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ABSTRACT MODELS OF STEP METHOD WHICH IMPLY THE CONVERGENCE OF SUCCESSIVE APPROXIMATIONS

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Abstract. This paper has three goals:

- to present two abstract models: forward step model and backward step model;
- to prove that the global operator which appear in these models are weakly Picard;
- to give applications to functional differential equations with retarded argument and to functional differential equations with advanced argument.

Key Words and Phrases: operators on cartesian product, step method, Picard operators, weakly Picard operators, functional differential equations, retarded argument, advanced argument.
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1. Introduction

We formulate our problem by the following example:

$$x'(t) = f(t, x(t), x(r - h)), \quad t \in [a, b],\ h > 0, \tag{1.1}$$

$$x(t) = \varphi(t), \quad t \in [a - h, a], \tag{1.2}$$

$$x \in C([a - h, b], \mathbb{B}) \cap C^1([a, b], \mathbb{B}),$$

and the conditions

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(C_1) $(\mathbb{B}, \|\cdot\|, \leq)$ is an ordered Banach space and $f \in C([a,b]\times\mathbb{B}\times\mathbb{B},\mathbb{B})$, $\varphi \in C([a-h,a],\mathbb{B})$;

(C_2) $\exists\ L_f > 0 :\ \|f(t,u_1,v_1) - f(t,u_2,v_2)\| \leq L_f(\|u_1-u_2\| + \|v_1-v_2\|)$, $\forall\ t \in [a,b]$, $\forall\ u_i, v_i \in \mathbb{B}$, $i = 1,2$;

(C_3) $\exists\ L_f > 0 :\ \|f(t,u_1,v) - f(t,u_2,v)\| \leq L_f\|u_1-u_2\|$, $\forall\ t \in [a,b]$, $\forall\ u_1, u_2, v \in \mathbb{B}$.

Let $m \in \mathbb{N}^*$ be such that:

$$a + (m-1)h < b \quad \text{and} \quad a + mh \geq b.$$

We denote $t_{-1} := a - h$, $t_0 := a$, $t_i := a + ih$, $i = \overline{1, m-1}$, $t_m := b$.

In what follow we consider on $C([t_{-1}, t_m], \mathbb{B})$ the Bielecki norm,

$$\|x\|_B := \max_{t_{-1}\leq t\leq t_m}(\|x(t)\|e^{-\tau|t-t_0|}),$$

and on $C([t_{i-1}, t_i], \mathbb{B})$ the norm

$$\|x_i\|_B := \max_{t_{i-1}\leq t\leq t_i}(\|x(t)\|e^{-\tau(t-t_{i-1})}).$$

The equation (1.1) is equivalent with the fixed point equation

$$x = E_f(x), \quad x \in C([t_{-1}, t_m], \mathbb{B}),$$

and the problem (1.1)+(1.2) is equivalent with

$$x = B_f(x), \quad x \in C([t_{-1}, t_m], \mathbb{B}),$$

where

$$E_f(x)(t) := \begin{cases} x(t), & t \in [t_{-1}, t_0] \\ x(t_0) + \displaystyle\int_{t_0}^{t} f(s, x(s), x(s-h))ds, & t \in [t_0, t_m] \end{cases}$$

and

$$B_f(x)(t) := \begin{cases} \varphi(t), & t \in [t_{-1}, t_0] \\ \varphi(t_0) + \displaystyle\int_{t_0}^{t} f(s, x(s), x(s-h))ds, & t \in [t_0, t_m]. \end{cases}$$

The following results are well known (see [1]-[7], [9]-[11], [15], [16]-[21], [27], [28]):

Theorem 1.1. *In the conditions* $(C_1) + (C_2)$ *we have:*

(i) the problem (1.1)+(1.2) has in $C([t_{-1}, t_m], \mathbb{B})$ *a unique solution* x^* *and*

$$x^* \in C([t_{-1}, t_m], \mathbb{B}) \cap C^1([t_0, t_m], \mathbb{B});$$

(ii) the successive approximations

$$x^{n+1}(t) = \begin{cases} \varphi(t), & t \in [t_{-1}, t_0] \\ \varphi(t_0) + \displaystyle\int_{t_0}^{t} f(s, x^n(s), x^n(s-h))ds, & t \in [t_0, t_m] \end{cases}$$

converges to x^*, for all $x^0 \in C([t_{-1}, t_m], \mathbb{B})$;

(iii) the operator E_f is weakly Picard operator and B_f is Picard operator (see [26]*).*

In what follow we suppose that we are in the conditions (C_1) and (C_3).

