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STRONG CONVERGENCE OF SOME EXPLICIT ITERATIVE PROCESSES WITH MEAN ERRORS FOR A CLASS OF QUASICONTRACTIVE OPERATORS

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Abstract. The purpose of this paper is to establish a strong convergence of two explicit iteration processes with mean errors to a common fixed point for a finite family of quasi-contractive operators in normed spaces or in generalized convex metric spaces. The results presented have generalize and improve the corresponding results of Berinde [1]-[2], Gu Feng [15], Rafiq [3]-[4], Rhoades [14], Şoltuz [10]-[11] and Zamfirescu [17].

Key Words and Phrases: Explicit iteration process with mean errors, common fixed point.

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1. Introduction and Preliminaries

Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be a *quasicontraction* if T has at least one fixed point ($F(T) \neq \emptyset$) and, for each fixed point q, we have

$$d(Tx, q) \leq hd(x, q) \tag{Q}$$

for all $x \in X$, where $h \in (0, 1)$.

Clearly, if T is a quasicontraction then T has a unique fixed point. Supposing that p is another fixed point of T we obtain by (Q) that $d(p, q) \leq hd(p, q)$, so $d(p, q) = 0$, which means $p = q$.

Zamfirescu [17] proved the following result.

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Theorem 1.1. *Let (X,d) be a complete metric space and $T : X \to X$ a mapping for which there exist real numbers a, b, c satisfying $a \in (0,1), b, c \in (0, 1/2)$ such that for each pair $x, y \in X$, at least one of the following conditions holds:*

(i) $d(Tx,Ty) \leq ad(x,y)$,
(ii) $d(Tx,Ty) \leq b[d(x,Tx) + d(y,Ty)]$,
(iii) $d(Tx,Ty) \leq c[d(x,Ty) + d(y,Tx)]$.

Then T has a unique fixed point p and the Picard iteration $\{x_n\}$ defined by

$$x_{n+1} = Tx_n$$

converges to p for any arbitrary fixed $x_0 \in X$.

An operator T satisfying the contractive conditions $(i) - (iii)$ in the above theorem is called a *Z-operator*. The conditions $(i) - (iii)$ can be written in the following equivalent form

$$d(Tx,Ty) \leq h \max\left\{d(x,y), \frac{d(x,Tx) + d(y,Ty)}{2}, \frac{d(x,Ty) + d(y,Tx)}{2}\right\},$$

so it results the class of Z-operators is a subclass of Ćirić mapping [6] satisfying the following condition

$$d(Tx,Ty) \leq h \max\left\{d(x,y), \frac{d(x,Tx) + d(y,Ty)}{2}, d(x,Ty), d(y,Tx)\right\} \quad \text{(CR)}$$

for all $x, y \in X$, considered recently by Rafiq [3]. Rafiq proved that a Ćirić operator satisfies the following condition

$$d(Tx,Ty) \leq hd(x,y) + L \min\{d(x,Tx), d(y,Ty)\} \quad \text{(OS)}$$

for all $x, y \in X$, where $h \in (0,1)$ and $L \geq 0$, introduced by Osilike in [8]

Berinde [1] proved that this class is wider than the class of Z-operators and used the Mann iteration process to approximate fixed points of this class of operators in a normed space. Taking $y = q$ in (OS) we get the condition (Q), so we obtain that Osilike operators, Ćirić operators and Z-operators are quasicontractions.

Takahashi [16] introduced the notion of convex metric spaces and studied the fixed point theory for nonexpansive mappings in such a setting.

Definition 1.1. [16] Let (X,d) be a metric space. A mapping $W : X \times X \times [0,1] \to X$ is said to be a *convex structure* on X if, for each $(x, y, \lambda) \in$

$X \times X \times [0,1]$ and $u \in X$

$$d(u, W(x, y, \lambda) \leq \lambda d(u, x) + (1 - \lambda)d(u, y) \tag{T}$$

The metric space X together with W is called a *convex metric space*.

Definition 1.2. [16] Let X be a convex metric space. A nonempty subset A of X is said to be *convex* if $W(x, y, \lambda) \in A$ whenever $(x, y, \lambda) \in A \times A \times [0, 1]$.

All normed spaces and their convex subsets are convex metric spaces with $W(x, y, \lambda) = \lambda x + (1 - \lambda)y$.

Rafiq [5] introduced the notion of generalized convex metric spaces.

