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ON BOUNDARY VALUE PROBLEMS OF SECOND ORDER CONVEX AND NONCONVEX DIFFERENTIAL INCLUSIONS

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Abstract. This paper presents sufficient conditions for the existence of solutions to boundary value problems of second order multi-valued convex as well as nonconvex differential inclusions.

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1. Introduction

Let $\mathbb{R}$ denote the real line and let $\mathcal{P}_f(\mathbb{R})$ denote the class of all non-empty subsets of $\mathbb{R}$ with a property f. In particular, $\mathcal{P}_{cl}(\mathbb{R}), \mathcal{P}_{bd}(\mathbb{R}), \mathcal{P}_{cv}(\mathbb{R})$, and $\mathcal{P}_{cp}(\mathbb{R})$ denote respectively the classes of closed, bounded, convex and compact subsets of $\mathbb{R}$. Similarly $\mathcal{P}_{cl,bd}(\mathbb{R})$ and $\mathcal{P}_{cp,cv}(\mathbb{R})$ denote respectively the classes of all closed-bounded and compact-convex subsets of $\mathbb{R}$. Let $J = [t_0, t_1]$ be a closed and bounded interval in $\mathbb{R}$ for some real numbers $t_0, t_1 \in \mathbb{R}$ with $t_0 < t_1$. Now consider the two point boundary value problem (in short BVP) of second order differential inclusions

$$-x''(t) \in F(t, x(t), x'(t)) \ \ a.e. \ \ t \in J \tag{1.1}$$

satisfying the boundary conditions

$$\left.\begin{aligned} a_0 x(t_0) - a_1 x'(t_1) = c_0 \\ b_0 x(t_0) + b_1 x'(t_1) = c_1 \end{aligned}\right\} \tag{1.2}$$

where the function and the constants involved in (1.1) and (1.2) satisfy the following properties:

(a) $F : J \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}_f(\mathbb{R})$,
(b) $a_0, a_1, b_0, b_1 \in \mathbb{R}^+$ satisfying $a_0 a_1(t_1 - t_0) + a_0 b_1 + a_1 b_0 > 0$ and
(c) $c_0, c_1 \in \mathbb{R}$.

By a solution of BVP (1.1)-(1.2) we mean a function $x \in AC^1(J, \mathbb{R})$ whose second derivative exists and is a member of $L^1(J, \mathbb{R})$ in $F(t, x, x')$, i.e. there exists a $v \in L^1(J, \mathbb{R})$ such that $v(t) \in F(t, x(t), x'(t))$ for a.e $t \in J$, and $-x''(t) = v(t)$ for all $t \in J$ satisfying (1.2), where $AC^1(J, \mathbb{R})$ is the space of continuous real-valued functions whose first derivative exists and is absolutely continuous on J.

The special cases of the BVP (1.1)-(1.2) have been discussed in the literature for existence of the solutions. The special case of the form

$$-x''(t) = f(t, x(t), x'(t)), \text{ a.e. } t \in J \tag{1.3}$$

satisfying the boundary conditions (1.2) where $f : J \times \mathbb{R} \to \mathbb{R}$, $a_0, a_1, b_0, b_1 \in \mathbb{R}_+$, $c_0, c_1 \in \mathbb{R}$ and $a_0 a_1(t_1 - t_0) + a_0 b_1 + a_1 b_0 > 0$ has been discussed in Bernfeld and Lakshmikantham [2] for the existence of solutions and in Heikkila [9] for the existence of the extremal solutions. Again when $c_0 = c_1, a_1 = 0 = b_1, a_0 = b_0$, and F not depending on x', the BVP (1.1)-(1.2) reduces to

$$y'' \in F(t, y) \text{ a.e } t \in J, \quad y(t_0) = y(t_1). \tag{1.4}$$

where $y = -x$. This is a BVP of second order differential inclusions considered in Benchohra and Ntouyas [3]. Finally, the special case of the BVP consisting of the equation

$$-y''(t) \in F(t, y(t)), \text{ a.e } t \in J \tag{1.5}$$

satisfying the boundary conditions (1.2) has been studied in Dhage [6] and Halidias and Papageorgiou [8] via the method of lower and upper solutions. Thus the BVP (1.1)-(1.2) is more general and so is its importance in the theory of differential inclusions. Here in the present paper, we discuss the BVP (1.1)-(1.2) via a Nonlinear Alternative of Leray-Schauder type ([7], [12]) and on a selection theorem for lower semicontinuous maps ([4]). The paper is organized as follows. In Section 2 we give some preliminaries needed in the sequel. In Section 3 we prove the main existence results for the BVP (1.1)-(1.2) when the right hand side has convex or nonconvex values.