The step method for the problem (1.1)+(1.2) consists in:

(e_0) $\quad x_0(t) = \varphi(t),\ t \in [t_{-1}, t_0]$

(e_1) $\quad x_1(t) = \varphi(t_0) + \displaystyle\int_{t_0}^{t} f(s, x_1(s), \varphi(s-h))ds,\ t \in [t_0, t_1]$

(e_2) $\quad x_2(t) = x_1^*(t_1) + \displaystyle\int_{t_1}^{t} f(s, x_2(s), x_1^*(s-h))ds,\ t \in [t_1, t_2]$

....

(e_m) $\quad x_m(t) = x_{m-1}^*(t_{m-1}) + \displaystyle\int_{t_{m-1}}^{t} f(s, x_m(s), x_{m-1}^*(s-h))ds,\ t \in [t_{m-1}, t_m]$

where $x_i^* \in C([t_{-1}, t_i], \mathbb{B})$ is the unique solution of the equation (e_i), $i = \overline{1, m}$.

We have

Theorem 1.2. *In the conditions (C_1) and (C_3) we have that:*

(i) the problem (1.1)+(1.2) has in $C([t_{-1}, t_m], \mathbb{B})$ a unique solution x^* where

$$x^*(t) = \begin{cases} \varphi(t), & t \in [t_{-1}, t_0], \\ x_1^*(t), & t \in [t_0, t_1], \\ \dots \\ x_m^*(t), & t \in [t_{m-1}, t_m]; \end{cases}$$

(ii) the functions x_i^* are the limit of the successive approximations

$$x_i^{n+1}(t) = x_{i-1}^*(t_{i-1}) + \int_{t_{i-1}}^{t} f(s, x_i^n(s), x_{i-1}^*(s-h))ds, \quad t \in [t_{i-1}, t_i]$$

in $(C([t_{i-1}, t_i], \mathbb{B}), \|\cdot\|_B)$, $i = \overline{1, m}$.

In this paper we shall study the following problem:

Problem 1.1. Can we put x_{i-1}^n instead of x_{i-1}^*, $i = \overline{2, m}$, in the conclusion (ii) of the above theorem?

For to study this problem we need some notions and results from weakly Picard operator theory.

2. Fibre weakly Picard operators

Let (X, d) be a metric space and $A : X \to X$ an operator.

Definition 2.1. (see [26]). The operator A is weakly Picard operator (WPO) if the sequence

$$(A^n(x))_{n\in\mathbb{N}}$$

converges, for all $x \in X$, and the limit (which may depend on x) is a fixed point of A.

Definition 2.2. (see [26]). If the operator A is WPO and $F_A = \{x^*\}$, then by definition A is Picard operator (PO).

Definition 2.3. (see [26]). If A is WPO, then we consider the operator A^∞ defined by

$$A^\infty : X \to X, \quad A^\infty(x) := \lim_{n\to\infty} A^n(x).$$

It is clear that $A^\infty(x) \in F_A$ and $A^\infty(X) = F_A$.

In this paper we need the following results (see [12], [23]-[26]):

Theorem 2.1. *Let (X, d) and (Y, ρ) be two metric spaces and $A = (B, C) : X \times Y \to Y \times Y$ a triangular operator, i.e., $B : X \to X$, $C : X \times Y \to Y$.*

We suppose that:

(i) (Y, ρ) is a complete metric space;

(ii) $B : X \to X$ is WPO;

(iii) there exists $\alpha \in (0, 1)$ such that $C(x, \cdot) : Y \to Y$ is α-contraction, for all $x \in X$;

(iv) if $(x^*, y^*) \in F_A$, then $C(\cdot, y^*)$ is continuous in x^*.

