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TIME PERIODIC SOLUTIONS FOR A CLASS OF NONLINEAR BOUNDARY VALUE PROBLEMS

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Abstract. Using some results from the theory of monotone operators and a fixed point theorem due to F.E. Browder and W.V. Petryshyn, we prove the existence of time periodic solutions to a class of nonlinear hyperbolic problems, on positive semi-axis of spatial variable, which have applications in integrated circuits modelling.

Key Words and Phrases: Hyperbolic system, boundary condition, Cauchy problem, monotone operator, periodic solution.

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1. Introduction

We consider the following hyperbolic partial differential system

$$\text{(S)} \qquad \begin{cases} \dfrac{\partial u}{\partial t}(t,x) + \dfrac{\partial v}{\partial x}(t,x) + \alpha(x,u) = f(t,x) \\ \dfrac{\partial v}{\partial t}(t,x) + \dfrac{\partial u}{\partial x}(t,x) + \beta(x,v) = g(t,x), \end{cases}$$

$$t > 0, \;\; x > 0,$$

with the boundary condition

$$\text{(BC)} \qquad \begin{pmatrix} u(t,0) \\ S(w'(t)) \end{pmatrix} \in -G \begin{pmatrix} v(t,0) \\ w(t) \end{pmatrix} + B(t), \;\; t > 0.$$

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The unknown functions u, v and also the functions f, g are the vectorial ones depending on $(t,x) \in I\!R_+ \times I\!R_+$ with values in $I\!R^n$, and the unknown function w is a vectorial one depending on $t \in I\!R_+$ with values in $I\!R^m$. The functions α and β are of the form $\alpha(x,u) = \mathrm{col}(\alpha_1(x,u_1),\dots,\alpha_n(x,u_n))$, $\beta(x,v) = \mathrm{col}(\beta_1(x,v_1),\dots,\beta_n(x,v_n))$, S is a positive diagonal matrix, G is an operator in the space $I\!R^{n+m}$, which satisfy some assumptions and $B(t) = \mathrm{col}(b_1(t),\dots,b_{n+m}(t)) \in I\!R^{n+m}$, for all $t > 0$.

This problem has applications in the theory of integrated circuits (see [7], [11], [12] and their references). The existence, uniqueness and asymptotic behavior of the strong and weak solutions of the problem (S)+(BC) with the initial data

$$\text{(IC)} \qquad \begin{cases} u(0,x) = u_0(x), \;\; v(0,x) = v_0(x), \;\; x > 0, \\ w(0) = w_0, \end{cases}$$

have been investigated in [10], [11]. The system (S) for $x \in (0,1)$ and $t > 0$, with the boundary condition

$$\begin{pmatrix} u(t,0) \\ -u(t,1) \\ S(w'(t)) \end{pmatrix} \in -G \begin{pmatrix} v(t,0) \\ v(t,1) \\ w(t) \end{pmatrix} + B(t), \;\; t > 0,$$

and the initial data

$$u(0,x) = u_0(x), \;\; v(0,x) = v_0(x), \;\; x \in (0,1), \quad w(0) = w_0,$$

has been investigated in [7], [11] for the existence, uniqueness and asymptotic behavior of the solutions, in [8], [11] for the existence of periodic solutions, and in [9] for the existence of almost-periodic solutions.

In this paper we shall present some existence results for the time periodic solutions of the problem (S)+(BC), in two different cases $B(t) = \text{const.}$ and $B(t) \neq \text{const.}$ We shall use several results from the theory of monotone operators and nonlinear evolution equations of monotone type (see the monographs [1], [2], [5], [6]), and also a fixed point theorem due to F.E. Browder and W.V. Petryshyn (see [3]).

We introduce the assumptions that we shall use in the sequel

(A1) a) The functions $x \to \alpha_k(x,p)$ and $x \to \beta_k(x,p)$ are measurable on $I\!R_+$, for any fixed $p \in I\!R$. Besides, the functions $p \to \alpha_k(x,p)$ and

$p \to \beta_k(x,p)$ are continuous and nondecreasing from $\mathbb{R}$ into $\mathbb{R}$, for a.a. $x \in \mathbb{R}_+$, $k = \overline{1,n}$.

b) There exist $a_k,\ b_k > 0,\ k = \overline{1,n}$ and the functions $\varphi_k,\ \psi_k \in L^2(\mathbb{R}_+;\mathbb{R})$, $k = \overline{1,n}$ such that

$$|\alpha_k(x,p)| \le a_k|p| + \varphi_k(x), \quad |\beta_k(x,p)| \le b_k|p| + \psi_k(x),$$

for a.a. $x \in \mathbb{R}_+$, for all $p \in \mathbb{R}$, $k = \overline{1,n}$.

c) There exist $c_k,\ d_k > 0,\ k = \overline{1,n}$ and the functions $\xi_k,\ \eta_k \in L^2(\mathbb{R}_+;\mathbb{R}_+)$, $k = \overline{1,n}$ such that

$$|\alpha_k(x,p)| \ge c_k|p| - \xi_k(x), \quad |\beta_k(x,p)| \ge d_k|p| - \eta_k(x),$$

for a.a. $x \in \mathbb{R}_+$, for all $p \in \mathbb{R}$, $k = \overline{1,n}$.

(A2) a) $G : D(G) \subset \mathbb{R}^{n+m} \to \mathbb{R}^{n+m}$ is a maximal monotone operator (possibly multivalued). Moreover, G can be split in

$$G = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix},$$

where $G_{11} : D(G_{11}) \subset \mathbb{R}^n \to \mathbb{R}^n$, $G_{12} : D(G_{12}) \subset \mathbb{R}^m \to \mathbb{R}^n$, $G_{21} : D(G_{21}) \subset \mathbb{R}^n \to \mathbb{R}^m$, $G_{22} : D(G_{22}) \subset \mathbb{R}^m \to \mathbb{R}^m$, and

$G(\text{col}(x^a, x^b)) = \text{col}(G_{11}(x^a) + G_{12}(x^b), G_{21}(x^a) + G_{22}(x^b))$, for all $x \in D(G)$, $x = \text{col}(x^a, x^b) \in \mathbb{R}^n \times \mathbb{R}^m$.

b) There exists $\zeta_1 > 0$ such that for all $x, y \in D(G)$, $x = \text{col}(x^a, x^b)$, $y = \text{col}(y^a, y^b) \in \mathbb{R}^n \times \mathbb{R}^m$ and for all $w_1 \in G(x)$, $w_2 \in G(y)$ we have

$$\langle w_1 - w_2,\ x - y\rangle_{\mathbb{R}^{n+m}} \ge \zeta_1 \parallel x^b - y^b \parallel^2_{\mathbb{R}^m} .$$

c) There exists $\zeta_2 > 0$ such that for all $x, y \in D(G)$ and all $w_1 \in G(x), w_2 \in G(y)$ we have

$$\langle w_1 - w_2,\ x - y\rangle_{\mathbb{R}^{n+m}} \ge \zeta_2 \parallel x - y \parallel^2_{\mathbb{R}^{n+m}} .$$

($\| \cdot \|_{\mathbb{R}^n}$ and $\langle \cdot, \cdot \rangle_{\mathbb{R}^n}$ are the euclidian norm and corresponding scalar product in $\mathbb{R}^n$).

