. Matkowski* [12]*) If* $f : X \times Y \to X \times Y$, $f = (f_1, f_2)$ *such that:*

(i) *there exist* $a_1, a_2, L_1, L_2 \in \mathbb{R}_+$ *such that (7) and (8) hold;*

(ii) *there exist* $r_1, r_2 \in \mathbb{R}_+^*$ *such that* $\begin{cases} a_1 r_1 + L_1 r_2 & < r_1 \\ L_2 r_1 + a_2 r_2 & < r_2 \end{cases}$.

Then f *has a unique fixed point.*

Proof. We denote by

$$L_{CM} = \max\left\{\frac{a_1 r_1 + L_1 r_2}{r_1}, \frac{L_2 r_1 + a_2 r_2}{r_2}\right\}$$

and $Z = X \times Y$. Now we consider the metric $\sigma_{CM} : Z \times Z \to \mathbb{R}_+$

$$\sigma_{CM}\left((x_1, y_1), (x_2, y_2)\right) = r_1 d\left(x_1, x_2\right) + r_2 \rho\left(y_1, y_2\right)$$

It is easy to check that $f : Z \to Z$, $f = (f_1, f_2)$ is $L_{CM}-$ contraction with respect to σ_{CM}. □

Remark 3.1.3. *Condition (ii) from Theorem 3.1.5* $\Longleftrightarrow$ *the matrix* A *has eigenvalues with* $|\lambda| < 1$.

3.2. **Remarks on contraction condition for operators on $X \times Y$.** Let (X, d) and (Y, ρ) two metric spaces. For the set $X \times Y$ we can define the following metrics:

$$\sigma_C : (X \times Y) \times (X \times Y) \to \mathbb{R}_+$$
$$\sigma_C\left((x_1, y_1), (x_2, y_2)\right) = \max\left\{d\left(x_1, x_2\right), \rho\left(y_1, y_2\right)\right\},$$

$$\sigma_M : (X \times Y) \times (X \times Y) \to \mathbb{R}_+$$
$$\sigma_M\left((x_1, y_1), (x_2, y_2)\right) = d\left(x_1, x_2\right) + \rho\left(y_1, y_2\right),$$

$$\sigma_E : (X \times Y) \times (X \times Y) \to \mathbb{R}_+$$
$$\sigma_E\left((x_1, y_1), (x_2, y_2)\right) = \sqrt{\left(d\left(x_1, x_2\right)\right)^2 + \left(\rho\left(y_1, y_2\right)\right)^2},$$

and generalized metric used in Perov Theorem:

$$\delta : (X \times Y) \times (X \times Y) \to \mathbb{R}_+^2 :$$
$$\delta\left((x_1, y_1), (x_2, y_2)\right) = \begin{pmatrix} d\left(x_1, x_2\right) \\ \rho\left(y_1, y_2\right) \end{pmatrix}.$$

Lemma 3.2.1. *Let* $a_1, a_2, L_1, L_2 \in \mathbb{R}_+$ *and the matrix*

$$A = \begin{pmatrix} a_1 & L_1 \\ L_2 & a_2 \end{pmatrix}.$$

The following statements are equivalent:

(i) *A is convergent to zero matrix;*

(ii) $I - A$ *is non-singular and*

$$(I - A)^{-1} = I + A + A^2 + ...$$

(iii) *the matrix* A *has eigenvalues with* $|\lambda| < 1$*;*

(iv) $I - A$ *is non-singular and* $(I - A)^{-1}$ *has nonnegative elements;*

(v) $\dfrac{L_1 L_2}{(1 - a_1)(1 - a_2)} < 1$*;*

Proof. The equivalence of (i), (ii), (iii), (iv) is well-known (see R. Precup [13], [14], I.A. Rus [24]).