Then the operator A is WPO.

If B is PO, then A is PO.

By induction, from the above results we have (see [26]):

Theorem 2.2. *Let (X_i, d_i), $i = \overline{0, m}$, $m \geq 1$ be some metric spaces. Let $A_i : X_0 \times \cdots \times X_i \to X_i$, $i = \overline{0, m}$ be some operator. We suppose that:*

(i) (X_i, d_i), $i = \overline{1, m}$, are complete metric spaces;

(ii) the operator A_0 is WPO;

(iii) there exist $\alpha_i \in (0, 1)$ such that

$$A_i(x_0, \ldots, x_{i-1}, \cdot) : X_i \to X_i, \quad i = \overline{1, m},$$

are α_i-contractions;

(iv) the operator A_i, $i = \overline{1,m}$ are continuous.

Then the operator $A : X_0 \times \cdots \times X_m \to X_0 \times \cdots \times X_m$,

$$A(x_0, \ldots, x_m) := (A_0(x_0), A_1(x_0, x_1), \ldots, A_m(x_0, \ldots, x_m)),$$

is WPO.

If A_0 is PO, then A is PO.

3. Forward step method

Let $t_i \in \mathbb{R}$, $i \in \{-1, 0, 1, \ldots, m\}$ be such that $t_{-1} < t_0 < t_1 < \cdots < t_m$. Let $(\mathbb{B}, \|\cdot\|, \leq)$ be an ordered Banach space. We consider on $X_i := C([t_{i-1}, t_i], \mathbb{B})$ a complete metric d_i, $i = \overline{0,m}$. Let $A_0 : X_0 \to X_0$, $A_i : X_{i-1} \times X_i \to X_i$, $i = \overline{1,m}$ be some operators and the operator

$$A : X_0 \times X_1 \times \cdots \times X_m \to X_0 \times X_1 \times \cdots \times X_m$$

be defined by

$$A(x_0, x_1, \ldots, x_m) := (A_0(x_0), A_1(x_0, x_1), \ldots, A_m(x_{m-1}, x_m)).$$

We consider the following subset of $X_0 \times \cdots \times X_m$,

$$U := \{(x_0, x_1, \ldots, x_m) \in X_0 \times X_1 \times \cdots \times X_m \mid x_i(t_i) = x_{i+1}(t_i),\ i = \overline{0, m-1}\}$$

and the operator

$$R : C([t_{-1}, t_m], \mathbb{B}) \to U$$

defined by

$$R(x) := (x|_{[t_{-1}, t_0]}, x|_{[t_0, t_1]}, \ldots, x|_{[t_{m-1}, t_m]}).$$

It is clear that R is an increasing bijection.

Remark 3.1. In general U is not an invariant subset of A.

First our abstract result is the following

Theorem 3.1. *We suppose that:*

(i) A_0 is WPO;

(ii) $A_i(x_{i-1}, \cdot) : X_i \to X_i$ is α_i-contraction, for all $x_{i-1} \in X_{i-1}$, $i = \overline{1,m}$;

(iii) $A_i(x_{i-1}, x_i)(t_{i-1}) = x_{i-1}(t_{i-1})$, $i = \overline{1,m}$.

Then:

(a) A is WPO;

(b) if A_0 is PO, then A is PO;

(c) if $(x_0^*, x_1^*, \ldots, x_m^*) \in F_A$*, then* $(x_0^*, x_1^*, \ldots, x_m^*) \in U$ *and*

$$R^{-1}(x_0^*, x_1^*, \ldots, x_m^*) \in C([t_{-1}, t_m], \mathbb{B}).$$

Proof. (a)+(b). This part of the theorem is a particular case of the Theorem 2.2. A direct proof follows from the fibre contraction theorem (Theorem 2.1; see also M.W. Hirsch and C.C. Pugh [12] and I.A. Rus [24] and [25]).

(c) From $A(x_0^*, \ldots, x_m^*) = (x_0^*, x_1^*, \ldots, x_m^*)$ it follows that

$$A_i(x_{i-1}^*, x_i^*)(t_{i-1}) = x_i^*(t_{i-1}).$$

So, by (iii) we have $x_i^*(t_{i-1}) = x_{i-1}^*(t_{i-1})$, $i = \overline{1, m}$.

