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FIXED POINTS FOR MULTIVALUED ĆIRIĆ-TYPE OPERATORS

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Abstract. The following open question was proposed by I.A. Rus.
OPEN QUESTION. *Let (X,d) be a metric space and $Y \subseteq X \times X$. The operator $T : X \to P_{cl}(X)$ is called a multivalued (Y,a)-contraction if $a \in]0,1[$ and for each $(x_1,x_2) \in Y$ we have:*

$$H(T(x_1),T(x_2)) \leq a \cdot d(x_1,x_2).$$

Construct a fixed point theory for multivalued (Y,a)-contractions.

The purpose of this paper is to give a partial answer to this problem in the frame of an ordered metric space. More precisely, we present some results on the existence and data dependence of the fixed points for multivalued operators of Ćirić type.

Key Words and Phrases: Fixed point, multivalued operator, Ćirić type generalized contraction.
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1. Introduction

Throughout this paper, the standard notations and terminologies in nonlinear analysis are used. For the convenience of the reader we recall some of them.

Let (X,d) be a metric space. We will use the following symbols:
$P(X) := \{Y \subset X |\ Y \text{ nonempty}\}$, $P_{cl}(X) := \{Y \in P(X) |\ Y \text{ is closed}\}$.

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Let A and B be nonempty subsets of the metric space (X, d). The gap between these sets is

$$D(A, B) = \inf\{d(a, b) |\ a \in A,\ b \in B\}.$$

In particular, $D(x_0, B) = D(\{x_0\}, B)$ (where $x_0 \in X$) is called the distance from the point x_0 to the set B.

The Pompeiu-Hausdorff generalized distance between the nonempty closed subsets A and B of the metric space (X, d) is defined by the following formula:

$$H(A, B) := \max\{\sup_{a\in A} \inf_{b\in B} d(a, b), \sup_{b\in B} \inf_{a\in A} d(a, b)\}.$$

The symbol $T : X \multimap Y$ means $T : X \to P(Y)$, i.e., T is a multivalued operator from X to Y. We will denote by $Graph(T) := \{(x, y) \in X \times Y | y \in T(x)\}$ is the graph of T. Recall that the multivalued operator is called closed if $Graph(T)$ is closed in $X \times Y$. For $T : X \to P(X)$ we denote by $FixT := \{x \in X |\ x \in T(x)\}$ the fixed point set of the multivalued operator T.

If (X, d) is a metric space, then an operator $T : X \to P_{cl}(X)$ is a multivalued weakly Picard operator (see [4]) if for each $x \in X$ and each $y \in T(x)$ there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in X such that:

i) $x_0 = x,\ x_1 = y$

ii) $x_{n+1} \in T(x_n)$, for all $n \in \mathbb{N}$

iii) the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent and its limit is a fixed point of T.

A sequence $(x_n)_{n\in\mathbb{N}}$ in X satisfying the condition (ii) from the previous definition is called the sequence of successive approximations of T starting from $x_0 \in X$.

1.1. **The aim of the paper.** The purpose of this work is to consider the following open question of I. A. Rus [1]:

Open question. *Let (X, d) be a metric space and $Y \subseteq X \times X$. The operator $T : X \to P_{cl}(X)$ is called a multivalued (Y, a)-contraction if:*

(1) $a \in]0, 1[$ *and* $H(T(x_1), T(x_2)) \le a \cdot d(x_1, x_2)$, *for each* $(x_1, x_2) \in Y$.

Construct a fixed point theory for multivalued (Y, a)-contractions.

Of course, in the previous definition, one can consider generalized type conditions for T, such as:

(2) there exists $\varphi : \mathbb{R}^5_+ \to \mathbb{R}_+$ such that for each $(x_1, x_2) \in Y$

$$H(T(x_1), T(x_2)) \leq$$

$$\leq \varphi(d(x_1, x_2), D(x_1, T(x_1)), D(x_2, T(x_2)), D(x_1, T(x_2)), D(x_2, T(x_1))).$$

2. Multivalued (Y, a)-contractions in ordered metric spaces

Let $(X, \leq)$ be an partially ordered set.

