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# THE OPEN MAPPING THEOREM AND THE FUNDAMENTAL THEOREM OF ALGEBRA

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**Abstract.** This note is devoted to two classical theorems: the open mapping theorem for analytic functions (OMT) and the fundamental theorem of algebra (FTA). We present a new proof of the first theorem, and then derive the second one by a simple topological argument. The proof is elementary in nature and does not use any kind of integration (neither complex nor real). In addition, it is also independent of the fact that the roots of an analytic function are isolated. The proof is based on either the Banach or Brouwer fixed point theorems. In particular, this shows that one can obtain a proof of the FTA (albeit indirect) which is based on the Brouwer fixed point theorem, an aim which was not reached in the past and later the possibility to achieve it was questioned. We close this note with a simple generalization of the FTA. A short review of certain issues related to the OMT and the FTA is also included.

**Key Words and Phrases**: Analytic function, fixed point, FTA, open mapping, OMT, polynomial.

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