$(iii) \Longleftrightarrow (v)$ The eigenvalues of matrix A are solutions of the equation

$$\lambda^2 - (a_1 + a_2) \cdot \lambda + a_1 a_2 - L_1 L_2 = 0,$$

so

$$\lambda_{1,2} = \frac{a_1 + a_2 \pm \sqrt{(a_1 - a_2)^2 + 4 \cdot L_1 L_2}}{2}.$$

We have

$$0 \leq |\lambda_{1,2}| \leq \frac{a_1 + a_2 + \sqrt{(a_1 - a_2)^2 + 4 \cdot L_1 L_2}}{2}$$

and

$$\frac{a_1 + a_2 + \sqrt{(a_1 - a_2)^2 + 4 \cdot L_1 L_2}}{2} < 1 \Longleftrightarrow L_1 L_2 < (1 - a_1)(1 - a_2).$$

□

Theorem 3.2.1. *Let* (X, d) *and* (Y, ρ) *two complete metric spaces and* $f : X \times Y \to X \times Y$, $f = (f_1, f_2)$ *such that there exist* $a_1, a_2, L_1, L_2 \in \mathbb{R}_+$ *such that (7) and (8) hold. Then:*

(i) f *is lipschitz with respect to* σ_C *with the lipschitz constant* $L_{\sigma_C} = \max\{a_1 + L_1, a_2 + L_2\}$*. If* $L_{\sigma_C} < 1$ *then the matrix* A *is convergent to zero matrix;*

(ii) *f is lipschitz with respect to σ_M with the lipschitz constant $L_{\sigma_M} = \max\{a_1 + L_2, a_2 + L_1\}$. If $L_{\sigma_M} < 1$ then the matrix A is convergent to zero matrix;*

(iii) *f is lipschitz with respect to σ_E with the lipschitz constant $L_{\sigma_E} = \sqrt{a_1^2 + a_2^2 + L_1^2 + L_2^2}$. If $L_{\sigma_E} < 1$ then the matrix A is convergent to zero matrix;*

Proof. (i) We have

$$\sigma_C\left(f\left(x_1, y_1\right), f\left(x_2, y_2\right)\right) \leq$$
$$\leq \max\left\{a_1 d(x_1, x_2) + L_1 \rho(y_1, y_2), L_2 d(x_1, x_2) + a_2 \rho(y_1, y_2)\right\} \leq$$
$$\leq \max\left\{a_1 + L_1, a_2 + L_2\right\} \cdot \sigma_C\left(\left(x_1, y_1\right), \left(x_2, y_2\right)\right)$$

If $L_{\sigma_C} < 1$ then

$$a_1 + L_1 < 1 \Longleftrightarrow L_1 < 1 - a_1$$

and

$$a_2 + L_2 < 1 \Longleftrightarrow L_2 < 1 - a_2$$

therefore

$$L_1 L_2 < \left(1 - a_1\right)\left(1 - a_2\right)$$

so from Lemma 3.2.1 we have that A is convergent to zero matrix.

(ii) In this case we have

$$\sigma_M\left(f\left(x_1, y_1\right), f\left(x_2, y_2\right)\right) \leq$$
$$\leq \left(a_1 + L_2\right) \cdot d(x_1, x_2) + \left(L_1 + a_2\right) \cdot \rho(y_1, y_2) \leq$$
$$\leq \max\left\{a_1 + L_2, a_2 + L_1\right\} \cdot \sigma_M\left(\left(x_1, y_1\right), \left(x_2, y_2\right)\right)$$

If $L_{\sigma_M} < 1$ then

$$a_1 + L_2 < 1 \Longleftrightarrow L_2 < 1 - a_1$$

and

$$a_2 + L_1 < 1 \Longleftrightarrow L_1 < 1 - a_2$$

therefore

$$L_1 L_2 < \left(1 - a_1\right)\left(1 - a_2\right)$$

so, again, from Lemma 3.2.1 we have that A is convergent to zero matrix.