Definition 1.3. [5] Let (X, d) be a metric space. A mapping $W : X \times X \times X \times [0, 1] \times [0, 1] \times [0, 1] \to X$ is said to be a *generalized convex structure* on X if, for each $(x, y, z, a, b, c) \in X \times X \times X \times [0, 1] \times [0, 1] \times [0, 1]$ and $u \in X$,

$$d(u, W(x, y, z; a, b, c) \leq ad(u, x) + bd(u, y) + cd(u, z) \tag{R}$$

where $a+b+c = 1$. The metric space X together with W is called a *generalized convex metric space*.

Definition 1.4. [5] Let X be a generalized convex metric space. A nonempty subset A of X is said to be *generalized convex* if $W(x, y, z; a, b, c) \in A$ whenever $(x, y, z; a, b, c) \in A \times A \times A \times [0, 1] \times [0, 1] \times [0, 1]$.

Clearly every generalized convex metric space is a convex space, every generalized convex set is a convex set. All normed spaces and their generalized convex subsets are generalized convex metric spaces with $W(x, y, z; a, b, c) = ax + by + cz$.

Let D be a nonempty closed generalized convex subset of a generalized convex metric space X, $T : D \to D$ and $T_i : D \to D$ a finite family of mappings $(i = 1, 2, ..., N)$.

Algorithm 1. The Xu-Ori [13] iteration with errors is defined by $x_0 \in D$ and

$$x_{n+1} = W(x_n, T_n x_n, u_n; \alpha_n, \beta_n, \gamma_n), n \geq 0,$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are sequences in [0,1] with $\alpha_n + \beta_n + \gamma_n = 1$ and $T_n = T_{n(modN)}$.

For $N = 1$ we obtained the Xu - Mann iteration with errors [12].

Algorithm 2. The Xu multistep procedure with errors is defined by $x_0 \in D$ and

$$\begin{aligned} x_{n+1} &= W(x_n, Ty_n^1, u_n^1; \alpha_n^1, \beta_n^1, \gamma_n^1), \\ y_n^i &= W(x_n, Ty_n^{i+1}, u_n^{i+1}; \alpha_n^{i+1}, \beta_n^{i+1}, \gamma_n^{i+1}), \\ y_n^{p-1} &= W(x_n, Tx_n, u_n^p; \alpha_n^p, \beta_n^p, \gamma_n^p), \end{aligned} \tag{1.1}$$

where $i = 1, 2, ..., p-2, \{\alpha_n^i\}, \{\beta_n^i\}, \{\gamma_n^i\} \subset [0, 1]$ such that $\alpha_n^i + \beta_n^i + \gamma_n^i = 1$, $i = 1, 2, ..., p$, and $\{u_n^i\}$ are bounded sequences in D.

If X is a normed space, taking $W(x, y, z; a, b, c) = ax + by + cz$ and $\gamma_n^i = 0$ we obtain the multistep procedure introduced by Rhoades and Şoltuz [14] in 2004. In this case, for $p = 3$ we get the Noor procedure, for $p = 2$ we have the Ishikawa procedure [7] and for $p = 1$ we obtain the Mann procedure [9]. If X is a normed space and $W(x, y, z; a, b, c) = ax + by + cz$ then for $p = 2$ we get the Xu - Ishikawa procedure with errors [12] and for $p = 1$ we obtain the Xu - Mann procedure with errors [12].

In order to prove the main results of this paper, we need the following Lemma:

Lemma 1.1. [1] *Suppose that $\{a_n\}, \{b_n\}, \{c_n\}$ are three nonnegative real sequences satisfying the following condition:*

$$a_{n+1} \leq (1 - \lambda_n)a_n + b_n + c_n$$

for all $n \geq n_0$, $\lambda_n \in [0, 1]$, $\sum_{n=0}^{\infty} \lambda_n = \infty$, $b_n = o(\lambda_n)$ and $\sum_{n=0}^{\infty} c_n < \infty$. Then $\lim_{n\to\infty} a_n = 0$.

2. Main results

We are now able to prove our main results in this paper.

Theorem 2.1. *Let D be a nonempty closed generalized convex subset of a generalized convex metric space X. Let $T_i : D \to D$ be a finite family of quasicontractions, $i = 1, 2, ..., N$, with $F = \bigcap F(T_i) \neq \emptyset$. Let $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ be sequences in [0,1] with $\alpha_n + \beta_n + \gamma_n = 1$ and $T_n = T_{n(modN)}$ such that*

(i) $\sum_{n=1}^{\infty} \beta_n = \infty$,
(ii) $\sum_{n=1}^{\infty} \gamma_n < \infty$ *or* $\gamma_n = o(\beta_n)$.