Remark 3.2. Let A be as in the Theorem 3.1. If A is increasing, then the operator

$$R^{-1}A^{\infty}R : C([t_{-1}, t_m], \mathbb{B}) \to C([t_{-1}, t_m], \mathbb{B})$$

is increasing. Indeed, from (c) we have that U is an invariant set of A^{∞}, i.e., $R^{-1}A^{\infty}R$ is defined. On the other hand R^{-1}, A^{∞}, R are increasing operators.

Theorem 3.2. (Gronwall lemma). *Let A be as in Theorem 3.1. We suppose that A is an increasing operator. Let $x \in C([t_{-1}, t_m], \mathbb{B})$ be such that $R(x) \leq AR(x)$. Then, $x \leq R^{-1}A^{\infty}R(x)$.*

Proof. A increasing WPO imply that

$$R(x) \leq AR(x) \leq A^2R(x) \leq \cdots \leq A^{\infty}R(x).$$

From $R(x) \leq A^{\infty}R(x)$, it follows that, $x \leq R^{-1}A^{\infty}R(x)$.

Theorem 3.3. (Comparison lemma). *Let $A, B, C : X_0 \times \cdots \times X_m \to X_0 \times \cdots \times X_m$ be as in Theorem 3.1. We suppose that:*

(1) B is increasing operator;

(2) $A \leq B \leq C$.

Then:

$$x, y, z \in C([t_{-1}, t_m], \mathbb{B}),\ x \leq y \leq z \ \Rightarrow$$

$$R^{-1}A^{\infty}R(x) \leq R^{-1}B^{\infty}R(y) \leq R^{-1}C^{\infty}R(z).$$

Proof. $x \leq y \leq z$ implies that $R(x) \leq R(y) \leq R(z)$. Since A, B, C are WPOs and B is increasing, it follows from Lemma 7.4 in [26], that $A^{\infty}R(x) \leq B^{\infty}R(y) \leq C^{\infty}R(z)$. But R^{-1} is an increasing operator. So, $R^{-1}A^{\infty}R(x) \leq R^{-1}B^{\infty}R(y) \leq R^{-1}C^{\infty}R(z)$.

In the next section we present an application of the above results.

4. Applications to the Problem (1.1)+(1.2)

From Theorem 3.1 we have

Theorem 4.1. *In the conditions* (C_1) *and* (C_3)*, the problem (1.1)+(1.2) has in* $C([t_{-1}, t_m], \mathbb{B})$ *a unique solution,* x^**,*

$$x^*(t) := \begin{cases} \varphi(t), & t \in [t_{-1}, t_0], \\ x_1^*(t), & t \in [t_0, t_1], \\ \dots \\ x_m^*(t), & t \in [t_{m-1}, t_m], \end{cases}$$

and the functions x_i^**,* $i = \overline{1, m}$*, are the limit of the successive approximations*

$$x_i^{n+1}(t) := x_{i-1}^n(t_{i-1}) + \int_{t_{i-1}}^t f(s, x_i^n(s), x_{i-1}^n(s-h))ds, \quad t \in [t_{i-1}, t_i],$$

in $(C([t_{i-1}, t_i], \mathbb{B}), \|\cdot\|_B)$*,* $i = \overline{1, m}$*.*

Proof. We consider the following operators

$$B_{0f} : C([t_{-1}, t_0], \mathbb{B}) \to C([t_{-1}, t_0], \mathbb{B}), \quad x_0 \mapsto \varphi$$

and

$$B_{if} : C([t_{i-2}, t_{i-1}], \mathbb{B}) \times C([t_{i-1}, t_i], \mathbb{B}) \to C([t_{i-1}, t_i], \mathbb{B})$$

defined by

$$B_{if}(x_{i-1}, x_i)(t) := x_{i-1}(t_{i-1}) + \int_{t_{i-1}}^t f(s, x_i(s), x_{i-1}(s-h))ds,$$
$$t \in [t_{i-1}, t_i], \ i = \overline{1, m}.$$