(iii) From (7), (8) and Cauchy inequality we get:

$$\sigma_E\left(f\left(x_1, y_1\right), f\left(x_2, y_2\right)\right) \leq$$
$$\leq \sqrt{\left(a_1 d(x_1, x_2) + L_1 \rho(y_1, y_2)\right)^2 + \left(L_2 d(x_1, x_2) + a_2 \rho(y_1, y_2)\right)^2} \leq$$

$$\leq \sqrt{\left(a_1^2 + L_1^2\right)\left(d(x_1,x_2)^2 + \rho(y_1,y_2)^2\right) + \left(L_2^2 + a_2^2\right)\left(d(x_1,x_2)^2 + \rho(y_1,y_2)^2\right)} \leq$$

$$\leq \sqrt{\left(a_1^2 + L_1^2 + L_2^2 + a_2^2\right)} \cdot \sigma_E\left(\left(x_1, y_1\right), \left(x_2, y_2\right)\right).$$

If $L_{\sigma_M} < 1$ then a_1, a_2, L_1, $L_2 \in [0;1[$ and

$$L_1 L_2 \leq 2 \cdot L_1 L_2 \leq L_1^2 + L_2^2 < 1 - a_1^2 - a_2^2 \leq$$
$$\leq 1 - a_1 - a_2 \leq 1 - a_1 - a_2 + a_1 a_2 = \left(1 - a_1\right)\left(1 - a_2\right)$$

thus from Lemma 3.2.1 we have that A is convergent to zero matrix. □

Theorem 3.2.1 shows that if $f : X \times Y \to X \times Y$, $f = (f_1, f_2)$ satisfies conditions (7) and (8) for $a_1, a_2, L_1, L_2 \in \mathbb{R}_+$ and it is contraction with respect to σ_C or σ_M or σ_E the f satisfies conditions from Perov Theorem, which means that Perov Theorem is weaker than Banach Theorem used in complete metric space $(X \times Y, \sigma_C)$ or $(X \times Y, \sigma_M)$ or $(X \times Y, \sigma_E)$. If f satisfies Perov Theorem then there exist $r_1, r_2 \in \mathbb{R}_+^*$ such that

$$\begin{cases} a_1 r_1 + L_1 r_2 & < r_1 \\ L_2 r_1 + a_2 r_2 & < r_2 \end{cases}$$

and we can always construct a complete metric on $X \times Y$,

$$\sigma_{CM}\left(\left(x_1, y_1\right), \left(x_2, y_2\right)\right) = r_1 d\left(x_1, x_2\right) + r_2 \rho\left(y_1, y_2\right)$$

such that f becomes L_{CM}-contraction ($L_{CM} = \max\left\{\frac{a_1 r_1 + L_1 r_2}{r_1}, \frac{L_2 r_1 + a_2 r_2}{r_2}\right\}$), due to Czerwik-Matkowski Theorem, Theorem 3.1.5.

3.3. **Generalization.** In this subsection we extend the Theorem 3.1.1 to the case of c-Picard operator. For the convenience of the reader we recall the following definitions:

Definition 3.3.1. *Let (X, d) be a metric space. $A : X \to X$ is called a Picard operator (briefly PO) if:*

(i) $F_A = \{x^*\}$;
(ii) $A^n(x) \to x^*$ *as* $n \to \infty$, *for all* $x \in X$.

Definition 3.3.2. *Let (X, d) be a metric space. A is c-Picard operator (briefly c-PO) if A is PO and there exists $c > 0$ such that*

$$d\left(x, x^*\right) \leq c \cdot d\left(x, A\left(x\right)\right), \quad \forall x \in X.$$

Example 3.3.1. *(S. Reich-I.A. Rus-L. Ćirić, (1971)) Let* (X, d) *be a complete metric space and* $f : X \to X$*. There exist* $\alpha_i \in \mathbb{R}_+$*,* $i = \overline{1,3}$ *with* $\alpha_1 + 2\alpha_2 + 2\alpha_3 < 1$ *such that*

$$\begin{aligned} d(f(x), f(y)) \leq \alpha_1\, d(x, y) + \alpha_2 \cdot [d(x, f(x)) + d(y, f(y))] +\\ +\alpha_3 \cdot [d(x, f(y)) + d(y, f(x))]\,, \end{aligned}$$

then f *is c-PO operator with* $c = \frac{1}{1-a}$ *where* $a = \frac{\alpha_1+\alpha_2+\alpha_3}{1-\alpha_2-\alpha_3}$.