(A3) $S = \text{diag}(s_1, \ldots, s_m)$ with $s_j > 0$, $j = \overline{1,m}$.

The above assumption (A2)a is a technical one and it generalizes the matrix case.

2. Preliminary results

We shall write our problem (S)+(BC) as an evolution equation in a certain Hilbert space. For this aim, let us consider the Hilbert *spaces* $X = (L^2(I\!R_+; I\!R^n))^2$, $I\!R^m$ and $Y = X \times I\!R^m$ with the corresponding scalar products

$$\langle f, g\rangle_X = \langle f_1, g_1\rangle_{L^2(I\!R_+;I\!R^n)} + \langle f_2, g_2\rangle_{L^2(I\!R_+;I\!R^n)},$$
$$f = \mathrm{col}(f_1, f_2),\ \ g = \mathrm{col}(g_1, g_2),$$
$$\langle x, y\rangle_s = \sum_{i=1}^{m} s_i x_i y_i,\ \ x,\, y \in I\!R^m,$$
$$\langle \begin{pmatrix} f \\ x \end{pmatrix}, \begin{pmatrix} g \\ y \end{pmatrix} \rangle_Y = \langle f, g\rangle_X + \langle x, y\rangle_s,\ \ \begin{pmatrix} f \\ x \end{pmatrix}, \begin{pmatrix} g \\ y \end{pmatrix} \in Y.$$

We define the operator $\mathcal{A} : D(\mathcal{A}) \subset Y \to Y$,

$$D(\mathcal{A}) = \{y = \mathrm{col}(u, v, w) \in Y;\ \ u,\, v \in H^1(I\!R_+; I\!R^n),\ \ \mathrm{col}(v(0), w) \in D(G),$$
$$u(0) \in -G_{11}(v(0)) - G_{12}(w)\},$$

$$\mathcal{A}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} v' \\ u' \\ S^{-1}G_{21}(v(0)) + S^{-1}G_{22}(w) \end{pmatrix},\ \ \begin{pmatrix} u \\ v \\ w \end{pmatrix} \in D(\mathcal{A}),$$

and the operator $\mathcal{B} : D(\mathcal{B}) \subset Y \to Y$, $D(\mathcal{B}) = \{y = \mathrm{col}(u, v, w) \in Y,\ \ \mathcal{B}(y) \in Y\}$,

$$\mathcal{B}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} \alpha(\cdot, u) \\ \beta(\cdot, v) \\ 0 \end{pmatrix}.$$

Under the assumptions (A2)a and (A3) we have $D(\mathcal{A}) \neq \emptyset$ and $\overline{D(\mathcal{A})} = X \times \overline{D(G_{12}) \cap D(G_{22})}$, and under assumptions (A1)ab we have $D(\mathcal{B}) = Y$.

Lemma 1. *If the assumptions* (A2)a *and* (A3) *hold, then the operator* $\mathcal{A}$ *is maximal monotone in the space* Y.

Lemma 2. *If the assumptions* (A1)ab *hold, then the operator* $\mathcal{B}$ *is maximal monotone in* Y.

In the first case, i.e., $B(t) = \text{const.}$, we can replace G by $\tilde{G}$ defined by $\tilde{G}w = Gw - b_0$, which is also, in the assumption (A2)a, a maximal monotone operator. So, we can suppose without loss of generality that $B(t) = 0$.

We present some existence and uniqueness results for the solutions of the problem (S)+(BC)+(IC), which are obtained in the paper [10].

Using the operators $\mathcal{A}$ and $\mathcal{B}$ the problem (S)+(BC)+(IC) can be equivalently expressed as the following Cauchy problem in the space Y

$$\text{(P)} \qquad \begin{cases} \dfrac{dy}{dt}(t) + (\mathcal{A}+\mathcal{B})(y(t)) \ni F(t,\cdot), \;\; t>0, \\ y(0) = y_0, \end{cases}$$

where

$$y(t) = \mathrm{col}(u(t), v(t), w(t)),$$
$$F(t,\cdot) = \mathrm{col}(f(t,\cdot), g(t,\cdot), 0),$$
$$y_0 = \mathrm{col}(u_0, v_0, w_0).$$

We shall say that $y = \mathrm{col}(u,v,w)$ is a strong (weak) solution of the problem (S)+(BC)+(IC) if y is a strong (respectively weak) solution of the problem (P), (see {[1], Chapter III, §2}).

Theorem 1. *Assume the assumptions* (A1)ab, (A2)a *and* (A3) *hold. If* f, $g \in W^{1,1}(0,T;L^2(\mathbb{R}_+;\mathbb{R}^n))$ *(with* $T>0$ *fixed),* u_0, $v_0 \in H^1(\mathbb{R}_+;\mathbb{R}^n)$, $\mathrm{col}(v_0(0), w_0) \in D(G)$, $u_0(0) \in -G_{11}(v_0(0)) - G_{12}(w_0)$, *then the problem* (P)$\Leftrightarrow$ (S)+(BC)+(IC) *has a unique strong solution* $y = \mathrm{col}(u,v,w) \in W^{1,\infty}(0,T;Y)$. *Moreover* u, $v \in L^\infty(0,T;H^1(\mathbb{R}_+;\mathbb{R}^n))$.

Theorem 2. *Assume the assumptions* (A1)ab, (A2)a *and* (A3) *hold. If* f, $g \in L^1(0,T;L^2(\mathbb{R}_+;\mathbb{R}^n))$ *(with* $T>0$ *fixed),* u_0, $v_0 \in L^2(\mathbb{R}_+;\mathbb{R}^n)$, $w_0 \in \overline{D(G_{12}) \cap D(G_{22})}$, *then the problem* (S)+(BC)+(IC) *has a unique weak solution* $y = \mathrm{col}(u,v,w) \in C([0,T];Y)$.

