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WEAK CONVERGENCE THEOREM BY A MODIFIED EXTRAGRADIENT METHOD FOR NONEXPANSIVE MAPPINGS AND MONOTONE MAPPINGS

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Abstract. In this paper, we introduce a modified extragradient method for finding the common element of the set of fixed points of a nonexpansive mapping and the set of solutions of the variational inequality problem for a monotone and Lipschitz continuous mapping. We obtain a weak convergent theorem for two sequences generated by this modified extragradient method.

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1. Introduction

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. Let C be a nonempty closed convex subset of H and let P_C be the metric projection of H onto C. A mapping $A : C \to H$ is called monotone if

$$\langle Au - Av, u - v\rangle \geq 0, \quad \forall u, v \in C.$$

The variational inequality problem is the problem of finding $u \in C$ such that

$$\langle Au, v - u\rangle \geq 0, \quad \forall v \in C.$$

The set of solutions of the variational inequality problem is denoted by Ω. A mapping $A : C \to H$ is called α-inverse-strongly-monotone if there exists a positive real number α such that

$$\langle Au - Av, u - v\rangle \geq \alpha\|Au - Av\|^2, \quad \forall u, v \in C;$$

see Refs. 1-2. It is obvious that each α-inverse-strongly-monotone mapping A is monotone and Lipschitz continuous. A mapping $S : C \to C$ is called nonexpansive if

$$\|Su - Sv\| \leq \|u - v\|, \quad \forall u, v \in C;$$

see Ref. 3. We denote by $F(S)$ the set of fixed points of S. For finding an element of $F(S) \cap \Omega$ under the assumption that a set $C \subset H$ is nonempty, closed and convex, a mapping $S : C \to C$ is nonexpansive and a mapping $A : C \to H$ is α-inverse-strongly-monotone, Takahashi and Toyoda (Ref. 4) introduced the following iterative scheme:

$$\begin{cases} x_0 = x \in C, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S P_C(x_n - \lambda_n A x_n), \ \forall n \geq 0, \end{cases} \tag{I}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$, and $\{\lambda_n\}$ is a sequence in $(0, 2\alpha)$. They proved that, if $F(S) \cap \Omega \neq \emptyset$, then the sequence $\{x_n\}$ generated by (I) converges weakly to some element of $F(S) \cap \Omega \neq \emptyset$. On the other hand, for solving the variational inequality problem in the finite-dimensional Euclidean space $\mathcal{R}^n$ under the assumption that a set $C \subset \mathcal{R}^n$ is nonempty, closed and convex, a mapping $A : C \to \mathcal{R}^n$ is monotone and k-Lipschitz continuous and Ω is nonempty, Korpelevich (Ref. 5) introduced the following so-called extragradient method:

$$\begin{cases} x_0 = x \in C, \\ \bar{x}_n = P_C(x_n - \lambda A x_n), \\ x_{n+1} = P_C(x_n - \lambda A \bar{x}_n), \ \forall n \geq 0, \end{cases} \tag{II}$$

where $\lambda \in (0, 1/k)$. He showed that the sequences $\{x_n\}$ and $\{\bar{x}_n\}$ generated by this extragradient method, converge to the same point $z \in \Omega$. Recently, motivated by the idea of Korpelevich's extragradient method (Ref. 5), Nadezhkina and Takahashi (see Ref. 10) introduced an iterative scheme for finding an element of $F(S) \cap \Omega$ and presented the following weak convergence result (see Theorem 3.1 in Ref. 10).

Theorem 1.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A : C \to H$ be a monotone, k-Lipschitz continuous mapping and $S : C \to C$ be a nonexpansive mapping such that $F(S) \cap \Omega \neq \emptyset$. Let $\{x_n\}, \{y_n\}$ be the sequences generated by*

$$\begin{cases} x_0 = x \in C, \\ y_n = P_C(x_n - \lambda_n A x_n), \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S P_C(x_n - \lambda_n A y_n), \ \forall n \geq 0, \end{cases} \tag{1}$$

where $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, 1/k)$ and $\{\alpha_n\} \subset [c, d]$ for some $c, d \in (0, 1)$.