For other examples of PO and c-PO see I.A. Rus [23], [26], M.A. Şerban [30].

Theorem 3.3.1. *Let* (X, d) *be a complete metric space,* (Y, ρ) *be a metric space and* $f : X \times Y \to X \times Y$*,* $f = (f_1, f_2)$*, such that:*

(i) $f_1(\cdot, y) : X \to X$ *is* c_1*-PO* $\forall y \in Y$;
(ii) $f_2(x, \cdot) : Y \to Y$ *is* c_2*-PO* $\forall x \in X$;
(iii) $f_1(x, \cdot) : Y \to X$ *is* L_1*-lipschitz* $\forall x \in X$;
(iv) $f_2(\cdot, y) : X \to Y$ *is* L_2*-lipschitz* $\forall y \in Y$;
(v) $c_1 L_1 c_2 L_2 < 1$.

Then f has a unique fixed point.

Proof. From (i) and (ii) we have:

$$d\,(x, P(y)) \leq c_1 d\,(x, f_1\,(x, y))\,, \quad \forall x \in X$$

$$\rho\,(y, Q\,(x)) \leq c_2 \rho\,(y, f_2\,(x, y))\,, \quad \forall y \in Y$$

therefore if we take $x = P\,(y_1)$ we get

$$\begin{aligned} & d\,(P\,(y_1)\,, P\,(y_2)) \leq c_1 d\,(P\,(y_1)\,, f_1\,(P\,(y_1)\,, y_2)) = \\ = \;& c_1 d\,(f_1\,(P\,(y_1)\,, y_1)\,, f_1\,(P\,(y_1)\,, y_2)) \leq c_1 L_1 \rho(y_1, y_2), \quad \forall y_1, y_2 \in Y \end{aligned}$$

In the same way we have:

$$\rho(Q(x_1), Q(x_2)) \leq c_2 L_2 d\,(x_1, x_2) \qquad \forall x_1, x_2 \in X.$$

□

From the Example 3.3.1 point of view we get the following corollary:

Corollary 3.3.1. *Let* (X, d) *and* (Y, ρ) *two complete metric spaces and* $f : X \times Y \to X \times Y$*,* $f = (f_1, f_2)$*, such that:*

(i) *there exist* $\alpha_i \in \mathbb{R}_+$, $i = \overline{1,3}$ *with* $\alpha_1 + 2\alpha_2 + 2\alpha_3 < 1$ *such that:*

$$d(f_1(x_1, y), f_1(x_2, y)) \leq \alpha_1\, d(x_1, x_2) + \alpha_2 \cdot [d(x_1, f_1(x_1, y)) + d(x_2, f_1(x_2, y))] + \\ + \alpha_3 \cdot [d(x_1, f_1(x_2, y)) + d(x_2, f_1(x_1, y_1))],$$

$\forall x_1, x_2 \in X, y \in Y$;

(ii) *there exist* $\beta_i \in \mathbb{R}_+$, $i = \overline{1,3}$ *with* $\beta_1 + 2\beta_2 + 2\beta_3 < 1$ *such that:*

$$\rho(f_2(x, y_1), f_2(x, y_2)) \leq \beta_1\, \rho(y_1, y_2) + \beta_2 \cdot [\rho(y_1, f_2(x, y_1)) + \rho(y_2, f_2(x, y_2))] + \\ + \beta_3 \cdot [\rho(y_1, f_2(x, y_2)) + \rho(y_2, f_2(x, y_1))] \ ,$$

$\forall x \in X, y_1, y_2 \in Y.$

(iii) $f_1(x, \cdot) : Y \to X$ *is* L_1*-lipschitz* $\forall x \in X$;

(iv) $f_2(\cdot, y) : X \to Y$ *is* L_2*-lipschitz* $\forall y \in Y$;

(v) $\frac{L_1}{1-a_1} \cdot \frac{L_2}{1-a_2} < 1$ *where* $a_1 = \frac{\alpha_1+\alpha_2+\alpha_3}{1-\alpha_2-\alpha_3}$ *and* $a_2 = \frac{\beta_1+\beta_2+\beta_3}{1-\beta_2-\beta_3}$.