For the proofs of Lemma 1, Lemma 2, Theorem 1 and Theorem 2 see [10].

Lemma 3. *Assume that* (A1)abc, (A2)ab *and* (A3) *hold. Then the operator* $\mathcal{A}+\mathcal{B}$ *is coercive with respect to any* $y^0 = \mathrm{col}(u^0, v^0, w^0) \in D(\mathcal{A})$, *that is*

$$\lim_{\substack{\|y\|_Y \to \infty \\ y \in D(\mathcal{A})}} \frac{\langle (\mathcal{A}+\mathcal{B})(y), y - y^0 \rangle_Y}{\|y\|_Y} = \infty. \tag{1}$$

Proof. We suppose without loss of generality that the operator G is single-valued. Let $y^0 = \mathrm{col}(u^0, v^0, w^0)$ be arbitrary, but fixed for the moment in $D(\mathcal{A})$. By (A2)b, for every $y = \mathrm{col}(u,v,w) \in D(\mathcal{A})$, $u = \mathrm{col}(u_1, \ldots, u_n)$, $v = \mathrm{col}(v_1, \ldots, v_n)$, $w = \mathrm{col}(w_1, \ldots, w_m)$, we have

$$E = \langle (\mathcal{A}+\mathcal{B})(y), y - y^0 \rangle_Y = \langle \mathcal{A}(y) - \mathcal{A}(y^0), y - y^0 \rangle_Y + \langle \mathcal{B}(y), y - y^0 \rangle_Y$$
$$+ \underbrace{\langle \mathcal{A}(y^0), y - y^0 \rangle_Y}_{E_0} = \langle G \begin{pmatrix} v(0) \\ w \end{pmatrix} - G \begin{pmatrix} v^0(0) \\ w^0 \end{pmatrix}, \begin{pmatrix} v(0) \\ w \end{pmatrix} - \begin{pmatrix} v^0(0) \\ w^0 \end{pmatrix} \rangle_{\mathbb{R}^{n+m}}$$

$$+\sum_{k=1}^{n}\int_0^\infty \alpha_k(x,u_k(x))(u_k(x)-u_k^0(x))dx+\sum_{k=1}^{n}\int_0^\infty \beta_k(x,v_k(x))(v_k(x)-v_k^0(x))dx$$

$$+E_0 \geq \zeta_1\|w-w^0\|^2_{I\!R^m}+\sum_{k=1}^{n}\int_0^\infty \alpha_k(x,u_k(x))(u_k(x)-u_k^0(x))dx$$

$$+\sum_{k=1}^{n}\int_0^\infty \beta_k(x,v_k(x))(v_k(x)-v_k^0(x))dx+E_0,$$

$(u^0=\mathrm{col}(u_1^0,\ldots,u_n^0),\ \ v^0=\mathrm{col}(v_1^0,\ldots,v_n^0))$.

For $y\neq 0$, we obtain

$$\frac{E}{\|y\|_Y}\geq \frac{\zeta_1\|w-w^0\|^2_{I\!R^m}}{\|y\|_Y}+\frac{E_0}{\|y\|_Y}+\frac{\displaystyle\sum_{k=1}^{n}\int_0^\infty \alpha_k(x,u_k(x))(u_k(x)-u_k^0(x))dx}{\|y\|_Y}$$

$$+\frac{\displaystyle\sum_{k=1}^{n}\int_0^\infty \beta_k(x,v_k(x))(v_k(x)-v_k^0(x))dx}{\|y\|_Y}\geq$$

$$\min\left\{\frac{\displaystyle\int_0^\infty \alpha_k(x,u_k(x))(u_k(x)-u_k^0(x))dx}{\|u_k\|_{L^2(I\!R_+)}},\ \frac{\displaystyle\int_0^\infty \beta_k(x,v_k(x))(v_k(x)-v_k^0(x))dx}{\|v_k\|_{L^2(I\!R_+)}},\right.$$

$$\left.\frac{\zeta_1\|w-w^0\|^2_{I\!R^m}}{\|w\|_s},\ \ k=\overline{1,n}\right\}+\frac{E_0}{\|y\|_Y}.$$

To prove (1) it is sufficient to show that

$$\lim_{\|u_k\|_{L^2(I\!R_+)}\to\infty}\frac{\displaystyle\int_0^\infty \alpha_k(x,u_k(x))(u_k(x)-u_k^0(x))dx}{\|u_k\|_{L^2(I\!R_+)}}=\infty,\ \ k=\overline{1,n},\qquad (2)$$

$$\lim_{\|v_k\|_{L^2(I\!R_+)}\to\infty}\frac{\displaystyle\int_0^\infty \beta_k(x,v_k(x))(v_k(x)-v_k^0(x))dx}{\|v_k\|_{L^2(I\!R_+)}}=\infty,\ \ k=\overline{1,n},\qquad (3)$$

and

$$\lim_{\|w\|_s\to\infty}\frac{\zeta_1\|w-w^0\|^2_{I\!R^m}}{\|w\|_s}=\infty.\qquad (4)$$

For the relations (2), using the assumptions (A1)abc we have

$\alpha_k(x,u_k(x))(u_k(x)-u_k^0(x))\geq|\alpha_k(x,u_k(x))|\cdot|u_k(x)-u_k^0(x)|$
$-2|\alpha_k(x,u_k^0(x))|\cdot|u_k(x)-u_k^0(x)|\geq(c_k|u_k(x)|-\xi_k(x))\cdot|u_k(x)-u_k^0(x)|$
$-2(a_k|u_k^0(x)|+\varphi_k(x))\cdot|u_k(x)-u_k^0(x)|\geq c_k|u_k(x)|(|u_k(x)|-|u_k^0(x)|)$

$-\xi_k(x)(|u_k(x)| + |u_k^0(x)|) - 2(a_k|u_k^0(x)| + |\varphi_k(x)|) \cdot (|u_k(x)| + |u_k^0(x)|)$
$= c_k|u_k(x)|^2 - |u_k(x)| \cdot (c_k|u_k^0(x)| + \xi_k(x) + 2a_k|u_k^0(x)| + 2|\varphi_k(x)|)$
$-(|u_k^0(x)|\xi_k(x) + 2a_k|u_k^0(x)|^2 + 2|\varphi_k(x)| \cdot |u_k^0(x)|) \geq c_k|u_k(x)|^2 - C_1|u_k(x)|^2$
$-C_2(\widetilde{a}_k|u_k^0(x)| + \xi_k(x) + 2|\varphi_k(x)|)^2 - \frac{1}{2}|u_k^0(x)|^2 - \frac{1}{2}\xi_k^2(x) - 2a_k|u_k^0(x)|^2$
$-\varphi_k^2(x) - |u_k^0(x)|^2 = \widetilde{c}_k|u_k(x)|^2 - C_3(|u_k^0(x)|^2 + \xi_k^2(x) + \varphi_k^2(x)), \;\; x > 0.$
We choose $C_1, \; C_2 > 0$ such that $C_1 < c_k, \; \widetilde{c}_k = c_k - C_1 > 0, \; C_3 > 0$, $\widetilde{a}_k = c_k + 2a_k > 0$.

