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MONOTONE ITERATIVE METHODS FOR SYSTEMS OF NONLINEAR EQUATIONS INVOLVING MIXED MONOTONE OPERATORS

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Abstract. In this paper, we study the existence and uniqueness of fixed points for systems of mixed monotone operators in a partially ordered Banach space. As an application of the results obtained, the existence and uniqueness of positive solutions for a class of systems of integral equations are presented.
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1. Introduction

The study of mixed monotone operators was started 20 years ago in 1987 by D. J. Guo and V. Lakshmikantham [2] and during this time it has proven to have not only an important theoretical meaning, but also a wide spread of applications in engineering, the nuclear industry, biological chemistry technology. From the abstract mathematical point of view, the mixed monotone operators have great significance for studying nonlinear functional, differential or integral equations.

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In this paper we make the connection between the theory of mixed monotone operators in two variables as they were introduced in [2] and the study of fixed points for systems of mixed monotone operators in several variables. We will use known results and techniques for mixed monotone operators. The study is made in the case of partially ordered real Banach spaces.

2. Preliminaries

Consider $(E, \|\cdot\|)$ a real Banach space, partially ordered by a cone P, i.e., $x \leq y$ if and only if $y - x \in P$. Recall that a non-empty closed convex set $P \subset E$ is a cone if it satisfies

$$x \in P,\ \lambda \geq 0 \Rightarrow \lambda x \in P$$

and

$$x, -x \in P \Rightarrow x = \theta$$

where θ denotes the zero element of E.

P is said to be:

i. *solid* if its interior $\overset{\circ}{P}$ is non-empty
ii. *normal* if there exists some positive constant c such that $\theta \leq x \leq y$ implies $\|x\| \leq c\,\|y\|$.
iii. *regular* if every nondecreasing and bounded above in order sequence is convergent.

Let $D \subset E$ and $A : D \times D \to E$. A is said to be *mixed monotone* if A is nondecreasing in the first variable and nonincreasing in the last variable, i.e. $x_1 \leq x_2$ $(x_1, x_2 \in D)$ implies $A(x_1, y) \leq A(x_2, y)$ for any $y \in D$ and $y_1 \leq y_2$ $(y_1, y_2 \in D)$ implies $A(x, y_1) \geq A(x, y_2)$ for any $x \in D$. A point (x^*, y^*) is called a *coupled fixed point* of A if $A(x^*, y^*) = x^*$ and $A(y^*, x^*) = y^*$. x^* is called a *fixed point* for A if $A(x^*, x^*) = x^*$. Clearly, x^* is a fixed point of A if and only if (x^*, x^*) is a coupled fixed point of A.

For any $x, y \in E$, the notation $x \gtreqless y$ means that there exists $0 < \lambda \leq \mu$ such that $\lambda x \leq y \leq \mu x$. Clearly, $\gtreqless$ is an equivalence relation and for any $h > \theta$ (i.e. $h \geq \theta$ and $h \neq \theta$) we denote by P_h the equivalence class of h. It is easy to see that $P_h \subset P$ and that $P_h = \overset{\circ}{P}$ if $h \in \overset{\circ}{P}$.

2.1. **Fixed point results for mixed monotone operators.** There are many results regarding the coupled fixed points and the fixed points of mixed monotone operators (see [2-5,7,8] and the references therein). We limit ourselves here to some of the most known fixed point and coupled fixed point theorems which use explicitly the monotone iteration technique (i.e. the fixed point is obtained as the limit of some sequence which is given recursively).

Theorem 2.1. [2] *Let $x_0, y_0 \in E$, $x_0 \leq y_0$ and $A : [x_0, y_0] \times [x_0, y_0] \to E$ be a mixed monotone operator such that*

$$x_0 \leq A(x_0, y_0), \quad A(y_0, x_0) \leq y_0. \tag{1}$$

Suppose that one of the following two conditions are satisfied:

(H$_1$) *P is normal and A is completely continuous;*

(H$_2$) *P is regular and A is demicontinuous, i.e. $x_n \to x$ and $y_n \to y$ strongly implies $A(x_n, y_n) \to A(x, y)$ weakly.*