2. Preliminaries

Let (X, d) be a metric space. For $x \in X$ and $Y, Z \in \mathcal{P}_{cl}(X)$ we denote by $D(x, Y) = \inf\{\|x - y\| \mid y \in Y\}$, and $\rho(Y, Z) = \sup_{a \in Y} D(a, Z)$. Define a function $H : \mathcal{P}_{bd,cl}(X) \times \mathcal{P}_{bd,cl}(X) \to \mathbb{R}^+$ by

$$H(A, B) = \max\{\rho(A, B)\ ,\ \rho(B, A)\}.$$

The function H is called a Hausdorff metric on X. Note that $\|Y\|_{\mathcal{P}} = H(Y, \{0\})$.

A map $T : X \to P_f(X)$ is called a multi-valued mapping on X into itself. A point $u \in X$ is called a fixed point of the multi-valued operator $T : X \to P_f(X)$ if $u \in T(u)$. The fixed points set of T will be denoted by $Fix(T)$.

Definition 2.1. *Let $T : X \to \mathcal{P}_f(X)$ be a multi-valued operator. Then T is called a multi-valued contraction if there exists a constant $\lambda \in (0, 1)$ such that for all $x, y \in X$ we have*

$$H(T(x), T(y)) \leq \lambda\|x - y\|.$$

The constant λ is called a contraction constant of T.

Theorem 2.2. (Covitz and Nadler [5]) *Let X be a complete metric space and let $T : X \to \mathcal{P}_{cl}(X)$ be a multi-valued contraction. Then the fixed point set $\mathcal{F}(T)$ of T is non-empty and closed set in X.*

A multi-valued map T is closed-valued (resp. compact-valued) if Tx is closed (resp. compact) subset of X for each $x \in X$. T is said to be bounded on bounded sets if $T(B) = \bigcup_{x \in B} T(x) = \bigcup T(B)$ is a bounded subset of X for all bounded sets B in X. T is called compact if $\cup T(B)$ is relatively compact for a bounded subset B of X. Finally T is called totally compact if $\overline{\cup T(X)}$ is a compact subset of X. T is called upper semi-continuous (u.s.c.) if for every open set $N \subset X$, the set $\{x \in X : Tx \subset N\}$ is open in X. Again T is called completely continuous if it is upper semi-continuous and totally bounded on X. It is known that if the multi-valued compact map T has non empty compact values, then T is upper semi-continuous if and only if T has a closed graph (that is $x_n \to x_*, y_n \to y_*, y_n \in Tx_n \Rightarrow y_* \in Tx_*$).

For more details on multivalued maps we refer the interested reader to the book of Hu and Papageorgiou [10].

We apply the following nonlinear alternative in the sequel.

Theorem 2.3. (O'Regan [12]) *Let U and $\overline{U}$ be the open and closed subsets in a normed linear space X such that $0 \in U$ and let $T : \overline{U} \to \mathcal{P}_{cp,cv}(X)$ be a completely continuous multi-valued map. Then either*

(i) *the operator inclusion $x \in Tx$ has a solution, or*

(ii) *there is an element $u \in \partial U$ such that $\lambda u \in Tu$ for some $\lambda > 1$, where ∂U is the boundary of U.*

Corollary 2.4. *Let $\mathcal{B}_r(0)$ and $\overline{\mathcal{B}_r(0)}$ be the open and closed balls in a normed linear space X centered at origin 0 of radius r and let $T : \overline{\mathcal{B}_r(0)} \to \mathcal{P}_{cp,cv}(X)$ be a completely continuous multi-valued map. Then either*

(i) *the operator inclusion $x \in Tx$ has a solution, or*

(ii) *there is an element $u \in X$ such that $\|u\| = r$ and $\lambda u \in Tu$ for some $\lambda > 1$.*

Corollary 2.5. *Let $\mathcal{B}_r(0)$ and $\overline{\mathcal{B}_r(0)}$ be the open and closed balls in a normed linear space X centered at origin 0 of radius r and let $T : \overline{\mathcal{B}_r(0)} \to X$ be a completely continuous single-valued map. Then either*

(i) *the operator inclusion $x = Tx$ has a solution, or*

(ii) *there is an element $u \in X$ such that $\|u\| = r$ and $u = \lambda Tu$ for some $\lambda < 1$.*

Let $\mathcal{A}$ be a subset of $J \times \mathbb{R}$. A is called a $\mathcal{L} \otimes \mathcal{B}$-measurable if A belongs to the σ-algebra generated by all sets of the form $\mathcal{J} \times \mathcal{D}$, where $\mathcal{J}$ is Lebesgue measurable set in J, $\mathcal{D}$ is Borel measurable set in $\mathbb{R}$. A subset $\mathcal{A}$ of $L^1(J, \mathbb{R})$ is called decomposable, if for all $u, v \in \mathcal{A}$ and $\mathcal{J} \subset J$ measurable, the function $u_{\chi_{\mathcal{J}}} + v_{\chi_{J \setminus \mathcal{J}}} \in \mathcal{A}$, where χ_A stands for the characteristic function of A.