Integrating over $[0, \infty)$ we obtain

$$\int_0^\infty \alpha_k(x, u_k(x)) \cdot (u_k(x) - u_k^0(x))dx \geq \widetilde{c}_k \int_0^\infty |u_k(x)|^2 dx - C_3 \int_0^\infty (|u_k^0(x)|^2$$
$$+\xi_k^2(x) + \varphi_k^2(x))dx = \widetilde{c}_k \|u_k\|^2_{L^2(I\!R_+)} - C_4, \;\; C_4 > 0, \;\; k = \overline{1,n},$$

(because $u_k^0, \; \xi_k, \; \varphi_k \in L^2(I\!R_+)$).

The above inequality implies the relations (2). In the same manner we deduce the relations (3). The last relation (4) is a simple consequence of the equivalence between the norms $\|\cdot\|_{I\!R^m}$ and $\|\cdot\|_s$. Q.E.D.

In the second case, i.e, $B(t) \neq$ const., the existence, uniqueness and some properties (regularity, asymptotic behavior) of the solutions of the problem (S)+(BC)+(IC) were studied in [10], where we used the change of functions $u_k = \widetilde{u}_k + \widetilde{\widetilde{u}}_k$, with $\widetilde{\widetilde{u}}_k(t,x) = \dfrac{1}{1+x} b_k(t), \; k = \overline{1,n}$. Then our problem was written as

$$(\widetilde{\mathrm{S}}) \qquad \begin{cases} \dfrac{\partial \widetilde{u}}{\partial t}(t,x) + \dfrac{\partial v}{\partial x}(t,x) + \alpha(x, \widetilde{u} + \widetilde{\widetilde{u}}(t,x)) = \widetilde{f}(t,x) \\ \dfrac{\partial v}{\partial t}(t,x) + \dfrac{\partial \widetilde{u}}{\partial x}(t,x) + \beta(x,v) = \widetilde{g}(t,x), \\ \qquad\qquad t > 0, \;\; x > 0, \end{cases}$$

with the boundary condition

$$(\widetilde{\mathrm{BC}}) \qquad \begin{pmatrix} \widetilde{u}(t,0) \\ S(w'(t)) \end{pmatrix} \in -G \begin{pmatrix} v(t,0) \\ w(t) \end{pmatrix} + \begin{pmatrix} 0 \\ B_2(t) \end{pmatrix}, \;\; t > 0$$

and the initial data

$$(\widetilde{\mathrm{IC}}) \qquad \begin{cases} \widetilde{u}(0,x) = \widetilde{u}_0(x), \;\; v(0,x) = v_0(x), \;\; x > 0, \\ w(0) = w_0, \end{cases}$$

where $\widetilde{f} = \mathrm{col}(\widetilde{f}_1, \ldots, \widetilde{f}_n)$, $\widetilde{g} = \mathrm{col}(\widetilde{g}_1, \ldots, \widetilde{g}_n)$, $\widetilde{f}_k(t,x) = f_k(t,x) - \dfrac{1}{1+x} b_k'(t)$, $\widetilde{g}_k(t,x) = g_k(t,x) + \dfrac{1}{(1+x)^2} b_k(t), \; x > 0, \; t > 0, \; k = \overline{1,n}, \; \widetilde{u}_0 =$

$\mathrm{col}(\widetilde{u}_{10}, \ldots, \widetilde{u}_{n0})$, $\widetilde{u}_{k0}(x) = u_{k0}(x) - \dfrac{1}{1+x} b_k(0)$, $x > 0$, $k = \overline{1,n}$, $B_2(t) = \mathrm{col}(b_{n+1}(t), \ldots, b_{n+m}(t))$, $(f = \mathrm{col}(f_1, \ldots, f_n)$, $g = \mathrm{col}(g_1, \ldots, g_n)$, $u_0 = \mathrm{col}(u_{10}, \ldots, u_{n0}))$.

Using once again the operators $\mathcal{A}$ and $\mathcal{B}$, the problem$(\widetilde{\mathrm{S}})$+$(\widetilde{\mathrm{BC}})$+$(\widetilde{\mathrm{IC}})$ can be equivalently formulated as a time dependent Cauchy problem in the space Y

$$(\widetilde{\mathrm{P}}) \qquad \begin{cases} \dfrac{d}{dt}\begin{pmatrix} \widetilde{u} \\ v \\ w \end{pmatrix} + \mathcal{A}\begin{pmatrix} \widetilde{u} \\ v \\ w \end{pmatrix} + \mathcal{B}\begin{pmatrix} \widetilde{u} + \widetilde{\widetilde{u}}(t) \\ v \\ w \end{pmatrix} \ni \begin{pmatrix} \widetilde{f}(t,\cdot) \\ \widetilde{g}(t,\cdot) \\ S^{-1}B_2(t) \end{pmatrix} \\ \begin{pmatrix} \widetilde{u}(0) \\ v(0) \\ w(0) \end{pmatrix} = \begin{pmatrix} \widetilde{u}_0 \\ v_0 \\ w_0 \end{pmatrix}. \end{cases}$$

Theorem 3. *Assume the assumptions* (A1)ab, (A2)ac, (A3) *hold,* f, $g \in W^{1,1}(0,T;\ L^2(I\!R_+; I\!R^n))$ $(T > 0$ *fixed*$)$, $b_k \in W^{1,2}(0,T)$, $k = \overline{1, n+m}$, u_0, $v_0 \in H^1(I\!R_+; I\!R^n)$, $w_0 \in I\!R^m$, $\mathrm{col}(v_0(0), w_0) \in D(G)$ *and* $B_1(0) \in u_0(0) + G_{11}(v_0(0)) + G_{12}(w_0)$. *Then the problem* $(\widetilde{\mathrm{P}}) \Leftrightarrow (\widetilde{\mathrm{S}})$+$(\widetilde{\mathrm{BC}})$+$(\widetilde{\mathrm{IC}})$ *has a unique strong solution* $y = \mathrm{col}(u, v, w) \in W^{1,\infty}(0,T;Y)$. *Moreover* u, $v \in L^\infty(0,T; H^1(I\!R_+; I\!R^n))$, $(B_1(t) = \mathrm{col}(b_1(t), \ldots, b_n(t)))$.