Condition (C_1) and (C_3) imply that we are in the conditions of the Theorem 3.1, where $A_i = B_{if}$ and

$$A = \widetilde{B}_f := (B_{0f}(x_0), B_{1f}(x_0, x_1), \dots, B_{mf}(x_{m-1}, x_m)).$$

Since B_{0f} is PO, hence that $\widetilde{B}_f$ is PO and $R^{-1}(\widetilde{B})^\infty(x_0^0, \dots, x_n^0)$ is the unique solution of the problem (1.1)+(1.2), for all $x_i^0 \in X_i$, $i = \overline{0, m}$.

Remark 4.1. If we take $E_{0f} := 1_{C([t_{-1}, t_0], \mathbb{B})}$ and

$$E_{if} : C([t_{i-2}, t_{i-1}], \mathbb{B}) \times C([t_{i-1}, t_i], \mathbb{B}) \to C([t_{i-1}, t_i], \mathbb{B})$$

defined by

$$E_{if}(x_{i-1}, x_i)(t) := x_{i-1}(t_{i-1}) + \int_{t_{i-1}}^t f(s, x_i(s), x_{i-1}(s-h))ds, \quad t \in [t_{i-1}, t_i],$$

then, in the conditions of the Theorem 4.1, the operator

$$\widetilde{E}_f(x_0, x_1, \ldots, x_m) := (E_{0f}(x_0), E_{1f}(x_0, x_1), \ldots, E_{mf}(x_{m-1}, x_m))$$

is WPO and $R^{-1}(\widetilde{E}_f)^\infty(x_0^0, \ldots, x_m^0)$ is a solution of the equation (1.1) and for each solution $x \in (C[t_{-1}, t_m], \mathbb{B})$ there exists $x_i^0 \in C([t_{i-1}, t_i], \mathbb{B})$, $i = \overline{0, m}$, such that

$$x = R^{-1}(\widetilde{E}_f)^\infty(x_0^0, x_1^0, \ldots, x_m^0).$$

Theorem 4.2. *We suppose that f is as in the Theorem 4.1 and $f(t, \cdot, \cdot) : \mathbb{B} \times \mathbb{B} \to \mathbb{B}$ is increasing for all $t \in [a, b]$. Then:*

$$x \in C([t_{-1}, t_m], \mathbb{B}), \quad R(x) \leq \widetilde{E}_f R(x) \;\Rightarrow\; x \leq R^{-1}(\widetilde{E}_f)^\infty R(x).$$

Proof. The proof follows from Remark 4.1 and Theorem 3.2.

Remark 4.2. From Theorem 4.2 we have that if $x^* \in C([a - h, b], \mathbb{B})$ is the solution of the problem (1.1)+(1.2) and $x \in C([a-h, b], \mathbb{B})$ is a solution of the differential inequality

$$x'(t) \leq f(t, x(t), x(t-h)), \quad t \in [a, b],$$

$$x(t) \leq \varphi(t), \quad t \in [a - h, a]$$

then, $x \leq x^*$.

Theorem 4.3. *Let f, g, h be as in the Theorem 4.1. We suppose that:*

(1) $g(t, \cdot, \cdot) : \mathbb{B} \times \mathbb{B} \to \mathbb{B}$ is increasing;

(2) $f \leq g \leq h$.

Let x be a solution of the equation (1.1), y a solution of the equation

$$y'(t) = g(t, y(t), y(t-h)), \quad t \in [a, b],$$

and z a solution of the equation

$$z'(t) = h(t, z(t), z(t-h)), \quad t \in [a, b].$$

Then:

$$x|_{[a-h,a]} \leq y|_{[a-h,a]} \leq z|_{[a-h,a]} \;\Rightarrow\; x \leq y \leq z.$$

Proof. Let

$$\widetilde{x}(t) := \begin{cases} x(t), & t \in [a-h, a] \\ x(a), & t \in [a, b]. \end{cases}$$

In a similar way we define $\widetilde{y}$, $\widetilde{z}$. It is clear that, $\widetilde{x} \leq \widetilde{y} \leq \widetilde{z}$ and

$$x = R^{-1}(\widetilde{E}_f)^\infty R(\widetilde{x}), \quad y = R^{-1}(\widetilde{E}_g)^\infty R(\widetilde{y}) \quad \text{and} \quad z = R^{-1}(\widetilde{E}_h)^\infty R(\widetilde{z}).$$

From Theorem 3.3 it follows that $x \le y \le z$.