We need the following definitions in the sequel.

Definition 2.6. *Let Y be a separable metric space and let $N : Y \to \mathcal{P}_f(L^1(J, \mathbb{R}))$ be a multi-valued operator. We say N has property (BC) if*

(i) *N is lower semi-continuous (l.s.c.), and*

(ii) *N has closed and decomposable values.*

Let $F : J \times \mathbb{R} \to \mathcal{P}_{cp}(\mathbb{R})$ be a multi-valued function. We assign to F, a multi-valued operator $S_F^1 : C(J,\mathbb{R}) \to \mathcal{P}_f(L^1(J,\mathbb{R}))$ defined by

$$S_F^1(x) = \{v \in L^1(J,\mathbb{R}) \mid v(t) \in F(t, x(t), x'(t)) \text{ a.e. } t \in J\}.$$

The multi-valued operator S_F^1 is called *Nemytskii* or *selection operator* associated with the multi-function F.

Definition 2.7. *Let $F : J \times \mathbb{R} \to \mathcal{P}_{cp}(\mathbb{R})$ be a multi-valued function. We say F is of lower semi-continuous type (l.s.c. type) if its associated Nemytskii operator S_F^1 is lower semi-continuous and has closed and decomposable values.*

Now we state a selection theorem due to Bressan and Colombo [4].

Theorem 2.8. *Let Y be a separable metric space and let $N : Y \to \mathcal{P}_f(L^1(J,\mathbb{R}))$ be a multi-valued operator which has property (BC). Then N has a continuous selection, i.e., there exists a continuous function (single-valued) $g : Y \to L^1(J,\mathbb{R})$ such that $g(y) \in N(y)$ for every $y \in Y$.*

3. Existence Results

Define a norm $\|\cdot\|$ in $AC^1(J,\mathbb{R})$ by

$$\|x\| = \max\left\{\sup_{t\in J}|x(t)|, \sup_{t\in J}|x'(t)|\right\}. \tag{3.1}$$

Before going to the main existence theorems of this section we give a useful result from the theory of boundary value problems of ordinary differential equations.

Lemma 3.1. [9, page 156] *If $f \in L^1(J,\mathbb{R})$, then the BVP*

$$-x''(t) = f(t) \ \text{ a.e. } t \in J \quad \text{and} \quad \begin{cases} a_0x(t_0) - a_1x'(t_1) = c_0 \\ b_0x(t_0) + b_1x'(t_1) = c_1 \end{cases} \tag{3.2}$$

has a unique solution x given by

$$x(t) = z(t) + \int_{t_0}^{t_1} G(t,s)f(s)\,ds, \quad t \in J, \tag{3.3}$$

where z is a unique solution of the homogeneous differential equation

$$-x''(t) = 0 \ \ a.e. \ \ t \in J \qquad and \qquad \begin{cases} a_0 x(t_0) - a_1 x'(t_1) = c_0 \\ b_0 x(t_0) + b_1 x'(t_1) = c_1 \end{cases} \tag{3.4}$$

given by

$$z(t) = \frac{c_0 a_1(t_1 - t) + c_0 b_1 + c_1 a_0 (t - t_0) + c_1 b_0}{a_0 a_1 (t_1 - t_0) + a_0 b_1 + a_1 b_0}, \qquad t \in J, \tag{3.5}$$

and $G(t, s)$ is the Green's function associated to the differential equation

$$-x''(t) = 0 \ \ a.e. \ \ t \in J \qquad and \qquad \begin{cases} a_0 x(t_0) - a_1 x'(t_1) = 0 \\ b_0 x(t_0) + b_1 x'(t_1) = 0 \end{cases} \tag{3.6}$$

given by

$$G(t, s) = \begin{cases} \dfrac{(a_1(t_1 - t) + b_1)(a_0(s - t_0) + b_0)}{a_0 a_1 (t_1 - t_0) + a_0 b_1 + a_1 b_0}, & t_0 \le s \le t \le t_1, \\ \dfrac{(a_1(t_1 - s) + b_1)(a_0(t - t_0) + b_0)}{a_0 a_1 (t_1 - t_0) + a_0 b_1 + a_1 b_0}, & t_0 \le t \le s \le t_1. \end{cases} \tag{3.7}$$

Remark 3.1. It is known that the function z belongs to the class $C^1(J, \mathbb{R})$. Therefore it is bounded on J and there is a constant $C_1 > 0$ with

$$C_1 = \max\left\{ \frac{c_0 a_1(t_1 - t_0) + c_0 b_1 + c_1 a_0 (t_1 - t_0) + c_1 b_0}{a_0 a_1 (t_1 - t_0) + a_0 b_1 + a_1 b_0}, \frac{c_0 b_1 - c_0 a_1 + c_1 a_0 + c_1 b_0}{a_0 a_1 (t_1 - t_0) + a_0 b_1 + a_1 b_0} \right\}$$

such that

$$\|z\| = \max\left\{ \sup_{t \in J} |z(t)|, \sup_{t \in J} |z'(t)| \right\} \le C_1.$$

