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FIXED POINTS FOR DIRECTIONAL CONTRACTIONS

PETRU TÜNDE PETRA

Department of Applied Mathematics
Babeş-Bolyai University
Kogălniceanu Str., 400084, Cluj-Napoca, Romania
E-mail: ptpetru@econ.ubbcluj.ro

Abstract. The aim of this note is to present a fixed point result for a multivalued directional φ-contraction. Our result extends the main theorem in Uderzo [4].
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1. Introduction

Let (E, d) be a metric space. Given points $x, y \in E$, the open segment (x, y) defined by x and y is the set of points z in E (if any) distinct from x and y and satisfying $d(x, z) + d(z, y) = d(x, y)$.

A single valued map $f : X \to X$ is said to be a *directional contraction* provided f is continuous and there exists a number $\sigma \in (0, 1)$ with the following property: whenever $v \in X$ is such that $f(v) \neq v$, there exists $w \in (v, f(v))$ such that

$$d(f(v), f(w)) \leq \sigma d(v, w).$$

The notion of directional contractions was introduced by Clarke (see [1]). The following result was given by Clarke too.

Theorem 1.1 (Clarke [1]). *Let (E, d) be a complete metric space. Then every directional single-valued contraction has a fixed point.*

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2. Directional φ -contractions

Let (E,d) be a metric space. Let $\mathcal{P}(E)$ be the set of all subsets of E, and $P(E)$ the set of all nonempty subsets of E, i.e. $P(E)=\{Y\,|\varnothing\neq Y\subseteq E\}$. We denote by $P_b(E)$ the set of all nonempty and bounded subsets, $P_{cl,b}(E)$ the set of all nonempty closed and bounded subsets, $P_{cp}(E)$ the set of all nonempty compact subsets.

Definition 2.1. *A multivalued operator $F:E\to P_{cl,b}(E)$ is said to be a* directional contraction *provided F is upper semicontinuous with respect to the Pompeiu-Hausdorff distance H and there exists a number $\sigma\in(0,1)$ with the following property: whenever $v\in E$ is such that $v\notin F(v)$ and $u\in F(v)$, there exists $w\in(v,u)$ such that*

$$H(F(v),F(w))\leq\sigma\cdot d(v,w).$$

Definition 2.2. *Given an increasing function $\varphi:\mathbb{R}_+\to\mathbb{R}_+$ with the property $\varphi(t)<t$ for each $t>0$, a multivalued operator F on a metric space (E,d) is said to be a* multivalued φ-contraction *if*

$$H(F(x),F(y))\leq\varphi(d(x,y)),\ \textit{for } x,y\in E.$$

Definition 2.3. *Let $K\in P_{cl}(E)$. A map $F:K\to P_{cl,b}(E)$ is called a **multivalued directional** $\varphi-$**contraction** if there exist a strictly increasing mapping $a:]0,+\infty[\to]0,+\infty[$ (with $a(0)=0$ and $\lim\limits_{t\to+\infty}a(t)=+\infty$) and a comparison function $\varphi:]0,+\infty[\to]0,+\infty[$ such that for every $x\in K$ with $x\notin F(x)$ there exists $y\in K\setminus\{x\}$ such that*

$$a(d(x,y))+D(y,F(x))\leq D(x,F(x))$$

and

$$\rho(F(y),F(x))\leq\varphi(d(x,y)).$$

For the proof of the main result we will need the following well-known theorem (see [2]):

Lemma 2.1 (Ekeland's ε-Variational Principle)**.** *Let (E,d) be a complete metric space and let $F:E\to\mathbb{R}\cup\{+\infty\}$ be a lower semicontinuous function which is bounded below. If u is a point in E satisfying*

$$F(u)<\inf_E F+\varepsilon$$

for some $\varepsilon > 0$, then, for every $\lambda > 0$, exists a point v in E such that

i) $F(v) \leq F(u)$;

ii) $d(u, v) \leq \lambda$;

iii) *For all $w \neq v$ in E, one has*

$$F(v) < F(w) + \frac{\varepsilon}{\lambda} \cdot d(v, w).$$

3. Main result

Our result extends the main theorem of Uderzo in [4].