Then f *has a unique fixed point.*

Proof. In this case we have the operators $f_1(\cdot, y) : X \to X$ and $f_2(x, \cdot) : Y \to Y$ satisfies the condition from Example 3.3.1 which means that $f_1(\cdot, y) : X \to X$ is c_1-PO for every $y \in Y$ with:

$$c_1 = \frac{1}{1 - a_1}$$

where $a_1 = \frac{\alpha_1+\alpha_2+\alpha_3}{1-\alpha_2-\alpha_3}$ and $f_2(x, \cdot) : Y \to Y$ is c_2-PO for every $x \in X$ with:

$$c_2 = \frac{1}{1 - a_2}$$

where $a_2 = \frac{\beta_1+\beta_2+\beta_3}{1-\beta_2-\beta_3}$. Now we apply the Theorem 3.3.1 and we get the conclusion. □

3.4. **Fibre generalized contractions.**

Definition 3.1. *$A : X \to X$ is said to be a weakly Picard operator (briefly WPO) if the sequence $(A^n(x))_{n\in N}$ converges for all $x \in X$ and the limit (which may depend on x) is a fixed point of A.*

If $A : X \to X$ is a WPO, then we may define the operator $A^\infty : X \to X$ by

$$A^\infty(x) := \lim_{n\to\infty} A^n(x).$$

Obviously $A^\infty(X) = F_A$. Moreover, if A is a PO and we denote by x^* its unique fixed point, then $A^\infty(x) = x^*$, for each $x \in X$.

The following open problem was posed, (see Problem 10.5, in [23]), by I. A. Rus:

Fibre Picard operator problem. *Let* $\left(X, \overset{1}{\rightarrow}\right)$ *and* $\left(Y, \overset{2}{\rightarrow}\right)$ *be two L-spaces. Let* $B : X \to X$ *be a WPO and* $C : X \times Y \to Y$ *be such that* $C(x, \cdot) : Y \to Y$ *is a WPO for every* $x \in X$. *Consider the triangular operator* A *defined as follows:*

$$A : X \times Y \to X \times Y, \ A(x, y) := (B(x), C(x, y))$$

In which conditions A *is a WPO ?*

By $(X, \to)$ we will denote an L-space. Actually, an L-space is any set endowed with a structure implying a notion of convergence for sequences. For example, Hausdorff topological spaces, metric spaces, generalized metric spaces in Perov' sense: $d(x, y) \in \mathbb{R}^m_+$, in Luxemburg-Jung' sense: $d(x, y) \in \mathbb{R}_+ \cup \{+\infty\}$, $d(x, y) \in K$, K a cone in an ordered Banach space, $d(x, y) \in E$, E an ordered linear space with a notion of linear convergence, etc.), 2-metric spaces, D-R-spaces, probabilistic metric spaces, syntopogenous spaces, are examples of L-spaces. For more details see Fréchet [10], Blumenthal [6] and I. A. Rus [23].

For results on fibre WPO's see S. András [2], C. Bacoţiu [4], I.A. Rus [19], [20], [21], M.A. Şerban [28], [30].

In this section we present a result in the case of $(X, \to)$ an L-space and (Y, ρ) a generalized metric space in the Luxemburg-Jung' sense, $\rho(x, y) \in \mathbb{R}_+ \cup \{+\infty\}$. This result generalize a result from M.A. Şerban [27] to the case of $\varphi-$contractions. First we recall the definition of $\varphi-$contraction in the generalized metric space:

Definition 3.4.1. *A function* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *is a strong comparison function if it satisfies the conditions:*

(i_φ) φ *is increasing;*

(ii_φ) $\sum\limits_{n=0}^{\infty} \varphi^n(t) < +\infty$, $\forall t \in \mathbb{R}_+$.