Example 4.1. Let us consider the following problem (see [15], p. 27):

$$x'(t) = p(t)x(t) + q(t)x(t-2)e^{-x(t-2)}, \quad t \in [0,5] \tag{4.1}$$

$$x(t) = \varphi(t), \quad t \in [-2,0]. \tag{4.2}$$

If $p, q \in C[0,5]$ and $\varphi \in C[-2,0]$, then by the Theorem 4.1 the problem (4.1)+(4.2) has a unique solution

$$x^*(t) = \begin{cases} \varphi(t), & t \in [-2,0] \\ x_1^*(t), & t \in [0,2] \\ x_2^*(t), & t \in [2,4] \\ x_3^*(t), & t \in [4,5] \end{cases}$$

and x_1^*, x_2^*, x_3^* are the limits of the following sequences, respectively

$$x_1^{n+1}(t) = \varphi(0) + \int_0^t [p(s)x_1^n(s) + q(s)\varphi(s-2)e^{-\varphi(s-2)}]ds, \quad t \in [0,2],$$

$$x_2^{n+1}(t) = x_1^n(2) + \int_2^t [p(s)x_2^n(s) + q(s)x_1^n(s-2)e^{-x_1^n(s-2)}]ds, \quad t \in [2,4],$$

$$x_3^{n+1}(t) = x_2^n(4) + \int_4^t [p(s)x_3^n(s) + q(s)x_2^n(s-2)e^{-x_2^n(s-2)}]ds, \quad t \in [4,5].$$

Remark 4.3. In the case of the equation

$$x'(t) = p(t)x(t) + q(t, x(t-h)), \quad t \in [a,b]$$

if $p \in C[a,b]$, $q \in C([a,b] \times \mathbb{R})$ then we are in the conditions of the Theorem 4.1.

5. Backward step method

Let $t_i \in \mathbb{R}$, $t_0 < t_1 < \cdots < t_m < t_{m+1}$ and

$$X_i := C([t_{i-1}, t_i], \mathbb{B}), \quad i = \overline{1, m+1}.$$

Let $A_i : X_i \times X_{i+1} \to X_i$, $i = \overline{1,m}$ and $A_{m+1} : X_{m+1} \to X_{m+1}$ be some operators and

$$A : X_1 \times \cdots \times X_{m+1} \to X_1 \times \cdots \times X_{m+1}$$

be defined by

$$A(x_1, \ldots, x_{m+1}) := (A_1(x_1,x_2), A_2(x_2,x_3), \ldots, A_m(x_m, x_{m+1}), A_{m+1}(x_{m+1})).$$

We consider the following subset of $X_1 \times \cdots \times X_{m+1}$,

$$U := \{(x_1, \ldots, x_{m+1}) \in X_1 \times \cdots \times X_{m+1} \mid x_i(t_i) = x_{i+1}(t_i),\ i = \overline{1,m}\}$$

and the operator $R : C([t_0, t_{m+1}], \mathbb{B}) \to U$ defined by

$$R(x) := (x|_{[t_0,t_1]}, \ldots, x|_{[t_m,t_{m+1}]}).$$

We remark that the operator R is an increasing bijection.

The second our abstract result is the following

Theorem 5.1. *We suppose that:*

(i) A_{m+1} is WPO;

(ii) $A_i(\cdot, x_{i+1}) : X_i \to X_i$ is α_i-contraction, $i = \overline{1,m}$;

(iii) $A_i(x_i, x_{i+1})(t_i) = x_{i+1}(t_i)$, $i = \overline{1,m}$.