Remark 3.2. It is easy to see that the Green's function $G(t, s)$ of Lemma 3.1 is continuous in $J \times J$ and $G_t(t, s)$ is continuous in $(a, b) \times (a, b) \setminus \{(t, t) \,|\, t \in J\}$ and satisfy the inequalities

$$|G(t, s)| = G(t, s) \le \frac{(a_1(t_1 - t_0) + b_1)(a_0(t_1 - t_0) + b_0)}{a_0 a_1 (t_1 - t_0) + a_0 b_1 + a_1 b_0} = K_1, \tag{3.8}$$

and

$$
\begin{aligned}
|G_t(t,s)| &= \begin{cases} \dfrac{|-a_1|(a_0(s-t_0)+b_0)}{a_0a_1(t_1-t_0)+a_0b_1+a_1b_0}, & t_0 < s < t < t_1, \\ \dfrac{(a_1(t_1-s)+b_1)a_0}{a_0a_1(t_1-t_0)+a_0b_1+a_1b_0} & t_0 < t < s < t_1 \end{cases} \\
&= \max\left\{\frac{a_1(a_0(t_1-t_0)+b_0)}{a_0a_1(t_1-t_0)+a_0b_1+a_1b_0}, \frac{(a_1(t_1-t_0)+b_1)a_0}{a_0a_1(t_1-t_0)+a_0b_1+a_1b_0}\right\} \\
&= K_2.
\end{aligned}
\tag{3.9}
$$

3.1. **Convex Case.** Consider first the case when F is a convex-valued multi-valued map. We need the following definitions in the sequel.

Definition 3.3. *A multi-valued map $F : J \to \mathcal{P}_{cp,cv}(\mathbb{R})$ is said to be measurable if for every $y \in \mathbb{R}$, the function $t \to d(y, F(t)) = \inf\{\|y-x\| : x \in F(t)\}$ is measurable.*

Definition 3.4. *A multi-valued map $F : J \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}_f(\mathbb{R})$ is called Carathéodory if*

(i) $t \mapsto F(t,x,y)$ is measurable for all $x, y \in \mathbb{R}$, and
(ii) $(x,y) \mapsto F(t,x,y)$ is upper semi-continuous for almost all $t \in J$.

Further a Carathéodory multi-valued function F on $J \times \mathbb{R}$ is called L^1-Carathéodory if

(iii) for each real number $k > 0$, there exists a function $h_k \in L^1(J,\mathbb{R})$ such that

$$\|F(t,x,y)\|_{\mathcal{P}} = \sup\big\{|v| : v \in F(t,x,y)\big\} \le h_k(t), \quad a.e.\ t \in J$$

for all $x, y \in \mathbb{R}$ with $|x| \le k, |y| \le k$.

Then we have the following lemmas due to Lasota and Opial [11].

Lemma 3.2. *If $\dim(X) < \infty$ and $F : J \times X \times X \to \mathcal{P}_{cp,cv}(X)$ L^1-Carathéodory, then $S^1_F(x) \ne \emptyset$ for each $x \in X$.*

Lemma 3.3. *Let X be a Banach space, F an L^1-Carathéodory multi-valued map with $S^1_F \ne \emptyset$ and $\mathcal{L} : L^1(J,X) \to C(J,X)$ be a linear continuous mapping. Then the operator*

$$\mathcal{L} \circ S^1_F : C(J,X) \longrightarrow \mathcal{P}_{cp,cv}(C(J,X))$$

is a closed graph operator in $C(J,X)\times C(J,X)$.

We list here the following assumptions:

(H_1) The multi $F(t,x,y)$ has compact and convex values for each $(t,x,y)\in J\times\mathbb{R}\times\mathbb{R}$.

(H_2) F is Carathéodory.

(H_3) There exists a function $\phi\in L^1(J,\mathbb{R})$ with $\phi(t)>0$ for a.e. $t\in J$ and there is a nondecreasing function $\psi:\mathbb{R}^+\to(0,\infty)$ such that

$$\|F(t,x,y)\|_{\mathcal{P}}=\sup\{|u|\ :\ u\in F(t,x,y)\}\le\phi(t)\psi\big(\max\{|x|,|y|\}\big)$$

for a.e. $t\in J$ and for all $x,y\in\mathbb{R}$.

Theorem 3.5. *Assume that (H_1)-(H_3) hold. Suppose that there is a real number* $r>0$ *such that*

$$r>C_1+\max\{K_1,\ K_2\}\|\phi\|_{L^1}\psi(r), \tag{3.10}$$

where C_1,K_1 *and* K_2 *are the constants defined in Remark 3.2. Then the BVP (1.1)- (1.2) has at least one solution* u *such that* $\|u\|\le r$.