Theorem 4. *Assume the assumptions* (A1)ab, (A2)ac *and* (A3) *hold. If* f, $g \in L^1(0,T; L^2(I\!R_+; I\!R^n))$ $(T > 0$ *fixed*$)$, $b_k \in L^2(0,T)$, $k = \overline{1, n+m}$, u_0, $v_0 \in L^2(I\!R_+)$, $w_0 \in \overline{D(G_{12}) \cap D(G_{22})}$, *then the problem* (S)+(BC)+(IC) *has a unique weak solution* $y = \mathrm{col}(u, v, w) \in C([0,T]; Y)$.

For the proofs of Theorem 3 and Theorem 4 see [10].

3. The existence of time periodic solutions

In the first case, i.e., $B(t) = \text{const.}$, in fact under our assumption, $B(t) = 0$, we have the following result.

Theorem 5. *Assume that* (A1)abc, (A2)ab, (A3) *hold and*

$$f,\ g \in L^1_{loc}(I\!R_+; L^2(I\!R_+; I\!R^n))$$

are T_0*-periodic in time, that is* $f(t + T_0, x) = f(t,x)$, $g(t + T_0, x) = g(t,x)$, *for a.a.* $(t,x) \in I\!R_+ \times I\!R_+$. *Then the problem* (S)+(BC) *has at least one* T_0*-periodic weak solution.*

Proof. Let $y^0 = \mathrm{col}(u^0, v^0, w^0) \in D(\mathcal{A})$ be fixed. We define the operator $\mathcal{C}$ by

$$D(\mathcal{C}) = \{y = \mathrm{col}(u, v, w) \in Y;\ \ y + y^0 \in D(\mathcal{A})\},\ \ \mathcal{C}(y) = (\mathcal{A} + \mathcal{B})(y + y^0).$$

Because the operators $\mathcal{A}$, $\mathcal{B}$ are maximal monotone (Lemma 1, Lemma 2), the operator $\mathcal{B}$ is single-valued and everywhere defined, by {[1], Theorem 1.7, Chapter II}, we deduce that the operator $\mathcal{A} + \mathcal{B}$, and also $\mathcal{C}$ are maximal monotone. Using now Lemma 3, we obtain that the operator $\mathcal{C}$ is coercive with respect to 0. With the change of functions $\delta_k(t, x) = u_k(t, x) - u_k^0(x)$, $\theta_k(t, x) = v_k(t, x) - v_k^0(x)$, $k = \overline{1, n}$, $\tau_j(t) = w_j(t) - w_j^0$, $j = \overline{1, m}$, the problem (S)+(BC) becomes

$$(\widetilde{\mathrm{E}}) \qquad \frac{d\omega}{dt} + \mathcal{C}(\omega) \ni F,$$

where $\omega = \mathrm{col}(\delta, \theta, \tau)$, $\delta = \mathrm{col}(\delta_1, \ldots, \delta_n)$, $\theta = \mathrm{col}(\theta_1, \ldots, \theta_n)$, $\tau = \mathrm{col}(\tau_1, \ldots, \tau_m)$.

Using now the periodicity of functions f, g, and {[4], Proposition 1, p.285}, we deduce that the solutions of the equation $(\widetilde{\mathrm{E}})$ are bounded on the positive half-axis. Therefore all the solutions of the equation $(\mathrm{P})_1$ are also bounded, that is $\sup\limits_{t\geq 0} \|y(t, \cdot)\|_Y < \infty$. We define the operator $\mathcal{L} : \overline{D(\mathcal{A})} \to \overline{D(\mathcal{A})}$, $\mathcal{L}(y^0) = y(T_0; y^0)$, where $y(t, y^0)$, $t \geq 0$ is the weak solution of the problem (S)+(BC) with the initial date y^0. This operator is nonexpansive and if $y^0 \in \overline{D(\mathcal{A})}$, the sequence $\{\mathcal{L}^n(y^0)\}_{n\geq 1}$ is bounded in Y, because $\mathcal{L}^n(y^0) = y(nT_0; y^0)$. Using a theorem due to F.E. Browder and W.V. Petryshyn (see [3]) we deduce that the operator $\mathcal{L}$ has at least one fixed point. This means that the problem (S)+(BC) has at least one time periodic weak solution with the period T_0. Q.E.D.

Remark. If $\alpha_k(x, \cdot)$ and $\beta_k(x, \cdot)$ are strongly monotone, a.a. $x \in \mathbb{R}_+$ and $f,\ g \in W^{1,1}_{loc}(\mathbb{R}; L^2(\mathbb{R}_+; \mathbb{R}^n))$ are T_0-periodic functions in the variable t, then the problem (S)+(BC) has a T_0-periodic strong solution.

In the second case, i.e., $B(t) \neq$ const., we shall firstly present some conditions for the boundedness of the solutions to problem (S)+(BC).

Theorem 6. *Assume that* (A1)abc, (A2)ac, (A3) *hold, and* $f, g \in L^2_{loc}(I\!R_+; L^2(I\!R_+; I\!R^n))$, $b_k \in L^2_{loc}(I\!R_+)$, $k = \overline{1, n+m}$, *verify the conditions*

$$\sup_{t\geq 0}\int_t^{t+1}\|f(\theta,\cdot)\|^2_{L^2(I\!R_+;I\!R^n)}d\theta \leq C_0,\ \sup_{t\geq 0}\int_t^{t+1}\|g(\theta,\cdot)\|^2_{L^2(I\!R_+;I\!R^n)}d\theta \leq C_0,$$
$$\sup_{t\geq 0}\int_t^{t+1}|b_k(\theta)|^2 d\theta \leq C_0,\ \ (C_0 > 0). \tag{5}$$

Then, every weak solution of the problem (S)+(BC) *is bounded on* $I\!R_+$.