For more informations about comparison functions see I.A. Rus [22] (p. 41-42), V. Berinde [5], M.A. Şerban [30] (p. 33-36) and J. Jachymski and I. Jóźwik [11].

Definition 3.4.2. *Let (Y,ρ) be a generalized metric space, ($\rho(x,y) \in \mathbb{R}_+ \cup \{+\infty\}$), $A : Y \to Y$ an operator and $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a strong comparison function. A is a $\varphi-$contraction if*

$$\rho\left(A\left(y_1\right), A\left(y_2\right)\right) \leq \varphi\left(y_1, y_2\right)$$

for all $y_1, y_2 \in Y$ with $\rho\left(y_1, y_2\right) < +\infty$.

Theorem 3.4.1. *Let $(X, \to)$ be an L-space, (Y,ρ) a complete generalized metric space, $B : X \to X$ and $C : X \times Y \to Y$. We suppose that:*

(i) *B is weakly Picard operator;*
(ii) *$C(x,\cdot) : Y \to Y$ is a $\varphi-$contraction for any $x \in X$, where φ is a subadditive strong comparison function;*
(iii) *C is continuous*
(iv) *$\forall y \in Y,\ \rho(y, C(x,y)) < +\infty,\ \forall x \in X$.*

Then $A = (B, C) : X \times Y \to X \times Y$ is WPO.

Proof. (Y,ρ) is a generalized metric space, thus we have a partition $Y = \bigcup_{i \in I} Y_i$ from the equivalence relation and

$$X \times Y = \bigcup_{i \in I} X \times Y_i.$$

Let $x_0 \in X$, $y_0 \in Y_i$, $i \in I$. We consider the following sequences

$$x_n = B^n\left(x_0\right),$$
$$y_n = C\left(x_{n-1}, y_{n-1}\right),\ n \in \mathbb{N}.$$

We have that

$$\left(x_n, y_n\right) = A^n\left(x_0, y_0\right),\ n \in \mathbb{N}.$$

Since $C\left(B^{\infty}\left(x_0\right), \cdot\right)$ is $\varphi-$contraction and $\rho\left(y_0, C\left(B^{\infty}\left(x_0\right), y_0\right)\right) < +\infty$ (condition (iv)) there exists an unique $y^* \in Y_i \cap F_{C\left(B^{\infty}\left(x_0\right), \cdot\right)}$ and therefore $\left(B^{\infty}\left(x_0\right), y^*\right) \in F_A$. Now we prove that $\left(x_n, y_n\right) \to \left(B^{\infty}\left(x_0\right), y^*\right)$ which will imply that A is WPO. From condition (i) we have that $x_n \to B^{\infty}\left(x_0\right) \in F_B$. It remains to prove that $y_n \to y^*$.

First we show that $y_n \in Y_i$. Using condition (iv) we get

$$\rho\left(y_0, y_1\right) = \rho\left(y_0, C\left(x_0, y_0\right)\right) < +\infty$$

which implies that $y_1 \in Y_i$.

$$\rho\left(y_1, y_2\right) = \rho\left(y_1, C\left(x_1, y_1\right)\right) < +\infty,$$

so $y_2 \in Y_i$ and by induction we obtain that $y_n \in Y_i$, $n \in \mathbb{N}$.