Then the sequences $\{x_n\}, \{y_n\}$ converge weakly to the same point $z \in F(S) \cap \Omega$ where $z = \lim_{n\to\infty} P_{F(S)\cap\Omega} x_n$.

Very recently, inspired by Nadezhkina and Takahashi's iterative scheme (Ref. 10), L.C. Zeng and J.C. Yao (see Ref. 12) introduced another iterative scheme for finding an element of $F(S) \cap \Omega$ and obtained the following strong convergence theorem (See Theorem 3.1 in Ref. 12).

Theorem 1.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A : C \to H$ be a monotone, k-Lipschitz continuous mapping and $S : C \to C$ be a nonexpansive mapping such that $F(S) \cap \Omega \neq \emptyset$. Let $\{x_n\}, \{y_n\}$ be the sequences generated by*

$$\begin{cases} x_0 = x \in C, \\ y_n = P_C(x_n - \lambda_n A x_n), \\ x_{n+1} = \alpha_n x_0 + (1 - \alpha_n) S P_C(x_n - \lambda_n A y_n), \ \forall n \geq 0, \end{cases}$$

where $\{\lambda_n\}$ *and* $\{\alpha_n\}$ *satisfy the conditions: (a)* $\{\lambda_n k\} \subset (0, 1-\delta)$ *for some* $\delta \in (0,1)$*, and (b)* $\{\alpha_n\} \subset (0,1)$*,* $\sum_{n=0}^{\infty} \alpha_n = \infty$*,* $\lim_{n\to\infty} \alpha_n = 0$*.*

Then the sequences $\{x_n\}, \{y_n\}$ *converge strongly to the same point* $P_{F(S)\cap\Omega}(x_0)$*, provided*

$$\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0.$$

In this paper, inspired by Takahashi and Toyoda (Ref. 4), Korpelevich (Ref. 5), Nadezhkina and Takahashi (Ref. 10) and L.C. Zeng and J.C. Yao (Ref. 12), we introduce and consider a modified extragradient method, as follows:

$$\begin{cases} x_0 = x \in C, \\ y_n = P_C[(1-\beta_n)(x_n - \lambda_n A x_n) + \beta_n P_C(x_n - \lambda_n A x_n)], \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n) S P_C(x_n - \lambda_n A y_n), \ \forall n \geq 0. \end{cases}$$

Suppose $\{\alpha_n\}, \{\beta_n\}$ are sequences in $[0,1]$ and $\{\lambda_n\}$ is a sequence in $(0, 1/k)$ such that

(i) $\limsup_{n\to\infty} \alpha_n < 1$ and $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < 1$;

(ii) $\sum_{n=0}^{\infty} \beta_n^2 < \infty$.

It is shown that the sequences $\{x_n\}, \{y_n\}$ generated by the above modified extragradient method converge weakly to the same point $z \in F(S) \cap \Omega$ where $z = \lim\limits_{n\to\infty} P_{F(S)\cap\Omega} x_n$. It is easy to see that if $\beta_n = 0$ for all $n \geq 0$ then the last iterative scheme reduces to Nadezhkina and Takahashi's one (1). Our main result improves and extends of Nadezhkina and Takahashi's Theorem 3.1 in Ref. 10.