Then A has a coupled fixed point $(x^*, y^*) \in [x_0, y_0] \times [x_0, y_0]$ which is minimal and maximal in the sense that $x^* \leq \overline{x} \leq y^*$ and $x^* \leq \overline{y} \leq y^*$ for any coupled fixed point $(\overline{x}, \overline{y}) \in [x_0, y_0] \times [x_0, y_0]$ of A. Moreover, we have

$$x^* = \lim_{n\to\infty} x_n, \quad y^* = \lim_{n\to\infty} y_n \tag{2}$$

where

$$x_n = A(x_{n-1}, y_{n-1}), \quad y_n = A(y_{n-1}, x_{n-1}) \quad (n = 1, 2, \ldots) \tag{3}$$

satisfy

$$x_0 \leq x_1 \leq \ldots \leq x_n \leq \ldots \leq y_n \leq \ldots \leq y_1 \leq y_0. \tag{4}$$

Theorem 2.2. [3] *Let P be a normal cone such that the norm is monotone with respect to P (i.e. $\theta \leq x \leq y$ implies $\|x\| \leq \|y\|$), $h > \theta$ and $A : P_h \times P_h \to P_h$ a mixed monotone operator for which there exists a lower semicontinuos function $\phi : (0,1) \to (0,1)$ such that $\phi(t) > t$ for every t and*

$$A(tx, t^{-1}x) \geq \phi(t) A(x, x) \tag{5}$$

holds for all $x \in P_h$ and $t \in (0,1)$. Then A has exactly one fixed point x^* in P_h and for any initial point $x_0 \in \overset{\circ}{P}$, the sequence

$$x_n = A(x_{n-1}, x_{n-1}) \quad (n = 1, 2, \ldots) \tag{6}$$

converges to x^*.

Theorem 2.3. [8] *Let P be a normal cone, $h > \theta$ and $A : P_h \times P_h \to P_h$ a mixed monotone operator for which there exists a function $\alpha : (0,1) \to (0,1)$ such that*

$$A(tx, t^{-1}y) \geq t^{\alpha(t)} A(x,y) \tag{7}$$

holds for all $x, y \in P_h$ and $t \in (0,1)$. Then A has exactly one fixed point x^* in P_h.

Moreover, constructing successively the sequences

$$x_n = A(x_{n-1}, y_{n-1}), \quad y_n = A(y_{n-1}, x_{n-1}) \quad (n = 1, 2, \dots) \tag{8}$$

for any initial point $x_0, y_0 \in P_h$, we have

$$x^* = \lim_{n\to\infty} x_n, \quad x^* = \lim_{n\to\infty} y_n. \tag{9}$$

2.2. **The fixed point problem for multi-mixed monotone operators.** Consider X a Banach space partially ordered by a cone K, $Y \subseteq X$ a non-empty subset of X, N a positive integer. We say that an operator $T : Y^N \to X$ is *multi-mixed monotone* on Y if T is monotone (nondecreasing or nonincreasing) with respect to each variable.

We would like to study the following system of equations

$$x_i = T_i(x_1, x_2, \dots, x_N), \quad i = 1, 2, \dots, N \tag{10}$$

where T_i are multi-mixed monotone operators. Obviously, if we take $E = X^N$ and $P = K^N$ then (E, P) is an ordered Banach space with $\mathbf{x} \leq \mathbf{y}$ if and only if $x_i \leq y_i$ in (X, K) for every $i = 1, 2, \dots, N$. Then (10) can be viewed as a fixed point problem on the set $D = Y^N \subset E$

$$\mathbf{x} = \mathbf{T}(\mathbf{x}), \quad \mathbf{x} \in D \tag{11}$$

where $\mathbf{x} = (x_1, x_2, \dots, x_N)$ and $\mathbf{T} = (T_1, T_2, \dots, T_N) : D \to E$ (we will say that $\mathbf{T}$ is a *multi-mixed monotone operator*). In general, $\mathbf{T}$ is not a monotone operator, so we can not apply directly any monotone iterative technique (see [1,6].

The idea is to find a suitable mixed monotone operator $A : D \times D \to E$ such that

$$A(\mathbf{x}, \mathbf{x}) = \mathbf{T}(\mathbf{x}), \quad \forall \mathbf{x} \in D \tag{12}$$

hence any fixed point result for A will give a solution to our original problem (10).