Proof. Because the operator $\mathcal{A}+\mathcal{B}$ is maximal monotone and coercive, it follows that $R(\mathcal{A}+\mathcal{B}) = Y$ and, hence $F = (\mathcal{A}+\mathcal{B})^{-1}(0) \neq \emptyset$. We suppose again that G is single-valued.

First, we show that if f, $g \in W^{1,1}_{loc}(I\!R_+; L^2(I\!R_+; I\!R^n))$ and $b_k \in W^{1,2}_{loc}(I\!R_+)$, $k = \overline{1, n+m}$, verify the conditions (5), then every strong solution of the problem (S)+(BC) is bounded on $I\!R_+$. Let $T > 0$ be arbitrary, but fixed for the moment, f, $g \in W^{1,1}(0,T; L^2(I\!R_+; I\!R^n))$, $b_k \in W^{1,2}(0,T)$ $k = \overline{1, n+m}$, verify the conditions (5), u_0, $v_0 \in H^1(I\!R_+; I\!R^n)$, $w_0 \in I\!R^m$, $\mathrm{col}(v_0(0), w_0) \in D(G)$ and $B_1(0) \in u_0(0) + G_{11}(v_0(0)) + G_{12}(w_0)$. Then the strong solution $y(t) = \mathrm{col}(u(t), v(t), w(t))$ of the problem (S)+(BC)+(IC) corresponding to above data satisfies

$$\begin{cases} \dfrac{dy}{dt}(t) + \mathcal{A}(y(t)) + \mathcal{B}(y(t)) = F_1(t,\cdot),\ \ 0 \leq t < T \\ u(t,0) = -G_{11}(v(t,0)) - G_{12}(w(t)) + B_1(t),\ \ 0 \leq t < T \\ y(0) = y_0, \end{cases} \tag{6}$$

where $F_1(t,\cdot) = \mathrm{col}(f(t,\cdot), g(t,\cdot), S^{-1}B_2(t))$.

Let $\gamma = \mathrm{col}(p, q, r) \in F$, that is

$$(\mathcal{A}+\mathcal{B})(\gamma) = 0. \tag{7}$$

We subtract from equation $(6)_1$ the relation (7) and we multiply the obtained relation by $y(t) - \gamma$ in the space Y. We obtain

$$\frac{1}{2}\frac{d}{dt}\|y(t)-\gamma\|^2_Y + \langle G\begin{pmatrix} v(t,0) \\ w(t)\end{pmatrix} - G\begin{pmatrix} q(0) \\ r\end{pmatrix}, \begin{pmatrix} v(t,0)-q \\ w(t)-r\end{pmatrix}\rangle_{I\!R^{n+m}}$$
$$+\sum_{k=1}^n\int_0^\infty(\alpha_k(x,u_k(t,x)) - \alpha_k(x,p_k(x)))\cdot(u_k(t,x)-p_k(x))dx$$
$$+\sum_{k=1}^n\int_0^\infty(\beta_k(x,v_k(t,x)) - \beta_k(x,q_k(x)))\cdot(v_k(t,x)-q_k(x))dx$$

$= \langle B_1(t), v(t,0) - q(0)\rangle_{I\!R^n} + \langle B_2(t), w(t) - r\rangle_{I\!R^m}$
$+\langle f(t,\cdot), u(t,\cdot) - p\rangle_{L^2(I\!R_+;I\!R^n)} + \langle g(t,\cdot), v(t,\cdot) - q\rangle_{L^2(I\!R_+;I\!R^n)}, \ \ 0 \le t < T.$

Therefore using the assumption (A2)c we get

$\frac{1}{2}\frac{d}{dt}\|y(t) - \gamma\|_Y^2 + \zeta_2\|v(t,0) - q(0)\|_{I\!R^n}^2 + \zeta_2\|w(t) - r\|_{I\!R^m}^2$
$+\sum_{k=1}^{n}\int_0^\infty (\alpha_k(x, u_k(t,x)) - \alpha_k(x, p_k(x))) \cdot (u_k(t,x) - p_k(x))dx$
$+\sum_{k=1}^{n}\int_0^\infty (\beta_k(x, v_k(t,x)) - \beta_k(x, q_k(x))) \cdot (v_k(t,x) - q_k(x))dx$
$\le \frac{1}{\zeta_0}\|B_1(t)\|_{I\!R^n}^2 + \zeta_0\|v(t,0) - q(0)\|_{I\!R^n}^2 + \frac{1}{\zeta_0}\|B_2(t)\|_{I\!R^m}^2 + \zeta_0\|w(t) - r\|_{I\!R^m}^2$
$+\|F_0(t,\cdot)\|_X\|y(t) - \gamma\|_Y, \ \ 0 \le t < T,$

where $F_0(t,\cdot) = \mathrm{col}(f(t,\cdot), g(t,\cdot))$.

We choose $0 < \zeta_0 < \zeta_2$; then the above inequality gives us

$$\frac{1}{2}\frac{d}{dt}\|y(t) - \gamma\|_Y^2 + \sum_{k=1}^{n}\int_0^\infty (\alpha_k(x, u_k(t,x)) - \alpha_k(x, p_k(x)))(u_k(t,x) - p_k(x))dx$$
$$+\sum_{k=1}^{n}\int_0^\infty (\beta_k(x, v_k(t,x)) - \beta_k(x, q_k(x))) \cdot (v_k(t,x)) - q_k(x))dx$$
$$+C_5\|w(t) - r\|_{I\!R^m}^2 \le C_6\|B(t)\|_{I\!R^{n+m}}^2 + \|F_0(t,\cdot)\|_X\|y(t) - \gamma\|_Y, \ \ 0 \le t < T, \tag{8}$$

where the positive constant C_5, C_6 are independent of T.