Throughout the rest of this paper, we denote by "$\rightarrow$" and "$\rightharpoonup$" the strong convergence and weak convergence, respectively.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. It is well known that there holds the identity

$$\|\lambda x + (1-\lambda) y\|^2 = \lambda \|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2, \quad \forall x, y \in H, \lambda \in [0,1].$$

Let C be a nonempty closed convex subset of H. Then, for any $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\|, \quad \forall y \in C.$$

Such a P_C is called the metric projection of H onto C. We know that P_C is a nonexpansive mapping of H onto C. It is also known that P_C is characterized by the following properties (see Ref. 3 for more details): $P_C x \in C$ and for all $x \in H, y \in C$,

$$\langle x - P_C, P_C x - y\rangle \geq 0, \tag{2}$$

and

$$\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2. \tag{3}$$

Let $A : C \to H$ be a mapping. It is easy to see from (2) that the following implications hold:

$$\bar{x} \in \Omega \;\Leftrightarrow\; \bar{x} = P_C(\bar{x} - \lambda A\bar{x}),\; \forall \lambda > 0. \tag{4}$$

It is also known that H satisfies the Opial property (Ref. 6); i.e., for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

holds for every $y \in H$ with $y \neq x$. A set-valued mapping $T : H \to 2^H$ is called monotone if, for all $x, y \in H$, $f \in Tx$ and $g \in Ty$ we have $\langle x - y, f - g\rangle \geq 0$. A monotone mapping $T : H \to 2^H$ is maximal if its graph $G(T)$ is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal if and only if for $(x, f) \in H \times H$, $\langle x - y, f - g\rangle \geq 0$ for all $(y, g) \in G(T)$, then $f \in Tx$. Let $A : C \to H$ be a monotone, k-Lipschitz continuous mapping and let $N_C v$ be the normal cone to C at $v \in C$; i.e., $N_C v = \{w \in H : \langle v - y, w\rangle \geq 0,\ \forall y \in C\}$. Define

$$Tv = \begin{cases} Av + N_C v, & \text{if } v \in C, \\ \emptyset, & \text{if } v \notin C. \end{cases}$$

Then T is maximal monotone and $0 \in Tv$ if and only if $v \in \Omega$; see Ref. 7.

In order to prove the main result in Section 3, we shall use the following lemmas in the sequel.

Lemma 2.1. *Let H be a real Hilbert space, let $\{\alpha_n\}$ be a sequence of real numbers such that $0 < a \leq \alpha_n \leq b < 1$ for all $n \geq 0$, and let $\{v_n\}$ and $\{w_n\}$ be sequences in H such that*

$$\limsup_{n\to\infty} \|v_n\| \leq c,\; \limsup_{n\to\infty} \|w_n\| \leq c \text{ and } \lim_{n\to\infty} \|\alpha_n v_n + (1 - \alpha_n) w_n\| = c$$

for some $c \geq 0$. Then, $\lim_{n\to\infty} \|v_n - w_n\| = 0$.

Lemma 2.2. *Let H be a real Hilbert space and let D be a nonempty closed convex subset of H. Let $\{x_n\}$ be a sequence in H. Suppose that, for all $u \in D$,*

$$\|x_{n+1} - u\| \leq \|x_n - u\|, \quad \forall n \geq 0.$$

Then the sequence $\{P_D x_n\}$ converges strongly to some $z \in D$.

Remark 2.1. Regarding the above two lemmas, let us notice that the first was proved by Schu (Ref. 8) in a uniformly convex Banach space and the second was proved by Takahashi and Toyoda (Ref. 4).

Lemma 2.3. (The demiclosedness principle, see Ref. 3.) *Assume that S is a nonexpansive self-mapping of a nonempty closed convex subset C of a real Hilbert space H. If $F(S) \neq \emptyset$, then $I - S$ is demiclosed; that is, whenever $\{x_n\}$ is a sequence in C weakly converging to some $x \in C$ and the sequence $\{(I - S)x_n\}$ strongly converges to some y, it follows that $(I - S)x = y$, where I stands for the identity operator of H.*

Lemma 2.4. (see Ref. 11.) *Let $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ be two sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq a_n + b_n, \quad \forall n \geq 0.$$

If $\sum_{n=0}^{\infty} b_n$ converges, then $\lim_{n\to\infty} a_n$ exists.