Theorem 3.1 (Uderzo [4])**.** *Let K be a closed nonempty subset of a complete metric space (E, d) and let $F : K \to P_{cl,b}(E)$ be an u.s.c. directional multivalued $k(\cdot)$-contraction. Assume that there exists $x_0 \in K$, $\delta > 0$ and $\alpha \in (0, 1]$ such that $D(x_0, F(x_0)) \leq \alpha\delta$ and*

$$\sup_{t \in (0,\delta]} k(t) < \inf_{t \in (0,\delta]} a(t)$$

where $a :]0, +\infty[\to [\alpha, 1]$ and $k : (0, +\infty) \to [0, 1)$ such that for every $x \in K$, with $x \notin F(x)$, there is $y \in K \setminus \{x\}$ satisfying the inequalities

$$a(d(x, y)) \cdot d(x, y) + D(y, F(x)) \leq D(x, F(x))$$

and

$$\rho(F(y), F(x)) \leq k(d(x, y)) \cdot d(x, y).$$

Then F admits a fixed point.

Our main result is:

Theorem 3.2. *Let (E, d) be a complete metric space, $K \in P_{cl}(E)$ and let $F : K \to P_{cl,b}(E)$ be an u.s.c. directional $\varphi-$contraction. Assume that there exists $x_0 \in K$, $\delta > 0$ and $\alpha \in]0, 1]$ such that $D(x_0, F(x_0)) \leq \delta\alpha$, $a(\delta) \geq \alpha\delta$ and there exists $\beta > 0$ such that*

$$\sup_{t \in]0,\delta]} \varphi(t) - \inf_{t \in]0,\delta]} a(t) \leq -\beta\delta.$$

Then $FixF \neq \emptyset$.

Proof. By hypothesis, there exists $\beta > 0$ and $\delta > 0$ such that

$$\sup_{t \in]0,\delta]} (\varphi(t) - a(t)) \leq \sup_{t \in]0,\delta]} \varphi(t) - \inf_{t \in]0,\delta]} a(t) \leq -\beta\delta \tag{3.1}$$

Since F is u.s.c., $f : K \to \mathbb{R}_+$, where $f(x) = D(x, F(x))$ is l.s.c. in K. Since K is complete is equipped with the metric induced by d, and $f(x_0) \leq \delta\alpha := \varepsilon$, then it is possible to apply Ekeland variational principle around x_0, to get for any $\lambda > 0$ the existance of $x_\lambda \in K$ such that

$$f(x_\lambda) \leq f(x_0), \tag{3.2}$$

$$d(x_0, x_\lambda) \leq \lambda, \tag{3.3}$$

$$f(x_\lambda) < f(x) + \frac{\alpha\delta}{\lambda} \cdot d(x_\lambda, x), \forall x \in K \backslash \{x_\lambda\}. \tag{3.4}$$

Suppose that $f(x_\lambda) > 0,\ \forall \lambda > 0$.

Since F is directional φ-contraction, we have that there exists $y \in K \backslash \{x_\lambda\}$ such that

$$a(d(x_\lambda, y)) + D(y, F(x_\lambda)) \leq D(x_\lambda, F(x_\lambda)) = f(x_\lambda) \tag{3.5}$$

and

$$\rho(F(y), F(x_\lambda)) \leq \varphi(d(x_\lambda, y)) \tag{3.6}$$

From (3.5) we have that $a(d(x_\lambda, y)) \leq f(x_\lambda) - D(y, F(x_\lambda)) \leq f(x_\lambda)$, thus

$$0 < d(x_\lambda, y) \leq a^{-1}(f(x_\lambda)) \leq a^{-1}(f(x_0)) \leq a^{-1}(\alpha\delta) \leq \delta$$

and

$$D(y, F(x_\lambda)) \leq f(x_\lambda) - a(d(x_\lambda, y)).$$

So

$$\begin{aligned} f(y) &= D(y, F(y)) \leq D(y, F(x_\lambda)) + \rho(F(y), F(x_\lambda)) \leq \\ &\leq f(x_\lambda) - a(d(x_\lambda, y)) + \varphi(d(x_\lambda, y)). \end{aligned}$$

Putting $x := y$ in (3.4) and $\lambda := \dfrac{2\alpha\delta}{\beta}$ we obtain

$$\begin{aligned} f(x_\lambda) &< f(y) + \frac{\alpha\delta}{\lambda} \cdot d(x_\lambda, y) \leq \\ &\leq f(x_\lambda) - a(d(x_\lambda, y)) + \varphi(d(x_\lambda, y)) + \frac{\beta}{2} \cdot d(x_\lambda, y) \leq \\ &\leq f(x_\lambda) - \beta\delta + \frac{\beta}{2} \cdot \delta < f(x_\lambda). \end{aligned}$$

Contradiction. Thus $f(x_\lambda) = 0$. □

Remark 3.1. *If we are in the following particular case* $a(t) = A(t) \cdot t$ *and* $\varphi(t) = k(t) \cdot t$ *we regain Uderzo's theorem in* [4].

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