Now, by assumptions (A1)abc we have

$(\alpha_k(x, u_k(t,x)) - \alpha_k(x, p_k(x))) \cdot (u_k(t,x) - p_k(x)) \ge (c_k|u_k(t,x)|$
$-\xi_k(x)) \cdot |u_k(t,x) - p_k(x)| - (a_k|p_k(x)| + \varphi_k(x))|u_k(t,x) - p_k(x)|$
$\ge c_k(|u_k(t,x) - p_k(x)| - |p_k(x)|) \cdot |u_k(t,x) - p_k(x)| - (\xi_k(x) + a_k|p_k(x)|$
$+|\varphi_k(x)|) \cdot |u_k(t,x) - p_k(x)| = c_k|u_k(t,x) - p_k(x)|^2 - (c_k|p_k(x)| + \xi_k(x)$
$+a_k|p_k(x)| + |\varphi_k(x)|) \cdot |u_k(t,x) - p_k(x)| \ge c_k|u_k(t,x) - p_k(x)|^2 - C_7|u_k(t,x)$
$-p_k(x)|^2 - C_8((c_k + a_k)^2 p_k^2(x) + \xi_k^2(x) + \varphi_k^2(x)) = \tilde{\tilde{c}}_k|u_k(t,x) - p_k(x)|^2$
$-C_9(p_k^2(x) + \xi_k^2(x) + \varphi_k^2(x)), \ \ x > 0, \ \ 0 \le t \le T, \ \ k = \overline{1,n},$

(we choose $C_7 > 0$ such that $C_7 < c_k$; $\tilde{\tilde{c}}_k = c_k - C_7$, $k = \overline{1,n}$, and C_8, $C_9 > 0$ are independent of T).

Integrating over $(0,\infty)$ the obtained inequality, we deduce

$$\int_0^\infty (\alpha_k(x, u_k(t,x)) - \alpha_k(x, p_k(x))) \cdot (u_k(t,x) - p_k(x))dx$$
$$\ge \tilde{\tilde{c}}_k\|u_k(t) - p_k\|_{L^2(I\!R_+)}^2 - C_{10}, \ \ 0 \le t \le T, \ \ k = \overline{1,n}. \tag{9}$$

In the same manner we obtain

$$\begin{aligned}&\int_0^\infty (\beta_k(x,v_k(t,x))-\beta_k(x,q_k(x)))\cdot(v_k(t,x)-q_k(x))dx\\&\geq \widetilde{\widetilde{d}}_k\|v_k(t)-q_k\|^2_{L^2(I\!R_+)}-C_{11},\ \ 0\leq t\leq T,\ \ k=\overline{1,n},\end{aligned} \tag{10}$$

where $\widetilde{\widetilde{d}}_k>0$, $k=\overline{1,n}$, and the constants $C_{10},\ C_{11}>0$ are independent of T.

From the inequalities (8)-(10) we deduce

$\frac{1}{2}\frac{d}{dt}\|y(t)-\gamma\|^2_Y+C_5\|w(t)-r\|^2_{I\!R^m}+\sum_{k=1}^n\widetilde{\widetilde{c}}_k\|u_k(t)-p_k\|^2_{L^2(I\!R_+)}$
$+\sum_{k=1}^n\widetilde{\widetilde{d}}_k\|v_k(t)-q_k\|^2_{L^2(I\!R_+)}\leq nC_{10}+nC_{11}+C_6\|B(t)\|^2_{I\!R^{n+m}}+\|F_0(t,\cdot)\|_X\|y(t)$
$-\gamma\|_Y,$
$\Rightarrow\ \frac{1}{2}\frac{d}{dt}\|y(t)-\gamma\|^2_Y+C_{12}\|y(t)-\gamma\|^2_Y\leq C_{13}+C_6\|B(t)\|^2_{I\!R^{n+m}}+C_{14}\|F_0(t,\cdot)\|^2_X$
$+C_{15}\|y(t)-\gamma\|^2_Y,\ \ 0\leq t<T,$
(we choose $C_i>0$, $i=\overline{12,15}$, such that $C_{15}<C_{12}$).

Therefore we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|y(t)-\gamma\|^2_Y+C_{16}\|y(t)-\gamma\|^2_Y&\leq C_{13}+C_6\|B(t)\|^2_{I\!R^{n+m}}\\&+C_{14}\|F_0(t,\cdot)\|^2_X,\ \ 0\leq t<T,\end{aligned} \tag{11}$$

with $C_{16}>0$ independent of T.

Because T is arbitrary, the inequality (11) is verified for a.a. $t\in[0,\infty)$. Now we shall use the following lemma from [4] (see {[4], Lemma 4, p.286}).

Lemma 4. *Let λ be a nondecreasing nonnegative function on $I\!R_+$, $\alpha>0$, $C>0$, $\zeta(t)$ be measurable nonnegative function, with $\int_t^{t+1}\zeta(\theta)d\theta\leq C$, for all $t\geq 0$. Let $V\in C([0,\infty),I\!R_+)$ be absolutely continuous on every compact interval of $I\!R_+$, such that $\frac{dV}{dt}+\lambda(V(t))\leq\zeta(t)$, for a.a. $t\in[0,\infty)$. Then, if $\alpha\geq V(0)$ and $\lambda(\alpha)\geq C$, we have $V(t)\leq\alpha+C$, for all $t\geq 0$.*

We consider $V(t)=\|y(t)-\gamma\|^2_Y$, $\lambda(u)=2C_{16}u$, $\zeta(t)=2C_{13}+2C_6\|B(t)\|^2_{I\!R^{n+m}}+2C_{14}\|F_0(t,\cdot)\|^2_X$. Using the conditions (5) we have

$$\sup_{t\geq 0}\int_t^{t+1}\zeta(\theta)d\theta\leq 2C_{13}+2(n+m)C_0C_6+4C_0C_{14}\overset{not}{=}\widetilde{C}.$$

Therefore, Lemma 4 gives us that if $\alpha\geq\max\left\{\|y_0-\gamma\|^2_Y,\frac{\widetilde{C}}{2C_{16}}\right\}$, then we obtain $\|y(t)-\gamma\|^2_Y\leq\alpha+\widetilde{C}$, for all $t\geq 0$. We deduce that the solution

$y(t)$ is bounded on $I\!R_+$. The extension to the case of weak solutions is then immediate, so we obtain the conclusion of the theorem. Q.E.D.

Theorem 7. *Assume that* (A1)ab, (A2)ac, (A3) *hold,*

$$f,\, g \in L^1_{loc}(I\!R_+; L^2(I\!R_+; I\!R^n))$$

are T_0-periodic in time and $b_k \in L^2_{loc}(I\!R_+)$, $k = \overline{1, n+m}$ are T_0-periodic functions. Then, if the problem (S)+(BC) *has at least one bounded solution on $I\!R_+$, then the problem has also a weak T_0-periodic solution.*

Proof. Let $y = \mathrm{col}(u, v, w)$ be a bounded solution on $I\!R_+$ of the problem (S)+(BC). Then using the following inequality for the solutions y and $\bar{y}$

$$\|y(t) - \bar{y}(t)\|_Y \leq \|y(0) - \bar{y}(0)\|_Y, \;\; t > 0,$$

we deduce that all the solutions of the problem (S)+(BC) are bounded on $I\!R_+$. Now, using the operator $\mathcal{L}$, defined as in the proof of Theorem 5 (for this case) and the same fixed point theorem due to F.E. Browder and W.V. Petryshyn, we conclude that the problem (S)+(BC) has at least one T_0-periodic weak solution. Q.E.D.