Denote $X_{\leq} := \{(x, y) \in X \times X | x \leq y \text{ or } y \leq x\}$.

Definition 2.1. Let X be a nonempty set. Then, by definition $(X, d, \leq)$ is an ordered metric space if and only if:

(i) (X, d) is a metric space

(ii) $(X, \leq)$ is a partially ordered set

(iii) $(x_n)_{n\in\mathbb{N}} \to x$, $(y_n)_{n\in\mathbb{N}} \to y$ and $x_n \leq y_n$, for each $n \in \mathbb{N} \Rightarrow x \leq y$.

The first main result of this paper is the following fixed point theorem.

Theorem 2.1 *Let $(X, d, \leq)$ be a complete ordered metric and $T : X \to P_{cl}(X)$ be a multivalued operator. Suppose that the following assertions hold:*

(i) there exists $x_0 \in X$ such that, if $y \in T(x_0)$ then $(x_0, y) \in X_{\leq}$;

(ii) for each $x, y \in X$ with $(x, y) \notin X_{\leq}$ there exists $c(x, y) \in X$ such that $(x, c(x, y)) \in X_{\leq}$ and $(y, c(x, y)) \in X_{\leq}$;

(iii) if $(x, y) \in X_{\leq}$ then ($u \in T(x)$ and $v \in T(y)$ imply $(u, v) \in X_{\leq}$);

(iv) T is closed;

(v) there exists $a \in]0, \frac{1}{2}[$ such that

$$H(T(x), T(y)) \leq$$

$$\leq a \cdot \max\{d(x, y), D(x, T(x)), D(y, T(y)), D(x, T(y)), D(y, T(x))\},$$

for all $x, y \in X$, with $x \leq y$.

Then, for each $x \in X$ there exists a sequence $(x_n)_{n\in\mathbb{N}}$ of successive approximations for T starting from x, that converges to a fixed point of T.

Proof. Let $q > 1$ and $x_1 \in T(x_0)$. Then there is $x_2 \in T(x_1)$ such that $d(x_1, x_2) \leq q \cdot H(T(x_0), T(x_1))$. From (i) we have $(x_0, x_1) \in X_{\leq}$. Using (v) we get

$$H(T(x_0), T(x_1)) \leq$$

$$\leq a \cdot \max\{d(x_0, x_1), D(x_0, T(x_0)), D(x_1, T(x_1)), D(x_0, T(x_1)), D(x_1, T(x_0))\} \leq$$

$$\leq a \cdot \max\{d(x_0, x_1), D(x_1, T(x_1)), d(x_0, x_1) + D(x_1, T(x_1))\} \leq$$

$$\leq a[d(x_0, x_1) + H(T(x_0), T(x_1))].$$

Hence

$$H(T(x_0), T(x_1)) \leq \frac{a}{1-a} \cdot d(x_0, x_1).$$

Then

$$d(x_1, x_2) \leq \frac{qa}{1-a} \cdot d(x_0, x_1).$$

Using (iii) we obtain $(x_1, x_2) \in X_{\leq}$. We can construct the sequence $(x_n)_{n\in\mathbb{N}}$, with the following properties:

(a) $x_{n+1} \in T(x_n)$, for any $n \in \mathbb{N}$

(b) $d(x_{n+1}, x_n) \leq (\dfrac{qa}{1-a})^n \cdot d(x_0, x_1)$, for any $n \in \mathbb{N}^*$.

Choosing $q \in]1, \frac{1-a}{a}[$ we obtain that $(x_n)_{n\in\mathbb{N}}$ is Cauchy, and hence convergent to $x^* \in X$. From (a) and (iv) we get $x^* \in F_T$. Thus $F_T \neq \emptyset$.

Finally, let us show that for each $x \in X$ there is a sequence $(z_n)_{n\in\mathbb{N}}$ of successive approximations for T starting from $z_0 = x$ such that (z_n) converges to a fixed point of T. We distinguish two cases.