Proof. Let $X=AC^1(J,\mathbb{R})$ and consider an open ball $\mathcal{B}_r(0)$ centered at origin of radius r, where r satisfies the condition given in (3.10). The problem of existence of a solution of BVP (1.1)-(1.2) reduces to finding the solution of the integral inclusion

$$x(t)\in z(t)+\int_{t_0}^{t_1}G(t,s)F(s,x(s),x'(s))\,ds,\ \ t\in J. \tag{3.11}$$

Define a multi-valued map $T:\overline{\mathcal{B}_r(0)}\to\mathcal{P}_f(AC^1(J,\mathbb{R}))$ by

$$Tx=\left\{u\in AC^1(J,\mathbb{R}):u(t)=z(t)+\int_{t_0}^{t_1}G(t,s)v(s)ds,\ \ v\in\overline{S^1_F}(x)\right\}. \tag{3.12}$$

We shall show that the multi T satisfies all the conditions of Corollary 2.4. The proof will be given in several steps.

Step I. We prove that Tx is a convex subset of $AC^1(J,\mathbb{R})$ for each $x\in AC^1(J,\mathbb{R})$. Let $u_1,u_2\in Tx$. Then there exist v_1 and v_2 in $S^1_F(x)$ such that

$$u_j(t)=z(t)+\int_{t_0}^{t_1}G(t,s)v_j(s)\,ds,\ \ \ j=1,2.$$

Since $F(t,x,y)$ has convex values for all $x,y \in \mathbb{R}$, one has for $0 \le k \le 1$

$$[kv_1 + (1-k)v_2](t) \in S^1_F(x)(t), \;\; \forall t \in J.$$

As a result we have

$$[ku_1 + (1-k)u_2](t) = z(t) + \int_{t_0}^{t_1} G(t,s)[kv_1(s) + (1-k)v_2(s)]\, ds.$$

Therefore $[ku_1 + (1-k)u_2] \in Tx$ and consequently T has convex values in $AC^1(J,\mathbb{R})$.

Step II. T maps bounded sets into bounded sets in $AC^1(J,\mathbb{R})$. To see this, let B be a bounded set in $AC^1(J,\mathbb{R})$. Then there exists a real number $q > 0$ such that $\|x\| \le q, \forall x \in B$.

Now for each $u \in Tx$, there exists a $v \in S^1_F(x)$ such that

$$u(t) = z(t) + \int_{t_0}^{t_1} G(t,s)v(s)ds.$$

Then for each $t \in J$,

$$\begin{aligned} |u(t)| &\le |z(t)| + \int_{t_0}^{t_1} |G(t,s)||v(s)|\, ds \\ &\le |z(t)| + \int_{t_0}^{t_1} |G(t,s)|\phi(s)\psi\big(\max\{|x(t)|, |x'(t)|\}\big)\, ds. \end{aligned}$$

Again,

$$\begin{aligned} |u'(t)| &\le |z'(t)| + \int_{t_0}^{t_1} |G_t(t,s)||v(s)|\, ds \\ &\le |z'(t)| + \int_{t_0}^{t_1} |G_t(t,s)|\phi(s)\psi\big(\max\{|x(t)|, |x'(t)|\}\big)\, ds. \end{aligned}$$

This further implies that

$$\begin{aligned} \|u\| &= \max_{t\in J}\{|x(t), \, |x'(t)|\} \\ &\le \max_{t\in J} \max\{|z(t)|, \, |z'(t)|\} \\ &\quad + \int_{t_0}^{t_1} \max_{t,s\in J}\{|G(t,s)|, \, |G_t(t,s)|\}\phi(s)\psi\big(\max\{|x(s)|, |x'(s)|\}\big)\, ds \\ &\le C_1 + \max\{K_1, \, K_2\}\|\phi\|_{L^1}\psi(q) \end{aligned}$$

for all $u \in Tx \subset \bigcup T(B)$. Hence $\bigcup T(B)$ is bounded.

Step III. Next we show that T maps bounded sets into equi-continuous sets. Let B be a bounded set as in step II, and $u \in Tx$ for some $x \in B$. Then there exists $v \in S_F^1(x)$ such that

$$u(t) = z(t) + \int_{t_0}^{t_1} G(t,s)v(s)\,ds.$$

Then for any $t, \tau \in J$, we have

$$\begin{aligned}
|u(t) \quad &- \quad u(\tau)| \\
&\leq |z(t) - z(\tau)| + \left| \int_{t_1}^{t_2} G(t,s)v(s)\,ds - \int_{t_0}^{t_2} G(\tau,s)v(s)\,ds \right| \\
&\leq |z(t) - z(\tau)| + \int_{t_0}^{t_1} |G(t,s) - G(\tau,s)|\,|v(s)|\,ds \\
&\leq |z(t) - z(\tau)| + \int_{t_0}^{t_1} |G(t,s) - G(\tau,s)|\,\phi(s)\psi\big(\max\{|x(s)|, |x'(s)|\}\big)\,ds \\
&\leq |z(t) - z(\tau)| + \int_{t_0}^{t_1} |G(t,s) - G(\tau,s)|\,\phi(s)\psi(q)\,ds.
\end{aligned}$$