Let $\{u_n\}$ be a bounded sequence in D, $x_0 \in D$ and $\{x_n\}$ the Xu-Ori iteration with errors defined by Algorithm 1. Then $\{x_n\}$ converges strongly to a common fixed point of $\{T_i\}$.

Proof. Let q be a common fixed point of T_i. By (Q) we have

$$\begin{aligned} d(x_{n+1},q) &= d(W(x_n,T_nx_n,u_n;\alpha_n,\beta_n,\gamma_n),q) \\ &\le \alpha_n d(x_n,q)+\beta_n d(T_nx_n,q)+\gamma_n d(u_n,q) \\ &\le \alpha_n d(x_n,q)+\beta_n h_n d(x_n,q)+\gamma_n d(u_n,q) \qquad (2.1) \\ &\le (\alpha_n+\beta_n h)d(x_n,q)+\gamma_n d(u_n,q) \\ &\le (1-\beta_n(1-h))d(x_n,q)+\gamma_n d(u_n,q), \end{aligned}$$

where h_i is the coefficient of quasicontractivity of T_i, $i=1,2,..,N$ and $h=\max\{h_1,h_2,...,h_N\}$.
From the conditions (i)-(ii), using the relation (2.1) and Lemma 1.1 we have $\lim_{n\to\infty} d(x_n,q)=0$, and so $\{x_n\}$ converges strongly to q. □

Corollary 2.1. (Theorem 2.1, [15]) *Let D be a nonempty closed convex subset of a normed space X. Let $T_i : D \to D(i=1,2,...,N)$ be a finite family of operators satisfying the condition (CR) with $F=\bigcap F(T_i)\neq\emptyset$. Let $\{\alpha_n\},\{\beta_n\},\{\gamma_n\}$ be sequences in [0,1] with $\alpha_n+\beta_n+\gamma_n=1$ and $T_n=T_{n(modN)}$ such that*

(i) $\sum_{n=1}^{\infty}\beta_n=\infty$,
(ii) $\sum_{n=1}^{\infty}\gamma_n<\infty$ *or* $\gamma_n=o(\beta_n)$.

Let $\{u_n\}$ be a bounded sequence in D, $x_0\in D$ and $\{x_n\}$ the Xu - Ori iteration with errors defined by

$$x_{n+1}=\alpha_n x_n+\beta_n T_n x_n+\gamma_n u_n, n\ge 1.$$

Then $\{x_n\}$ converges strongly to a common fixed point of $\{T_i\}$.

Proof. Since every Ćirić operator is a quasicontraction, the conclusion of Corollary 2.1 can be obtained from Theorem 2.1 immediately. □

Corollary 2.2. (Theorem 2.2, [15]) *Let D be a nonempty closed convex subset of a normed space X. Let $T_i : D \to D(i=1,2,...,N)$ be a finite family of operators satisfying the condition (CR) with $F=\bigcap F(T_i)\neq\emptyset$. Let $\{\alpha_n\},\{\beta_n\}$ be sequences in [0,1] with $\alpha_n+\beta_n=1$ and $T_n=T_{n(modN)}$ such that*

$$\sum_{n=1}^{\infty}\beta_n=\infty.$$

Let $x_0\in D$ and $\{x_n\}$ the sequence defined by

$$x_{n+1}=\alpha_n x_n+\beta_n T_n x_n, n\ge 1.$$

Then $\{x_n\}$ converges strongly to a common fixed point of $\{T_i\}$.

Proof. Taking $\gamma_n = 0$ in the previous Corollary, the conclusion of Corollary 2.2 can be obtained immediately. □

Corollary 2.3. (Corollary 2.2, [15]) *Let D be a nonempty closed convex subset of a normed space X. Let $T_i : D \to D$ $(i = 1, 2, ..., N)$ be a finite family of operators satisfying the condition (OS) with $F = \bigcap F(T_i) \neq \emptyset$. Let $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ be sequences in [0,1] with $\alpha_n + \beta_n + \gamma_n = 1$ and $T_n = T_{n(modN)}$ such that*

(i) $\sum_{n=1}^{\infty} \beta_n = \infty$,
(ii) $\sum_{n=1}^{\infty} \gamma_n < \infty$ *or* $\gamma_n = o(\beta_n)$.