3. Main results

3.1. The connection between systems of multi-mixed monotone operators and mixed monotone operators. For every multi-mixed monotone operator $T : D = Y^N \to X$ we consider the vector $\mu^T \in \{0,1\}^N$ defined by

$$\mu_j^T = \begin{cases} 1, & \text{if } T \text{ is nondecreasing in the } j\text{-th variable} \\ 0, & \text{if } T \text{ is nonincreasing in the } j\text{-th variable} \end{cases} \tag{13}$$

If T is constant with respect to some variable, then μ_j^T can be chosen either 0 or 1. Remark that

$$\mu_j^{-T} = 1 - \mu_j^T, \quad \forall j = 1, 2, \ldots, N. \tag{14}$$

Consider also the following two operators σ^T and λ^T associated to T and defined by:

$$\sigma^T : D \times D \to E, \quad \sigma^T(\mathbf{x}, \mathbf{y}) = \left(\mu_j^T x_j + \mu_j^{-T} y_j\right)_{j=1,2,\ldots,N} \tag{15}$$

and

$$\lambda^T : (0, \infty) \times D \to E, \quad \lambda^T(t; \mathbf{x}) = \sigma^T(t\mathbf{x}, \mathbf{x}) = \left(t^{\mu_j^T} x_j\right)_{j=1,2,\ldots,N} \tag{16}$$

Remark that if Y is convex, then $\sigma^T : D \times D \to D$.

It is easy to see that

$$\sigma^{-T}(\mathbf{x}, \mathbf{y}) = \sigma^T(\mathbf{y}, \mathbf{x}) \tag{17}$$

and

$$\lambda^{-T}(t; \mathbf{x}) = t\lambda^T(t^{-1}; \mathbf{x}) \tag{18}$$

for every $\mathbf{x}, \mathbf{y} \in D$ and $t \in (0, \infty)$. Also, if $\mathbf{y} = \lambda^T(t; \mathbf{x})$ then $\mathbf{x} = \lambda^T(t^{-1}; \mathbf{y})$.

There is one more identity which will provide useful:

$$\sigma^T(t\mathbf{x}, \mathbf{y}) = \lambda^T(t; \sigma^T(\mathbf{x}, \mathbf{y})), \quad \mathbf{x}, \mathbf{y} \in D, t \in (0, \infty) \tag{19}$$

which is easy to prove on components: if $\mu_j^T = 1$, then

$$\sigma^T(t\mathbf{x}, \mathbf{y})_j = tx_j = t^{\mu_j^T} \sigma^T(\mathbf{x}, \mathbf{y})_j = \lambda^T(t; \sigma^T(\mathbf{x}, \mathbf{y}))_j$$

and if $\mu_j^T = 0$, then

$$\sigma^T(t\mathbf{x}, \mathbf{y})_j = y_j = t^{\mu_j^T} \sigma^T(\mathbf{x}, \mathbf{y})_j = \lambda^T(t; \sigma^T(\mathbf{x}, \mathbf{y}))_j$$

and this is true for every $j = 1, 2, \ldots, N$ which concludes our argument.

For example, if T is a mixed monotone operator ($N = 2$), then $\mu^T = (1, 0)$, $\sigma^T(\mathbf{x}, \mathbf{y}) = (x_1, y_2)$, $\lambda^T(t; \mathbf{x}) = (tx_1, x_2)$ and $\mu^{-T} = (0, 1)$, $\sigma^{-T}(\mathbf{x}, \mathbf{y}) = (x_2, y_1)$, $\lambda^{-T}(t; \mathbf{x}) = (x_1, tx_2)$. Another example, if T is a nondecreasing operator, then $\sigma^T(\mathbf{x}, \mathbf{y}) = \mathbf{x}$ and $\lambda^T(t; \mathbf{x}) = t\mathbf{x}$.

The following lemma is straightforward, but fundamental:

Lemma 3.1. *Assume that Y is convex and consider $\mathbf{T} = (T_1, T_2, \ldots, T_N) : D \to E$ a multi-mixed monotone operator. Then the operator $A : D \times D \to E$, $A = (A_1, A_2, \ldots, A_N)$ whose components are defined by*

$$A_i = T_i \sigma^{T_i}, \quad i = 1, 2, \ldots, N \tag{20}$$

is mixed monotone with respect to the cone P in E and the fixed point sets of operators A and $\mathbf{T}$ coincide.