Similarly we have

$$|u'(t) - u'(\tau)| \leq |z'(t) - z'(\tau)| + \int_{t_0}^{t_1} |G_t(t,s) - G_t(\tau,s)|\,.$$

Therefore from the above two estimates, it follows that

$$\max\{|u(t) - u(\tau)|,\, |u'(t) - u'(\tau)|\} \to 0, \text{ as } t \to \tau.$$

As a result $\bigcup T(B)$ is an equi-continuous set in $AC^1(J,\mathbb{R})$. Now an application of Arzelá-Ascoli theorem yields that the multi T is completely continuous operator on $AC^1(J,\mathbb{R})$.

Step IV. Next we prove that T has a closed graph. Let $\{x_n\} \subset AC^1(J,\mathbb{R})$ be a sequence such that $x_n \to x_*$ and let $\{y_n\}$ be a sequence defined by $y_n \in Tx_n$ for each $n \in \mathbb{N}$ such that $y_n \to y_*$. We must show that $y_* \in Tx_*$. Since $y_n \in Tx_n$, there exists a $v_n \in S_F^1(x_n)$ such that

$$y_n(t) = z(t) + \int_{t_0}^{t_1} G(t,s)v_n(s)\,ds.$$

Consider the linear and continuous operator $\mathcal{L} : L^1(J,\mathbb{R}) \to AC^1(J,\mathbb{R})$ defined by

$$\mathcal{L}v(t) = \int_{t_0}^{t_1} G(t,s)v(s)\,ds.$$

Now

$$\begin{aligned}\max_{t\in J}\{|y_n(t) - z(t) - (y_*(t) - z(t))|\,,\,&|y_n'(t) - z'(t) - (y_*'(t) - z'(t))|\} \\ &\leq \max_{t\in J}\{|y_n(t) - y_*(t)|,\, |y_n'(t) - y_*'(t)|\} \\ &= \|y_n - y_*\| \to 0 \ \text{ as } \ n \to \infty.\end{aligned}$$

From Lemma 3.2 it follows that $(\mathcal{K} \circ S_F^1)$ is a closed graph operator and from the definition of $\mathcal{L}$ one has

$$y_n - z \in (\mathcal{L} \circ \overline{S_F^1}(x_n)).$$

As $x_n \to x_*$ and $y_n \to y_*$, there is a $v_* \in S_F^1(x_*)$ such that

$$y_*(t) = z(t) + \int_{t_0}^{t_1} G(t,s)v_*(s)ds.$$

Hence the multi T is an upper semi-continuous operator on $\overline{\mathcal{B}_r(0)}$.

Thus, T is an upper semi-continuous and compact operator on $\overline{\mathcal{B}_r(0)}$. Now an application of Corollary 2.4 yields that either (i) the operator inclusion $x \in Tx$ has a solution in $\overline{\mathcal{B}_r(0)}$, or (ii) there is an element $u \in X$ with $\|u\| = r$ such that $\lambda u \in Tu$ for some $\lambda > 1$. We show that the assertion (ii) is not possible. Assume the contrary. Then proceeding with the arguments as in Step II, we obtain

$$r = \|u\| \leq C_1 + \max\{K_1,\, K_2\}\|\phi\|_{L^1}\psi(r),$$

which is a contradiction to (3.10). Hence BVP (1.1) -(1.2) has a solution u on J such that $\|u\| \leq r$. □

3.2. **Nonconvex Case.** Now, we study the case when F is not necessarily convex valued. We give two results. The first, Theorem 3.6, based on Covitz and Nadler fixed point theorem, and the second, Theorem 3.7, based on the Leray-Schauder Alternative for single valued maps combined with a selection theorem due to Bressan and Colombo [4] for lower semicontinuous multivalued operators with decomposable values.

The following assumptions will be needed in the sequel.

(H_4) The multi-valued function $t \mapsto F(t,x,y)$ is measurable and integrably bounded for all $x, y \in \mathbb{R}$.

(H_5) The multi-function $F : J \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}_{cl}(\mathbb{R})$ satisfies

$$H(F(t,x_1,y_1), F(t,x_2,y_2)) \leq \ell_1(t)|x_1 - y_1| + \ell_2(t)|x_2 - y_2| \text{ a.e. } t \in J$$

for all $x_1, x_2, y_1, y_2 \in \mathbb{R}$, where ℓ_1, ℓ_2 are integrable functions.