Then:

(a) A is WPO;

(b) if A_{m+1} is PO, then A is PO;

(c) if $(x_1^*, \ldots, x_{m+1}^*) \in F_A$, then $(x_1^*, \ldots, x_{m+1}^*) \in U$ and

$$R^{-1}(x_1^*, \ldots, x_{m+1}^*) \in C([t_0, t_{m+1}], \mathbb{B}).$$

Proof. The proof is similar with that of Theorem 3.1.

Remark 5.1. Let A be as in Theorem 5.1. If A is increasing operator, then the operator $R^{-1}A^{\infty}R : C([t_0, t_{m+1}], \mathbb{B}) \to C([t_0, t_{m+1}], \mathbb{B})$ is increasing.

In a similar way as in section 3 we have:

Theorem 5.2. *Let A as in Theorem 5.1. We suppose that A is increasing operator. Then:*

$$x \in C([t_0, t_{m+1}], \mathbb{B}), \quad R(x) \leq AR(x) \ \Rightarrow\ x \leq R^{-1}A^{\infty}R(x).$$

Theorem 5.3. *Let $A, B, C : X_1 \times \cdots \times X_{m+1} \to X_1 \times \cdots \times X_{m+1}$ be as in Theorem 5.1. We suppose that*

(1) B is increasing operator;

(2) $A \leq B \leq C$.

Then:

$$x, y, z \in C([t_1, t_{m+1}], \mathbb{B}), \quad x \leq y \leq z \ \Rightarrow$$

$$R^{-1}A^{\infty}R(x) \leq R^{-1}B^{\infty}R(y) \leq R^{-1}C^{\infty}R(z).$$

In what follow we shall give some applications of the above results.

6. Applications to differential equations with advanced argument

We consider the following Cauchy problem for a functional differential equation with advanced argument (see [8], [14], [15], [22],...)

$$x'(t) = f(t, x(t), x(t+h)), \quad t \in [a, b],\ h > 0; \tag{6.1}$$

$$x(t) = \varphi(t), \quad t \in [b, b+h]; \tag{6.2}$$

in the following conditions:

(C_1') $f \in C([a, b] \times \mathbb{B} \times \mathbb{B}, \mathbb{B})$, $\varphi \in C([b, b+h], \mathbb{B})$;

(C_3') $\exists\, L_f > 0 :\ \|f(t, u_1, v) - f(t, u_2, v)\| \leq L_f \|u_1 - u_2\|$,

$$\forall\, t \in [a, b],\ \forall\, u_1, u_2, v \in \mathbb{B}.$$

Let $m \in \mathbb{N}^*$ be such that

$$b - (m-1)h > a \quad \text{and} \quad b - mh \leq a.$$

We denote

$$t_0 := a, \quad t_1 := b - (m-1)h, \ldots, t_m = b, \quad t_{m+1} := b + h,$$

and $X_i := C([t_{i-1}, t_i], \mathbb{B})$, $i = \overline{1, m+1}$.

The equation (6.1) is equivalent with the fixed point equation

$$x = E_f(x), \quad x \in C([t_0, t_{m+1}], \mathbb{B})$$

and the problem (6.1)+(6.2) is equivalent with

$$x = B_f(x), \quad x \in C([t_0, t_{m+1}), \mathbb{B}),$$

where

$$E_f(x)(t) := \begin{cases} x(t), & t \in [t_m, t_{m+1}] \\ x(t_m) + \displaystyle\int_{t_m}^{t} f(s, x(s), x(s+h))ds, & t \in [t_s, t_m] \end{cases}$$

and

$$B_f(x)(t) := \begin{cases} \varphi(t), & t \in [t_m, t_{m+1}] \\ \varphi(t_m) + \displaystyle\int_{t_m}^{t} f(s, x(s), x(s+h))ds, & t \in [t_0, t_m] \end{cases}$$

The step method for the problem (6.1)+(6.2) consists in the following:

$$x_{m+1}(t) = \varphi(t), \quad t \in [t_m, t_{m+1}],$$

$$x_m(t) = \varphi(t_m) + \int_{t_m}^{t} f(s, x_m(s), \varphi(s+h))ds, \ t \in [t_{m+1}, t_m],$$

$$x_{m-1}(t) = x_m^*(t_{m-1}) + \int_{t_{m-1}}^{t} f(s, x_{m-1}(s), x_m^*(s+h)), \ t \in [t_{m-2}, t_{m-1}],$$

$$\dots\dots\dots\dots\dots\dots$$

$$x_1(t) = x_2^*(t_1) + \int_{t_1}^{t} f(s, x_1(s), x_2^*(s+h)), \ t \in [t_0, t_1]$$

where x_{m-i}^* is the unique solution of the integral equation in the i-step.