Proof. It is not hard to see, by the definition of the operators σ^{T_i}, that for every $i = 1, 2, \ldots, N$, $T_i(\sigma^{T_i}(\cdot, \mathbf{y})) : D \to X$ is nondecreasing for every $\mathbf{y} \in D$. This takes place because for a fixed $\mathbf{y} \in D$, $\sigma^{T_i}(\mathbf{x}, \mathbf{y})$ equals $\mathbf{x}$ on the components where T_i is nondecreasing and is constant on the components where T_i is nonincreasing.

Also $T_i(\sigma^{T_i}(\mathbf{x}, \cdot)) : D \to X$ is nonincreasing for every $\mathbf{y} \in D$ because for a fixed $\mathbf{x} \in D$, $\sigma^{T_i}(\mathbf{x}, \mathbf{y})$ equals $\mathbf{y}$ on the components where T_i is nonincreasing and is constant on the components where T_i is nondecreasing.

Because $\sigma^{T_i}(\mathbf{x}, \mathbf{x}) = \mathbf{x}$ for every $\mathbf{x} \in D$ and $i = 1, 2, \ldots, N$, we obtain

$$A(\mathbf{x}, \mathbf{x}) = \mathbf{T}(\mathbf{x}), \quad \forall \mathbf{x} \in D \tag{21}$$

which concludes our proof. □

In the next section we will use this lemma, together with the fixed point results for mixed monotone operators presented in the Section 2, in order to obtain existence results for system (10).

3.2. **Fixed point results for multi-mixed monotone operators.** If $\mathbf{T} = (T_1, T_2, \ldots, T_N) : D \to E$ is a multi-mixed monotone operator, we will say that $(\mathbf{x}^*, \mathbf{y}^*)$ is a coupled fixed point of $\mathbf{T}$ if

$$\begin{cases} x_i^* = T_i \sigma^{T_i}(\mathbf{x}^*, \mathbf{y}^*) \\ y_i^* = T_i \sigma^{T_i}(\mathbf{y}^*, \mathbf{x}^*) \end{cases}, \quad \forall i = 1, 2, \ldots, N. \tag{22}$$

Theorem 3.2. *Let* $\mathbf{x}^0, \mathbf{y}^0 \in E$, $\mathbf{x}^0 \leq \mathbf{y}^0$ *and* $\mathbf{T} : [\mathbf{x}^0, \mathbf{y}^0] \to E$ *be a multi-mixed monotone operator such that*

$$\begin{cases} x_i^0 \leq T_i \sigma^{T_i}(\mathbf{x}^0, \mathbf{y}^0) \\ y_i^0 \geq T_i \sigma^{T_i}(\mathbf{y}^0, \mathbf{x}^0) \end{cases}, \quad \forall i = 1, 2, \ldots, N \tag{23}$$

Suppose that one of the following two conditions are satisfied:

(H$_1$) K *is normal and* T_i *are completely continuous for every* $i = 1, 2, \ldots, N$*;*

(H$_2$) K *is regular and* T_i *are demicontinuous for every* $i = 1, 2, \ldots, N$*, i.e.* $\mathbf{x}^n \to \mathbf{x}$ *strongly implies* $T_i(\mathbf{x}^n) \to T_i(\mathbf{x})$ *weakly.*

Then $\mathbf{T}$ *has a coupled fixed point* $(\mathbf{x}^*, \mathbf{y}^*) \in [\mathbf{x}^0, \mathbf{y}^0] \times [\mathbf{x}^0, \mathbf{y}^0]$ *which is minimal and maximal in the sense that* $\mathbf{x}^* \leq \overline{\mathbf{x}} \leq \mathbf{y}^*$ *and* $\mathbf{x}^* \leq \overline{\mathbf{y}} \leq \mathbf{y}^*$ *for any coupled fixed point* $(\overline{\mathbf{x}}, \overline{\mathbf{y}}) \in [\mathbf{x}^0, \mathbf{y}^0] \times [\mathbf{x}^0, \mathbf{y}^0]$ *of* $\mathbf{T}$*. Moreover, we have*

$$\mathbf{x}^* = \lim_{n\to\infty} \mathbf{x}^n, \quad \mathbf{y}^* = \lim_{n\to\infty} \mathbf{y}^n \tag{24}$$

where

$$\begin{cases} x_i^n = T_i \sigma^{T_i}(\mathbf{x}^{n-1}, \mathbf{y}^{n-1}) \\ y_i^n = T_i \sigma^{T_i}(\mathbf{y}^{n-1}, \mathbf{x}^{n-1}) \end{cases}, \quad \forall i = 1, 2, \ldots, N \tag{25}$$

satisfy

$$\mathbf{x}^0 \leq \mathbf{x}^1 \leq \ldots \leq \mathbf{x}^n \leq \ldots \leq \mathbf{y}^n \leq \ldots \leq \mathbf{y}^1 \leq \mathbf{y}^0. \tag{26}$$