(H_6) The multi-function $F : J \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}_{cp}(\mathbb{R})$ satisfies:
  (a) $(t,x,y) \mapsto F(t,x,y)$ is $(\mathcal{L} \otimes \mathcal{B} \otimes \mathcal{B})$-measurable, and
  (b) $(x,y) \mapsto F(t,x,y)$ is lower semi-continuous for almost every $t \in J$.

Lemma 3.4. *Let $F : J \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}_{cp}(\mathbb{R})$ be an integrably bounded multi-valued function satisfying (H_6). Then F is of lower semi-continuous type.*

First, we prove an existence result for BVP (1.1)-(1.2) under a Lipschitz condition on multi-valued function F.

Theorem 3.6. *Assume that the hypotheses (H_4) and (H_5) hold and suppose that*

$$(\|\ell_1\|_{L^1} + \|\ell_2\|_{L^1}) \max\{K_1,\, K_2\} < 1,$$

where K_1 and K_2 are given in Remark 3.2. Then the BVP (1.1)-(1.2) has at least one solution on J.

Proof. First, we transform the BVP (1.1)-(1.2) into a fixed point inclusion problem in a suitable Banach space. Let $X = C^1(J, \mathbb{R})$ be equipped with the norm given by (3.2). Then X is a Banach space with this norm. Define a multi-valued operator T on X by (3.12). Then the BVP (1.1)-(1.2) is equivalent to the operator inclusion

$$x(t) \in Tx(t), \; t \in J. \tag{3.13}$$

We will show that the multi-valued operator T satisfies all the conditions of Theorem 2.2. Clearly the operator T is well defined since $S^1_F(x) \neq \emptyset$ for each $x \in X$.

First we show that Tx is closed subset of X for each $x \in X$. This follows easily if we show the values of Nemytskii operator S^1_F has closed values in $L^1(J, \mathbb{R})$. Let $\{w_n\}$ be a sequence in $L^1(J, \mathbb{R})$ converging to a point w. Then $w_n \to w$ in measure, and so, there exists a subsequence S of positive integers with w_n converging a.e. to w as $n \to \infty$ through S. Now since (H_4) holds, the values of S^1_F are closed in $L^1(J,\mathbb{R})$. Thus for each $x \in X$, we have that Tx is non-empty and closed subset of X.

Next we show that T is a multi-valued contraction on X. Let $x, y \in X$ and let $u_1 \in T(x)$. Then $u_1 \in X$ and $u_1(t) = z(t) + \int_{t_0}^{t_1} k(t,s)v_1(s)\,ds$ for some $v_1 \in S^1_F(x)$. From hypothesis (H_5) it follows that

$$H(F(t,x(t),x'(t)),F(t,y(t),y'(t)) \leq \ell_1(t)|x(t)-y(t)| + \ell_2(t)|x'(t)-y'(t)|.$$

Hence there is $w \in F(t,y(t),y'(t))$ such that

$$|v_1(t) - w| \leq \ell_1(t)|x(t)-y(t)| + \ell_2(t)|x'(t)-y'(t)|.$$

Thus the multi-valued operator U defined by $U(t) = S^1_F(y)(t) \cap K(t)$ $t \in J$, where $K(t)$ is given by

$$K(t) = \big\{w|\ |v_1(t)-w| \leq \ell_1(t)|x(t)-y(t)| + \ell_2(t)|x'(t)-y'(t)|\big\},$$

has nonempty values and is measurable. Let v_2 be a measurable selection for U (which does exist by Kuratowski-Ryll-Nardzewski's selection theorem. See [1]). Then $v_2 \in F(t,y(t),y'(t))$ and

$$|v_1(t)-v_2(t)| \leq \ell_1(t)|x(t)-y(t)| + \ell_2(t)|x'(t)-y'(t)| \quad \text{for} \quad \text{a.e.} \quad t \in J.$$

Define $u_2(t) = z(t) + \int_{t_0}^{t_1} k(t,s)v_2(s)\,ds$. It follows that $u_2 \in Tx$ and

$$\begin{aligned}
|u_1(t)-u_2(t)| &\leq \left|\int_{t_0}^{t_1} k(t,s)v_1(s)\,ds - \int_{t_0}^{t_1} k(t,s)v_2(s)\,ds\right| \\
&\leq \int_{t_0}^{t_1} |k(t,s)||v_1(s)-v_2(s)|\,ds \\
&\leq \int_{t_0}^{t_1} k(t,s)\big[\ell_1(t)|x(t)-y(t)| + \ell_2(t)|x'(t)-y'(t)|\big]\,ds \\
&\leq K_1\,(\|\ell_1\|_{L^1} + \|\ell_2\|_{L^1})\|x-y\|.
\end{aligned}$$