Case A. Let $z_0 \in X$ such that $(x_0, z_0) \in X_{\leq}$. Then there exists $z_1 \in T(z_0)$ such that $d(x_1, z_1) \leq q \cdot H(T(x_0), T(z_0)) \leq \frac{qa}{1-a} \cdot d(x_0, z_0)$. Since $(x_1, z_1) \in X_{\leq}$ we obtain that there exists a sequence $(z_n)_{n\in\mathbb{N}}$ with $z_{n+1} \in T(z_n)$, for $n \in \mathbb{N}$ and $d(x_{n+1}, z_{n+1}) \leq \frac{qa}{1-a} \cdot d(x_n, z_n)$. Since $\frac{qa}{1-a} < 1$, we obtain $(z_n) \to x^* \in T(x^*)$.

Case B. Let $z_0 \in X$ be such that $(x_0, z_0) \notin X_{\leq}$. Then, from (ii) it follows that there is $c(x_0, z_0) \in X$ such that:

(c)$(x_0, c(x_0, z_0)) \in X_{\leq}$

and

(d) $(z_0, c(x_0, z_0)) \in X_{\leq}$.

From (c), in a similar way to Case A., we can construct a sequence of succssive approximations for T starting from $c(x_0, z_0)$, which converges to x^*. From (d), the above conclusion and in a similar way to Case A., we deduce that there is a sequence (z_n) of successive approximations for T starting from z_0 such that $(z_n) \to x^*$. The proof is complete. □

In a similar way to [4], we obtain the second result of the paper, which is a data dependence theorem for Ćirić type multivalued operators in complete ordered metric spaces.

Theorem 2.2 *Let $(X, d, \leq)$ be a complete ordered metric space and $T_1, T_2 : X \to P_{cl}(X)$ two multivalued operators. Suppose that the following assertions hold:*

(i) there exists $x_0 \in X$ such that, if $y \in T_1(x_0)$ then $(x_0, y) \in X_{\leq}$;

(ii) there exists $y_0 \in X$ such that, if $y \in T_2(y_0)$ then $(y_0, y) \in X_{\leq}$;

(iii) for each $x, y \in X$ with $(x, y) \notin X_{\leq}$ there exists $c(x, y) \in X$ such that $(x, c(x, y)) \in X_{\leq}$ and $(y, c(x, y)) \in X_{\leq}$;

(iv) if $(x, y) \in X_{\leq}$ then ($u \in T_i(x)$ and $v \in T_i(y)$, imply $(u, v) \in X_{\leq}$), for $i \in \{1, 2\}$;

(v) T_i is closed, for $i \in \{1, 2\}$;

(vi) there exist $a_i \in]0, \frac{1}{2}[$ such that

$$H(T_i(x), T_i(y)) \leq$$

$$\leq a_i \cdot \max\{d(x, y), D(x, T(x)), D(y, T(y)), D(x, T(y)), D(y, T(x))\},$$

for all $x, y \in X$ with $x \leq y$, for $i \in \{1, 2\}$;

(vii) if $x_0^* \in Fix(T_1)$ then, for each $y \in T_2(x_0^*)$ we have $(x_0^*, y) \in X_{\leq}$;

(viii) if $y_0^* \in Fix(T_2)$ then, for all $y \in T_1(y_0^*)$ we have $(y_0^*, y) \in X_{\leq}$;

(ix) there exists $\eta > 0$ such that $H(T_1(x), T_2(x)) \leq \eta$, for each $x \in X$.

Then:

(a) $Fix(T_i) \neq \emptyset$, for $i \in \{1, 2\}$

(b) $H(Fix(T_1), Fix(T_2)) \leq \frac{1-\min\{a_1,a_2\}}{1-2\max\{a_1,a_2\}} \cdot \eta$.

Open question. Is an open problem if the multivalued operators in Theorem 2.1-2.2 are multivalued weakly Picard.

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