Let $\{u_n\}$ be a bounded sequence in D, $x_0 \in D$ and $\{x_n\}$ the sequence defined by

$$x_{n+1} = \alpha_n x_n + \beta_n T_n + \gamma_n u_n, n \geq 1.$$

Then $\{x_n\}$ converges strongly to a common fixed point of $\{T_i\}$.

Proof. Since every Osilike operator is a quasicontraction, the conclusion of Corollary 2.2 can be obtained from Theorem 2.1 immediately. □

Theorem 2.2. *Let D be a nonempty closed generalized convex subset of a generalized convex metric space X. Let $T : D \to D$ be a quasicontraction. Let $\{\alpha_n^i\}, \{\beta_n^i\}, \{\gamma_n^i\}$ be sequences in [0,1] with $\alpha_n^i + \beta_n^i + \gamma_n^i = 1$, $i = 1, 2, ..., p$ and $\{u_n^i\}$ bounded sequence in D, $i = 1, 2, ..., p$. Suppose further that $x_0 \in D$ and $\{x_n\}$ is the multistep procedure with errors defined by Algorithm 2. If the following conditions*

(i) $\sum_{n=1}^{\infty} \beta_n^1 = \infty$,
(ii) $\sum_{n=1}^{\infty} \gamma_n^1 < \infty$ *or* $\gamma_n^1 = o(\beta_n^1)$,
(iii) $\gamma_n^2 \to 0, \beta_n^2 \to 0$ *or* $\gamma_n^i \to 0, i = 1, 2, ..., s, \beta_n^s \to 0, s \leq p$,

hold, then $\{x_n\}$ converges strongly to the unique fixed point of T.

Proof. Since T is a quasicontraction we have by (R), (Q) and Algorithm 2:

$$\begin{aligned} d(y_n^{p-1}, q) &\leq \alpha_n^p d(x_n, q) + \beta_n^p d(Tx_n, q) + \gamma_n^p d(u_n^p, q) \\ &\leq (\alpha_n^p + h\beta_n^p) d(x_n, q) + \gamma_n^p d(u_n^p, q) \qquad (2.2) \\ &\leq d(x_n, q) + \gamma_n^p d(u_n^p, q) \end{aligned}$$

Also

$$
\begin{aligned}
d(y_n^{p-2}, q) &\le \alpha_n^{p-1} d(x_n, q) + \beta_n^{p-1} d(Ty_n^{p-1}, q) + \gamma_n^{p-1} d(u_n^{p-1}, q) \\
&\le \alpha_n^{p-1} d(x_n, q) + h\beta_n^{p-1} d(y_n^{p-1}, q) + \gamma_n^{p-1} d(u_n^{p-1}, q) \\
&\le (\alpha_n^{p-1} + h\beta_n^{p-1}) d(x_n, q) + h\beta_n^{p-1}\gamma_n^p d(u_n^p, q) + \gamma_n^{p-1} d(u_n^{p-1}, q) \\
&\le d(x_n, q) + \beta_n^{p-1}\gamma_n^p d(u_n^p, q) + \gamma_n^{p-1} d(u_n^{p-1}, q).
\end{aligned} \tag{2.3}
$$

Inductively, we obtain that

$$
\begin{aligned}
d(y_n^1, q) \le d(x_n, q) + \beta_n^2\beta_n^3 ... \beta_n^{p-1}\gamma_n^p d(u_n^p, q) + \\
\beta_n^2\beta_n^3 ... \beta_n^{p-2}\gamma_n^{p-1} d(u_n^{p-1}, q) + ... + \gamma_n^2 d(u_n^2, q),
\end{aligned} \tag{2.4}
$$

so we have

$$
\begin{aligned}
d(x_{n+1}, q) &\le \alpha_n^1 d(x_n, q) + \beta_n^1 d(Ty_n^1, q) + \gamma_n^1 d(u_n^1, q) \\
&\le \alpha_n^1 d(x_n, q) + h\beta_n^1 d(y_n^1, q) + \gamma_n^1 d(u_n^1, q) \\
&\le (\alpha_n^1 + h\beta_n^1) d(x_n, q) + h\beta_n^1 [\beta_n^2\beta_n^3 ... \beta_n^{p-1}\gamma_n^p d(u_n^p, q) \\
&+ \beta_n^2\beta_n^3 ... \beta_n^{p-2}\gamma_n^{p-1} d(u_n^{p-1}, q) + ... + \gamma_n^2 d(u_n^2, q)] + \gamma_n^1 d(u_n^1, q) \\
&\le [1 - \beta_n^1(1-h)] d(x_n, q) + \gamma_n^1 d(u_n^1, q) + h\beta_n^1 [\beta_n^2\beta_n^3 ... \beta_n^{p-1}\gamma_n^p d(u_n^p, q) \\
&+ \beta_n^2\beta_n^3 ... \beta_n^{p-2}\gamma_n^{p-1} d(u_n^{p-1}, q) + ... + \gamma_n^2 d(u_n^2, q)].
\end{aligned} \tag{2.5}
$$