We have

$$\rho\left(y_{n+1}, y^*\right) \leq \rho\left(C\left(x_n, y_n\right), C\left(x_n, y^*\right)\right) + \rho\left(C\left(x_n, y^*\right), C\left(B^\infty\left(x_0\right), y^*\right)\right) \leq$$
$$\leq \varphi\left(\rho\left(y_n, y^*\right)\right) + \rho\left(C\left(x_n, y^*\right), C\left(B^\infty\left(x_0\right), y^*\right)\right) \leq$$
$$\leq \varphi^2\left(\rho\left(y_{n-1}, y^*\right)\right) + \varphi\left(\rho\left(C\left(x_{n-1}, y^*\right), C\left(B^\infty\left(x_0\right), y^*\right)\right)\right) +$$
$$+\rho\left(C\left(x_n, y^*\right), C\left(B^\infty\left(x_0\right), y^*\right)\right) \leq$$
$$\leq ... \leq$$
$$\leq \varphi^{n+1}\left(\rho\left(y_0, y^*\right)\right) + \varphi^n\left(\rho\left(C\left(x_0, y^*\right), C\left(B^\infty\left(x_0\right), y^*\right)\right)\right) + ...+$$
$$+\varphi\left(\rho\left(C\left(x_{n-1}, y^*\right), C\left(B^\infty\left(x_0\right), y^*\right)\right)\right) + \rho\left(C\left(x_n, y^*\right), C\left(B^\infty\left(x_0\right), y^*\right)\right).$$

We take

$$a_n = \rho\left(C\left(x_n, y^*\right), C\left(B^\infty\left(x_0\right), y^*\right)\right)$$

Using conditions (ii) and (iii) we have that $a_n \to 0$. Applying the convergence Lemma 3.1 from M.A. Şerban [28] we obtain that $\sum_{k=0}^{n} \varphi^{n-k}\left(a_k\right) \to 0$, as $n \to +\infty$, which implies that $\rho\left(y_{n+1}, y^*\right) \to 0$, as $n \to +\infty$, and the theorem is proved. □

4. Operators on cartesian product of topological spaces

Definition 4.1. *A topological space (X, τ) has the fixed point property (shortly fpp) if any continuous map $A : X \to X$ has a fixed point.*

It is well known that the Kuratowski problem (1930) stated as follows:

Kuratowski Problem. If spaces X and Y have the fixed point property, does their cartesian product $X \times Y$ have the fixed point property?

has a negative answer even for *Peano continuum* (compact, connected and locally connected metric spaces). The study of behavior of fixed point property under cartesian product was suggested by the Brouwer Fixed Point Theorem which states that I^n has the fpp, where I is the unit interval from $\mathbb{R}$, but in 1967 E. Fadell and W. Lopez presented an example of Peano continuum X with the fpp such that $X \times I$ doesn't have the fpp. For details see R.F. Brown [7], [8].

In this section we consider the case of (X, d) a metric space and (Y, τ) a Hausdorff topological space with the fpp. A general principle for the existence of the fixed point of operator $f = (f_1, f_2)$ in this case can be formulated as follows:

Theorem 4.1. *(I.A. Rus [17]) Let (X, τ_1), (Y, τ_2) be two Hausdorff topological spaces and $f : X \times Y \to X \times Y$, $f = (f_1, f_2)$. Suppose that:*

(i) *$f_1(\cdot, y) : X \to X$ satisfies condition $(H1')$;*
(ii) *the operator $P : Y \to X$ defined by (4) is continuous;*
(iii) *$f_2 : X \times Y \to Y$ is continuous;*
(iv) *the topological space (Y, τ_2) has the fixed point property.*

Then the operator f has a fixed point.

Proof. We consider the operator $H : Y \to Y$ defined by (6). From (ii) and (iii) we have that H is continuous and using the fixed point property of the topological space (Y, τ_2) we get that $F_H \neq \emptyset$. Applying the Theorem 1.2 we obtain that $F_f \neq \emptyset$. □

In order to give some applications of the Theorem 4.1 we present an auxiliary result which gives sufficient conditions for the continuity of the operator $P : Y \to X$ defined by (4).

Lemma 4.1. *Let (X, d) be a metric space, (Y, τ) a Hausdorff topological space and $f : X \times Y \to X$ such that*

(i) *$f(\cdot, y) : X \to X$ is c-PO for every $y \in Y$;*
(ii) *$f(x, \cdot) : Y \to X$ is continuous for every $x \in X$.*

Then the operator $P : Y \to X$ defined by (4) is continuous.