Similarly we have

$$|u_1'(t) - u_2'(t)| \leq K_2(\|\ell_1\|_{L^1} + \|\ell_2\|_{L^1})\|x-y\|.$$

Therefore,

$$\|u_1 - u_2\| \leq (\|\ell_1\|_{L^1} + \|\ell_2\|_{L^1})\max\{K_1,\ K_2\}\|x-y\|.$$

From this, and the analogous inequality obtained by interchanging the roles of x and y we obtain

$$H(T(x), T(y)) \leq (\|\ell_1\|_{L^1} + \|\ell_2\|_{L^1}) \max\{K_1,\ K_2\} \|x - y\|,$$

for all $x, y \in X$. This shows that T is a multi-valued contraction since $(\|\ell_1\|_{L^1} + \|\ell_2\|_{L^1}) \max\{K_1,\ K_2\} < 1$. Now an application of Theorem 2.2 yields that T has a fixed point which further implies that the BVP (1.1)-(1.2) has a solution on J. □

Now, we prove our second existence result for BVP (1.1)-(1.2).

Theorem 3.7. *Assume that the hypotheses (H_3)-(H_6) hold and there exists a real number $r > 0$ satisfying*

$$r > C_1 + \max\{K_1,\ K_2\} \|\phi\|_{L^1} \psi(r), \tag{3.14}$$

where C_1, K_1 and K_2 are the constants defined in Remark 3.2. Then the BVP (1.1)-(1.2) has at least one solution on J.

Proof. First, we transform the BVP (1.1)-(1.2) into a fixed point problem in a suitable normed linear space. The problem of existence of a solution of BVP (1.1)- (1.2) reduces to finding a solution of the integral equation

$$x(t) = z(t) + \int_{t_0}^{t_1} k(t,s) f(x(s))\, ds, \ \ t \in J, \tag{3.15}$$

where $f(x(\cdot)) \in L^1$ with $f(x(t)) \in F(t, x(t), x'(t))$ a.e. $t \in J$ (this is a consequence of (H_3), (H_6) and Theorem 2.8). We study the integral equation (3.15) in the space $AC^1(J, \mathbb{R})$. Let $X = AC^1(J, \mathbb{R})$ and define an open ball $\mathcal{B}_r(0)$ in X centered at origin 0 of radius r, where the real number $r > 0$ satisfies the inequality (3.12). Define the operator T on $\overline{\mathcal{B}_r(0)}$ by

$$Tx(t) = z(t) + \int_{t_0}^{t_1} k(t,s) f(x((s)))\, ds. \tag{3.16}$$

Then the integral equation (3.15) is equivalent to the operator equation

$$x(t) = Tx(t), \ \ t \in J. \tag{3.17}$$

We will show that the multi-valued operator T satisfies all the conditions of Corollary 2.5.

First, we show that T is continuous on $\overline{\mathcal{B}_r(0)}$. Since (H_3) holds, we have

$$|f(x(t))| \leq \phi(t)\psi(\max\{|x(t)\,,\,|x'(t)|\}) \text{ a.e. } t \in J$$

for all $x \in AC^1(J,\mathbb{R})$. Let $\{x_n\}$ be a sequence in $\overline{\mathcal{B}_r(0)}$ converging to a point $x \in \overline{\mathcal{B}_r(0)}$. Then

$$|f(x_n(t))| \leq \phi(t)\psi(r) \text{ a.e. } t \in J.$$

Hence by the dominated convergence theorem and continuity of f, we obtain

$$\begin{aligned}\lim_{n\to\infty} Tx_n(t) &= z(t) + \int_{t_0}^{t_1} G(t,s)f(x_n((s)))\,ds\\ &= z(t) + \int_{t_0}^{t_1} G(t,s)f(x((s)))\,ds\\ &= Tx(t)\end{aligned}$$

and

$$\begin{aligned}\lim_{n\to\infty} (Tx_n)'(t) &= z'(t) + \int_{t_0}^{t_1} G_t(t,s)f(x_n((s)))\,ds\\ &= z'(t) + \int_{t_0}^{t_1} G_t(t,s)f(x((s)))\,ds\\ &= (Tx)'(t)\end{aligned}$$

for all $t \in J$. As a result, T is continuous on $\overline{\mathcal{B}_r(0)}$. Next following the arguments as in the proof of Theorem 3.5 with appropriate modifications, it is shown that T is a compact operator on $\overline{\mathcal{B}_r(0)}$. Now an application of Corollary 2.5 yields that either (i) the operator equation $x = Tx$ has a solution in $\overline{\mathcal{B}_r(0)}$, or (ii) there is an element $u \in X$ such that $\|u\| = r$ and $u = \lambda Tu$ for some $\lambda \in (0,1)$. If the assertion (ii) holds, then we obtain a contradiction to (3.12). Hence assertion (i) holds and the BVP (1.1)-(1.2) has a solution $u \in AC^1(J,\mathbb{R})$ such that $\|u\| \leq r$. This completes the proof. □

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