From the condition (i) - (iii), using Lemma 1.1 and the relation from above we get $\lim_{n\to\infty} d(x_n, q) = 0$, and so $\{x_n\}$ converges strongly to q. □

Corollary 2.4. (Theorem 3, [3]) *Let D be a nonempty closed generalized convex subset of a generalized convex metric space X. Let $T : D \to D$ be an operator satisfying the condition (CR). The sequence $\{x_n\}$ defined by $x_0 \in D$ and*

$$x_{n+1} = W(x_n, Tx_n, u_n; \alpha_n, \beta_n, \gamma_n), n \ge 0,$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are sequences in [0,1] with $\alpha_n + \beta_n + \gamma_n = 1$ and $\{u_n\}$ is a bounded sequence in D, converges strongly to the unique fixed point of T provided that

(i) $\sum_{n=1}^{\infty} \beta_n = \infty$,
(ii) $\gamma_n = o(\beta_n)$.

Proof. Taking $p = 1$ in Theorem 2.2 we obtain immediately the conclusion of Corollary 2.4. □

Corollary 2.5. *Let D be a nonempty closed convex subset of a normed space X. Let $T : D \to D$ be an operator with $F(T) \neq \emptyset$ and satisfying the condition (Z). Let $\{\alpha_n^i\}$, $\{\beta_n^i\} \subset [0,1]$ be such that $\alpha_n^i + \beta_n^i = 1$, $i = 1, 2, ..., p$ and $\{u_n^i\}$ bounded sequences in D. Suppose that $x_0 \in D$ and $\{x_n\}$ is the multistep procedure defined by*

$$\begin{aligned} x_{n+1} &= \alpha_n^1 x_n + \beta_n^1 T y_n^1, \\ y_n^i &= \alpha_n^{i+1} x_n + \beta_n^{i+1} T y_n^{i+1}, \\ y_n^{p-1} &= \alpha_n^p x_n + \beta_n^p T x_n. \end{aligned} \tag{2.6}$$

If the following condition

$$\sum_{n=1}^{\infty} \beta_n^1 = \infty,$$

holds, then $\{x_n\}$ converges strongly to the unique fixed point of T.

Proof. Taking $W(x, y, z; a, b, c) = ax + by + cz$ and $\gamma_n^i = 0, i = 1, 2, ..., p$ in Theorem 2.2 we get the conclusion of Corollary 2.5. □

Corollary 2.6. (Theorem 2, [2]) *Let D be a nonempty closed convex subset of a normed space X. Let $T : D \to D$ be an operator with $F(T) \neq \emptyset$ and satisfying the condition (Z). Let $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$, $x_0 \in D$ and $\{x_n\}$ the Ishikawa procedure defined by*

$$\begin{aligned} x_{n+1} &= (1 - \alpha_n) x_n + \alpha_n T y_n, \\ y_n &= (1 - \beta_n) x_n + \beta_n T x_n. \end{aligned} \tag{2.7}$$

If the following condition

$$\sum_{n=1}^{\infty} \alpha_n = \infty,$$

holds, then $\{x_n\}$ converges strongly to the unique fixed point of T.

Proof. Taking $p = 1$ in Corollary 2.5 we get the conclusion of Corollary 2.6. □

Remark 2.1. Under conditions of Corollary 2.6 Şoltuz [10] proved that the Mann iteration(Ishikawa iteration with $\beta_n = 0$) converges strongly to the unique fixed point of T if and only if the Ishikawa iteration converges strongly to the unique fixed point of T. But, by Corollary 2.6 it results that both always converge to the unique fixed point of T.

Remark 2.2. Under conditions of Corollary 2.5 Şoltuz [11] proved that the multistep procedure converges strongly to the unique fixed point of T if and only if the Mann iteration converges strongly to the unique fixed point of T. But, by Corollary 2.5 and 2.6 it results that both always converge to the unique fixed point of T.

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