Proof. From condition (i) we have that

$$d(x, x^*(y)) = d(x, P(y)) \leq c \cdot d(x, f(x, y)), \quad \forall x \in X,\ y \in Y. \tag{9}$$

Let $y \in Y$ and $(y_n)_{n \in N} \subset Y$ such that $y_n \to y$. Applying (9) for $x = P(y)$ and $x^*(y_n) = P(y_n)$ we obtain:

$$d(P(y), P(y_n)) \leq c \cdot d(P(y), f(P(y), y_n)).$$

Making $y_n \to y$ and using condition (ii) we have that $f(P(y), y_n) \to f(P(y), y) = P(y)$ therefore $d(P(y), P(y_n)) \to 0$ which shows the continuity of P. □

Theorem 4.2. *Let (X, d) be a metric space and (Y, τ_2) be a Hausdorff topological spaces and $f : X \times Y \to X \times Y$, $f = (f_1, f_2)$. Suppose that:*

(i) $f_1(\cdot, y) : X \to X$ *is c-PO for every* $y \in Y$ *;*
(ii) $f_2 : X \times Y \to Y$ *is continuous;*
(iii) *the topological space* (Y, τ_2) *has the fixed point property.*

Then the operator f *has a fixed point.*

Proof. From (i) we have that $f_1(\cdot, y) : X \to X$ satisfies condition $(H1')$. From (i), (ii) and Lemma 4.1 we get that $P : Y \to X$, defined by (4), is continuous and thus all the conditions of Theorem 4.1 are satisfied, therefore we have the conclusion. □

To get consequences of this result we just combine results which imply that $f_1(\cdot, y) : X \to X$ is c-PO for every $y \in Y$ with results which imply that (Y, τ_2) has the fixed point property. For example we have the following corollary:

Corollary 4.1. *Let* (X, d) *be a complete metric space,* Y *a Hausdorff locally convex space and* $f : X \times Y \to X \times Y$*,* $f = (f_1, f_2)$*. Suppose that:*

(i) $Z \subset Y$ *is a compact convex nonempty set and* $f(X \times Z) \subseteq X \times Z$*;*
(ii) *there exist* $\alpha_i \in \mathbb{R}_+$*,* $i = \overline{1,3}$ *with* $\alpha_1 + 2\alpha_2 + 2\alpha_3 < 1$ *such that:*

$$d(f_1(x_1, y), f_1(x_2, y)) \leq \alpha_1 \, d(x_1, x_2) + \alpha_2 \cdot [d(x_1, f_1(x_1, y)) + d(x_2, f_1(x_2, y))] +$$

$$+\alpha_3 \cdot [d(x_1, f_1(x_2, y)) + d(x_2, f_1(x_1, y_1))],$$

$\forall x_1, x_2 \in X, y \in Z$*;*
(iii) $f_1(x, \cdot) : Z \to X$ *is continuous for every* $x \in X$*;*
(iv) $f_2 : X \times Z \to Z$ *is continuous.*

Then the operator f *has a fixed point.*

Proof. From (ii) we have that $f_1(\cdot, y) : X \to X$ is c-PO for every $y \in Y$ with:

$$c = \frac{1}{1 - a}$$

where $a = \frac{\alpha_1 + \alpha_2 + \alpha_3}{1 - \alpha_2 - \alpha_3}$ (see Example 3.3.1). From conditions (iii) and (iv) we have that $H : Z \to Z$, defined by (6), is continuous and Z has the fixed point property due the Theorem of Tihonov, therefore we get the conclusion. □

If in condition (ii) of Corollary 4.1 we take $\alpha_2 = \alpha_3 = 0$ we obtain a result given by C. Avramescu in [3]. Similar results with Corollary 4.1 can be found also in I.A. Rus [17], M. A. Şerban [29], [30].

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