The following result is well known ([6], [8], [14], [15],...).

Theorem 6.1. *In the conditions* $(C_1') + (C_3')$ *we have that:*

(i) the problem (6.1)+(6.2) has in $C([t_0, t_{m+1}], \mathbb{B})$ *a unique solution* x^* ($x^* \in C([t_0, t_{m+1}], \mathbb{B}) \cap C^1([t_0, t_m], \mathbb{B})$)*, where*

$$x^*(t) := \begin{cases} \varphi(t), & t \in [t_m, t_{m+1}] \\ x_m^*(t), & t \in [t_{m-1}, t_m] \\ \dots & \\ x_1^*(t), & t \in [t_0, t_1] \end{cases}$$

(ii) the functions x_i^* *are the limits of the successive approximations*

$$x_{m+1}^{n+1}(t) = \varphi(t), \ t \in [t_m, t_{m+1}],$$

$$x_m^{n+1}(t) = \varphi(t_m) + \int_{t_m}^{t} f(s, x_m^n(s), \varphi(s+h))ds, \ t \in [t_{m-1}, t_m],$$

$$x_{m-1}^{n+1}(t) = x_m^*(t_{m-1}) + \int_{t_{m-1}}^{t} f(s, x_{m-1}^n(s), x_m^*(s+h))ds, \ t \in [t_{m-2}, t_{m-1}],$$

$$\dots\dots\dots\dots\dots\dots$$

$$x_1^{n+1}(t) = x_2^*(t_1) + \int_{t_1}^{t} f(s, x_1^n(s), x_2^*(s+h))ds, \ t \in [t_0, t_1]$$

In this section we shall study the following problem:

Problem 6.1. Can we put x_{i+1}^n instead x_{i+1}^*, $i = \overline{1, m}$, in the conclusion (ii) of the Theorem 6.1?

We have

Theorem 6.1. *In the conditions* (C_1') *and* (C_3') *the problem (6.1)+(6.2) has in* $C([t_0, t_{m+1}], \mathbb{B})$ *a unique solution* x^*,

$$x^*(t) := \begin{cases} \varphi(t), & t \in [t_m, t_{m+1}] \\ x_m^*, & t \in [t_{m-1}, t_m] \\ \dots & \\ x_1^*, & t \in [t_0, t_1] \end{cases}$$

and the functions x_i^* *are the limits of the successive approximations*

$$x_{m+1}^{n+1}(t) = \varphi(t),\ t \in [t_m, t_{m+1}],$$

$$x_m^{n+1}(t) = \varphi(t_m) + \int_{t_m}^{t} f(s, x_m^n(s), \varphi(s+h))ds,\ t \in [t_{m-1}, t_m],$$

$$x_{m-1}^{n+1}(t) = x_m^n(t_{m-1}) + \int_{t_{m-1}}^{t} f(s, x_{m-1}^n(s), x_m^n(s+h))ds,\ t \in [t_{m-2}, t_{m-1}],$$

$$\dots\dots\dots\dots\dots\dots$$

$$x_1^{n+1}(t) = x_2^n(t_1) + \int_{t_1}^{t} f(s, x_1^n(s), x_2^n(s+h))ds,\ t \in [t_0, t_1].$$

Proof. The proof follows from the Theorem 5.1. See the proof of the Theorem 4.1.

Remark 6.1. $f(t, \cdot, \cdot) : \mathbb{B} \times \mathbb{B} \to \mathbb{B}$ increasing do not imply that the operators $\widetilde{B}_f$ and $\widetilde{E}_f$ are increasing.

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