3. Weak Convergence Theorem

In this section, we deal with an iterative scheme by the modified extragradient method for finding the common element of the set of fixed points of a nonexpansive mapping and the set of solutions of the variational inequality problem for a monotone, Lipschitz continuous mapping in a Hilbert space.

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A : C \to H$ be a monotone k-Lipschitz continuous mapping and let $S : C \to C$ be a nonexpansive mapping such that $F(S) \cap \Omega \neq \emptyset$. Let $\{x_n\}, \{y_n\}$ be the sequences generated by*

$$\begin{cases} x_0 = x \in C, \\ y_n = P_C[(1 - \beta_n)(x_n - \lambda_n A x_n) + \beta_n P_C(x_n - \lambda_n A x_n)], \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S P_C(x_n - \lambda_n A y_n), \ \forall n \geq 0. \end{cases}$$

Suppose $\{\alpha_n\}, \{\beta_n\}$ are sequences in $[0, 1]$ and $\{\lambda_n\}$ is a sequence in $(0, 1/k)$ such that

(i) $\limsup_{n\to\infty} \alpha_n < 1$ *and* $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < 1/k$;

(ii) $\sum_{n=0}^{\infty} \beta_n^2 < \infty$.

Then, the sequences $\{x_n\}, \{y_n\}$ *converge weakly to the same point* $z \in F(S) \cap \Omega$, *where* $z = \lim\limits_{n\to\infty} P_{F(S)\cap\Omega} x_n$, *provided* $\{Ax_n\}$ *is bounded.*

Proof. First, we claim that $\{x_n\}$ is bounded. Indeed, put $t_n = P_C(x_n - \lambda_n Ay_n)$ for all $n \geq 0$. Let $x^* \in F(S) \cap \Omega$. Then $x^* = P_C(x^* - \lambda_n Ax^*)$. Taking $x = x_n - \lambda_n Ay_n$ and $y = x^*$ in (3), we obtain

$$\begin{aligned}
\|t_n - x^*\|^2 &\leq \|x_n - \lambda_n Ay_n - x^*\|^2 - \|x_n - \lambda_n Ay_n - t_n\|^2 \\
&= \|x_n - x^*\|^2 - 2\lambda_n \langle Ay_n, x_n - x^* \rangle + \lambda_n^2 \|Ay_n\|^2 \\
&\quad - \|x_n - t_n\|^2 + 2\lambda_n \langle Ay_n, x_n - t_n \rangle - \lambda_n^2 \|Ay_n\|^2 \\
&= \|x_n - x^*\|^2 + 2\lambda_n \langle Ay_n, x^* - t_n \rangle - \|x_n - t_n\|^2 \\
&= \|x_n - x^*\|^2 - \|x_n - t_n\|^2 - 2\lambda_n \langle Ay_n - Ax^*, y_n - x^* \rangle \\
&\quad - 2\lambda_n \langle Ax^*, y_n - x^* \rangle + 2\lambda_n \langle Ay_n, y_n - t_n \rangle \\
&\leq \|x_n - x^*\|^2 - \|x_n - y_n\|^2 - \|y_n - t_n\|^2 \\
&\quad + 2\langle x_n - \lambda_n Ay_n - y_n, t_n - y_n \rangle.
\end{aligned}$$