Proof. Consider the operator A from Lemma 3.1 corresponding to the operator $\mathbf{T}$ and apply directly Theorem 2.1. Notice that σ^{T_i} are linear and bounded which assures the conditions (H_1) and (H_2) in Theorem 2.1. □

Theorem 3.3. *Let* K *be normal cone,* $h_i > \theta$ *(*$i = 1, 2, \ldots, N$*),* $\mathbf{h} = (h_1, h_2, \ldots, h_N)$*,* K_i *the class of equivalence for* h_i *in* K *and* $P_{\mathbf{h}} = K_1 \times K_2 \times \cdots \times K_N$ *the class of equivalence of* $\mathbf{h}$ *in* P*. Consider* $\mathbf{T} = (T_1, T_2, \ldots, T_N)$ *a multi-mixed monotone operator such that* $T_i : K_i \to K_i$ *for every* $i = 1, 2, \ldots, N$ *and assume that there exists a function* $\alpha : (0, 1) \to (0, 1)$ *such that*

$$T_i\left(\lambda^{T_i}(t; \mathbf{x})\right) \geq t^{\alpha(t)} T_i(\lambda^{-T_i}(t; \mathbf{x})) \tag{27}$$

holds for all $\mathbf{x} \in P_{\mathbf{h}}$ *and* $t \in (0, 1)$*. Then* $\mathbf{T}$ *has exactly one fixed point* $\mathbf{x}^*$ *in* $P_{\mathbf{h}}$*.*

Moreover, constructing successively the sequences $(\mathbf{x}^n)_{n\geq 0}$ *and* $(\mathbf{y}^n)_{n\geq 0}$

$$\begin{cases} x_i^n = T_i \sigma^{T_i}(\mathbf{x}^{n-1}, \mathbf{y}^{n-1}) \\ y_i^n = T_i \sigma^{T_i}(\mathbf{y}^{n-1}, \mathbf{x}^{n-1}) \end{cases}, \quad \forall i = 1, 2, \ldots, N$$

for any initial point $\mathbf{x}^0, \mathbf{y}^0 \in P_{\mathbf{h}}$*, we have*

$$\mathbf{x}^* = \lim_{n\to\infty} \mathbf{x}^n, \quad \mathbf{y}^* = \lim_{n\to\infty} \mathbf{y}^n. \tag{28}$$

Proof. Consider the operator A from Lemma 3.1 corresponding to the operator $\mathbf{T}$ and apply directly Theorem 2.3. We just have to prove that (27) guarantees (7).

First, note that (7) can be written in the more symmetrical form

$$A(t\mathbf{x}, \mathbf{y}) \geq t^{\alpha(t)} A(\mathbf{x}, t\mathbf{y}), \quad \forall \mathbf{x}, \mathbf{y} \in P_{\mathbf{h}}, \forall t \in (0,1) \tag{29}$$

which in this case means

$$T_i\left(\sigma^{T_i}(t\mathbf{x}, \mathbf{y})\right) \geq t^{\alpha(t)} T_i(\sigma^{T_i}(\mathbf{x}, t\mathbf{y})), \quad \forall \mathbf{x}, \mathbf{y} \in P_{\mathbf{h}}, \forall t \in (0,1), \forall i = 1, 2, \ldots, N$$

By applying the properties of σ^{T_i} and λ^{T_i} and denoting $\mathbf{z} = \sigma^{T_i}(\mathbf{x}, \mathbf{y})$, we have from identity (19) that

$$\sigma^{T_i}(t\mathbf{x}, \mathbf{y}) = \lambda^{T_i}(t; \sigma^{T_i}(\mathbf{x}, \mathbf{y})) = \lambda^{T_i}(t; \mathbf{z})$$
$$\sigma^{T_i}(\mathbf{x}, t\mathbf{y}) = t\sigma^{T_i}\left(t^{-1}\mathbf{x}, \mathbf{y}\right) = t\lambda^{T_i}(t^{-1}; \mathbf{z}) = \lambda^{-T_i}(t; \mathbf{z})$$