Now, combining Theorem 6 and Theorem 7, and by using the fact that a periodic function from space L^2 belongs to the Stepanov space of index 2, we obtain sufficient conditions for the existence of time periodic weak solutions for our problem, formulated in the following corollary.

Corollary. *Assume that* (A1)abc, (A2)ac, (A3), *hold*,

$$f,\, g \in L^2_{loc}(I\!R_+; L^2(I\!R_+; I\!R^n))$$

are T_0-periodic in time, and *$b_k \in L^2_{loc}(I\!R_+)$, $k = \overline{1, n+m}$ are T_0-periodic functions. Then the problem* (S)+(BC) *has at least a time periodic weak solution with period T_0.*

4. Some remarks in the case $x \in I\!R$

If the spatial variable x belongs to $I\!R$, then from the boundary condition (BC) it only remains $w'(t) \in -S^{-1}G_{22}(w(t)) + S^{-1}B_2(t)$. This equation with the initial date $w(0) = w_0$ give by integration the function $w(t)$. Therefore, for u and v we obtain the problem

$$(\overline{\mathrm{S}})\qquad \begin{cases} \dfrac{\partial u}{\partial t}(t,x)+\dfrac{\partial v}{\partial x}(t,x)+\alpha(x,u)=f(t,x)\\ \dfrac{\partial v}{\partial t}(t,x)+\dfrac{\partial u}{\partial x}(t,x)+\beta(x,v)=g(t,x),\end{cases}$$

$$t>0,\ \ x\in I\!\!R,$$

with the initial data

$$(\overline{\mathrm{IC}})\qquad u(0,x)=u_0(x),\ \ v(0,x)=v_0(x),\ \ x\in I\!\!R,$$

under the assumptions $(\widetilde{\mathrm{A1}})$abc which are (A1)abc with $I\!\!R$ instead of $I\!\!R_+$.

We consider the space $Z=(L^2(I\!\!R;I\!\!R^n))^2$ with the standard scalar product and the *operators*

$\mathcal{C}:D(\mathcal{C})\subset Z\to Z,\ \ D(\mathcal{C})=(H^1(I\!\!R;I\!\!R^n))^2,\ \ \mathcal{C}(\mathrm{col}(u,v))=\mathrm{col}(v',u'),$

$\mathcal{D}:D(\mathcal{D})\subset Z\to Z,\ \mathcal{D}(\mathrm{col}(u,v))=\mathrm{col}(\alpha(\cdot,u),\beta(\cdot,v)).$

If $(\widetilde{\mathrm{A1}})$ab hold, then $\mathcal{C}$ is maximal monotone in Z, and $\mathcal{D}$ is everywhere defined $(D(\mathcal{D})=Z)$ and maximal monotone. By using the operators $\mathcal{C}$ and $\mathcal{D}$ the problem $(\overline{\mathrm{S}})$+$(\overline{\mathrm{IC}})$ can be written as

$$(\overline{\mathrm{P}})\qquad \begin{cases} \dfrac{dz}{dt}(t)+\mathcal{C}(z(t))+\mathcal{D}(z(t))=\overline{F}(t,\cdot),\ \ t>0,\ \ \text{in } Z\\ z(0)=z_0,\end{cases}$$

where $z(t)=\mathrm{col}(u(t),v(t))$, $z_0=\mathrm{col}(u_0,v_0)$, $\overline{F}(t,\cdot)=\mathrm{col}(f(t,\cdot),g(t,\cdot))$.

The existence, uniqueness and asymptotic behavior of the strong and weak solutions of the problem $(\overline{\mathrm{S}})$ + $(\overline{\mathrm{IC}})$ have been investigated in [10]. We shall only recall the existence results.

Theorem 8. a) *Assume that* $(\widetilde{\mathrm{A1}})$ab *hold. If* $f,\ g\in W^{1,1}(0,T;L^2(I\!\!R;I\!\!R^n))$ $(T>0$ *fixed*$)$, $u_0,\ v_0\in H^1(I\!\!R;I\!\!R^n)$, *then the problem* $(\overline{\mathrm{P}})\Leftrightarrow(\overline{\mathrm{S}})+(\overline{\mathrm{IC}})$ *has a unique strong solution* $z=\mathrm{col}(u,v)\in W^{1,\infty}(0,T;Z)$. *Moreover* $u,\ v\in L^\infty(0,T;H^1(I\!\!R;I\!\!R^n))$.

b) *Assume that* $(\widetilde{\mathrm{A1}})$ab *hold. If* $f,\ g\in L^1(0,T;L^2(I\!\!R;I\!\!R^n))$ $(T>0$ *fixed*$)$, $u_0,\ v_0\in L^2(I\!\!R;I\!\!R^n)$, *then the problem* $(\overline{\mathrm{S}})$+$(\overline{\mathrm{IC}})$ *has a unique weak solution* $z=\mathrm{col}(u,v)\in C([0,T];\,Z)$.

Using similar arguments as in the case $x\in I\!\!R_+$ we obtain for this problem the following results.

Lemma 5. *Assume that* $(\widetilde{\mathrm{A1}})\mathrm{abc}$ *hold. Then the operator* $\mathcal{C}+\mathcal{D}$ *is coercive with respect to any* $z^0 = \mathrm{col}(u^0, v^0) \in D(\mathcal{C})$, *that is*

$$\lim_{\substack{\|z\|_Z\to\infty \\ z\in D(\mathcal{C})}} \frac{\langle(\mathcal{C}+\mathcal{D})(z), z-z^0\rangle_Z}{\|z\|_Z} = \infty.$$

Theorem 9. *Assume that* $(\widetilde{\mathrm{A1}})\mathrm{abc}$ *hold and* $f,\, g \in L^1_{loc}(I\!\!R_+; L^2(I\!\!R; I\!\!R^n))$ *are* T_0*-periodic in time, that is* $f(t+T_0, x) = f(t,x)$, $g(t+T_0,x) = g(t,x)$, *for a.a.* $(t,x) \in I\!\!R_+ \times I\!\!R$. *Then the system* $(\overline{\mathrm{S}})$ *has at least one* T_0*-periodic weak solution.*

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