Now, observe that

$$\begin{aligned}
&\langle x_n - \lambda_n Ax_n - P_C(x_n - \lambda_n Ax_n), P_C(x_n - \lambda_n Ax_n) - y_n \rangle \\
&\leq \|x_n - \lambda_n Ax_n - P_C(x_n - \lambda_n Ax_n)\| \|P_C(x_n - \lambda_n Ax_n) - y_n\| \\
&\leq \{\lambda_n \|Ax_n\| + \|x_n - P_C(x_n - \lambda_n Ax_n)\|\} \|x_n - \lambda_n Ax_n \\
&\quad - [(1 - \beta_n)(x_n - \lambda_n Ax_n) + \beta_n P_C(x_n - \lambda_n Ax_n)]\| \\
&= \{\lambda_n \|Ax_n\| + \|P_C x_n - P_C(x_n - \lambda_n Ax_n)\|\} \beta_n \|x_n - \lambda_n Ax_n \\
&\quad - P_C(x_n - \lambda_n Ax_n)\| \\
&\leq \{2\lambda_n \|Ax_n\|\} \beta_n \{\lambda_n \|Ax_n\| + \|P_C x_n - P_C(x_n - \lambda_n Ax_n)\|\} \\
&\leq \{2\lambda_n \|Ax_n\|\} \beta_n \{2\lambda_n \|Ax_n\|\} \\
&= 4\beta_n \lambda_n^2 \|Ax_n\|^2.
\end{aligned}$$

Further, from (2) we have

$$
\begin{aligned}
&\langle x_n - \lambda_n A y_n - y_n, t_n - y_n\rangle \\
&= \langle x_n - \lambda_n A x_n - y_n, t_n - y_n\rangle + \langle \lambda_n A x_n - \lambda_n A y_n, t_n - y_n\rangle \\
&= \langle (1-\beta_n)(x_n - \lambda_n A x_n) + \beta_n P_C(x_n - \lambda_n A x_n) - y_n, t_n - y_n\rangle \\
&\ + \beta_n\langle x_n - \lambda_n A x_n - P_C(x_n - \lambda_n A x_n), t_n - y_n\rangle \\
&\ + \langle \lambda_n A x_n - \lambda_n A y_n, t_n - y_n\rangle \\
&\le \beta_n\langle x_n - \lambda_n A x_n - P_C(x_n - \lambda_n A x_n), t_n - y_n\rangle \\
&\ + \langle \lambda_n A x_n - \lambda_n A y_n, t_n - y_n\rangle \\
&= \beta_n\langle x_n - \lambda_n A x_n - P_C(x_n - \lambda_n A x_n), t_n - P_C(x_n - \lambda_n A x_n)\rangle \\
&\ + \beta_n\langle x_n - \lambda_n A x_n - P_C(x_n - \lambda_n A x_n), P_C(x_n - \lambda_n A x_n) - y_n\rangle \\
&\ + \langle \lambda_n A x_n - \lambda_n A y_n, t_n - y_n\rangle \\
&\le \beta_n\langle x_n - \lambda_n A x_n - P_C(x_n - \lambda_n A x_n), P_C(x_n - \lambda_n A x_n) - y_n\rangle \\
&\ + \langle \lambda_n A x_n - \lambda_n A y_n, t_n - y_n\rangle \\
&\le 4\beta_n^2\lambda_n^2\|Ax_n\|^2 + \lambda_n k\|x_n - y_n\|\|t_n - y_n\| \\
&\le 4\beta_n^2\lambda_n^2\|Ax_n\|^2 + \lambda_n^2 k^2\|x_n - y_n\|^2 + \|y_n - t_n\|^2.
\end{aligned}
$$

Since $\limsup_{n\to\infty}\alpha_n < 1$ and $0 < \liminf_{n\to\infty}\lambda_n \le \limsup_{n\to\infty}\lambda_n < 1/k$, there exists $N_0 \ge 1$ such that

$$\{\alpha_n\}_{n=N_0}^{\infty} \subset [0,a] \text{ and } \{\lambda_n\}_{n=N_0}^{\infty} \subset [b,d]$$

for some $a \in (0,1)$ and $b, d \in (0, 1/k)$. Thus, we deduce that for all $n \ge N_0$