Hence (7) is equivalent to

$$T_i\left(\lambda^{T_i}(t; \mathbf{z})\right) \geq t^{\alpha(t)} T_i(\lambda^{-T_i}(t; \mathbf{z}))$$

which is guaranteed by (27). This concludes our proof. □

Remark 3.4. *By applying Theorem 2.2, we obtain a similar result but which is less general because of the additional continuity condition for the function* ϕ *and the monotonicity of the norm. It spite of the fact that the condition (19) is more demanding then (5), for our case where* A *is defined by Lemma 3.1 the two conditions are similar. By using the same properties of* σ^{T_i} *and* λ^{T_i}*, it is easy to prove that (5) is equivalent to (27), with* $\alpha(t) = \frac{\ln \phi(t)}{\ln t}$*.*

4. Application

Let $\Omega \subset \mathbb{R}^m$ be an open and bounded set and $X = C\left(\overline{\Omega}; \mathbb{R}^m\right)$ the Banach space of continuous functions on $\overline{\Omega}$ with values in $\mathbb{R}^m$ ordered by the cone K of the nonnegative functions ($u \in K$ if and only if $u(x) \in \mathbb{R}^m_+$ for every $x \in \Omega$). It is a well known and easy to check result that K is regular. Take also a family of real valued functions $k_i \in L^1\left(\Omega^2\right)$ such that k_i are positive a.e. on Ω^2 ($i = 1, 2, \ldots, N$) and $(a_{ij})_{i,j=1}^N$ a matrix whose elements lie in $(-1, 1)$.

Consider the following system of equations

$$u_i(x) = \int_\Omega k_i(x,s)\textstyle\sum_{j=1}^N u_j(s)^{a_{ij}}\,\mathrm{d}s, \quad x \in \overline{\Omega} \tag{30}$$

with the unknowns $u_i \in X$, where y^r denotes $(y_1^r, y_2^r, \ldots, y_N^r)$ for $y \in \mathbb{R}^N$ and $r \in \mathbb{R}$.

By applying Theorem 3.3, it is easy to check that the operators T_i

$$T_i(u_1, u_2, \ldots u_N) = \int_\Omega k_i(\cdot,s)\textstyle\sum_{j=1}^N u_j(s)^{a_{ij}}\,\mathrm{d}s$$

are multi-mixed monotone and that the condition (27) is easily satisfied by taking $\alpha \geq \max\left(|a_{ij}|\right)_{i,j=1}^N$. By the positivity of the kernels k_i, we have that $T_i : \overset{\circ}{K} \to \overset{\circ}{K}$, hence the system (30) has a unique positive solution vector $\mathbf{u}^* = (u_1^*, u_2^*, \ldots, u_N^*)$ which can be obtained recursively as the limit in X of the sequences $(\mathbf{w}^n)_{n\geq 0}$ and $(\boldsymbol{\omega}^n)_{n\geq 0}$ from X^N given by

$$\left\{\begin{array}{l} w_i^n = \displaystyle\int_\Omega k_i(\cdot,s)\textstyle\sum_{j=1}^N \widetilde{w}_{ij}^{n-1}(s)^{a_{ij}}\,\mathrm{d}s \\ \omega_i^n = \displaystyle\int_\Omega k_i(\cdot,s)\textstyle\sum_{j=1}^N \widetilde{\omega}_{ij}^{n-1}(s)^{a_{ij}}\,\mathrm{d}s \end{array}\right., \quad \forall i = 1, 2, \ldots, N \tag{31}$$

for any initial positive functions w_i^0 and ω_i^0 in X, where $\widetilde{w}_{ij}$ and $\widetilde{\omega}_{ij}$ denote

$$\widetilde{w}_{ij}^k = \left\{\begin{array}{ll} w_j^k & \text{if } a_{ij} \geq 0 \\ \omega_j^k & \text{if } a_{ij} < 0 \end{array}\right. \tag{32}$$

$$\widetilde{\omega}_{ij} = \left\{\begin{array}{ll} \omega_j^k & \text{if } a_{ij} \geq 0 \\ w_j^k & \text{if } a_{ij} < 0 \end{array}\right. \tag{33}$$

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