$$
\begin{aligned}
\|t_n - x^*\|^2 &\le \|x_n - x^*\|^2 - \|x_n - y_n\|^2 - \|y_n - t_n\|^2 \\
&\quad +4\beta_n^2\lambda_n^2\|Ax_n\|^2 + \lambda_n^2 k^2\|x_n - y_n\|^2 + \|y_n - t_n\|^2 \\
&= \|x_n - x^*\|^2 + (\lambda_n^2 k^2 - 1)\|x_n - y_n\|^2 + 4\beta_n^2\lambda_n^2\|Ax_n\|^2 \\
&\le \|x_n - x^*\|^2 + 4\beta_n^2\lambda_n^2\|Ax_n\|^2,
\end{aligned}
\tag{5}
$$

and hence

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|\alpha_n x_n + (1-\alpha_n)St_n - x^*\|^2 \\
&= \|\alpha_n(x_n - x^*) + (1-\alpha_n)(St_n - x^*)\|^2 \\
&\le \alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)\|St_n - x^*\|^2 \\
&\le \alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)\|t_n - x^*\|^2 \\
&\le \alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)\{\|x_n - x^*\|^2 \\
&\quad +(\lambda_n^2 k^2 - 1)\|x_n - y_n\|^2 + 4\beta_n^2\lambda_n^2\|Ax_n\|^2\} \\
&= \|x_n - x^*\|^2 + (1-\alpha_n)\{(\lambda_n^2 k^2 - 1)\|x_n - y_n\|^2 \\
&\quad +4\beta_n^2\lambda_n^2\|Ax_n\|^2\} \\
&\le \|x_n - x^*\|^2 + 4\beta_n^2\lambda_n^2\|Ax_n\|^2.
\end{aligned}$$

Note that $\{Ax_n\}$ is bounded and $\sum_{n=0}^{\infty}\beta_n^2$ is convergent. Therefore, according to Lemma 2.4, there exists

$$c = \lim_{n\to\infty}\|x_n - x^*\|$$

and hence the sequences $\{x_n\}, \{t_n\}$ are bounded. From the last relations, we also obtain

$$\begin{aligned}
(1-\alpha_n)(1-\lambda_n^2 k^2)\|x_n - y_n\|^2 &\le \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\
&\quad +(1-\alpha_n)4\beta_n^2\lambda_n^2\|Ax_n\|^2.
\end{aligned}$$

So we have for all $n \ge N_0$

$$\begin{aligned}
\|x_n - y_n\|^2 &\le \tfrac{1}{(1-\alpha_n)(1-\lambda_n^2 k^2)}(\|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2) \\
&\quad +\tfrac{1}{1-\lambda_n^2 k^2}\cdot 4\beta_n^2\lambda_n^2\|Ax_n\|^2.
\end{aligned}$$

Since there exists $N_0 \ge 1$ such that

$$\{\alpha_n\}_{n=N_0}^{\infty} \subset [0,a] \text{ and } \{\lambda_n\}_{n=N_0}^{\infty} \subset [b,d]$$

for some $a \in (0,1)$ and $b, d \in (0, 1/k)$, so we have

$$x_n - y_n \to 0 \quad \text{as } n \to \infty.$$

Further, we obtain

$$
\begin{aligned}
\|y_n - t_n\|^2 &= \|P_C[(1-\beta_n)(x_n - \lambda_n Ax_n) + \beta_n P_C(x_n - \lambda_n Ax_n)] - P_C(x_n - \lambda_n Ay_n)\|^2 \\
&\le \|(1-\beta_n)(x_n - \lambda_n Ax_n) + \beta_n P_C(x_n - \lambda_n Ax_n) - (x_n - \lambda_n Ay_n)\|^2 \\
&= \|(1-\beta_n)\lambda_n(Ay_n - Ax_n) + \beta_n[P_C(x_n - \lambda_n Ax_n) - (x_n - \lambda_n Ay_n)]\|^2 \\
&\le (1-\beta_n)\lambda_n^2\|Ay_n - Ax_n\|^2 + \beta_n\|P_C(x_n - \lambda_n Ax_n) - (x_n - \lambda_n Ay_n)\|^2 \\
&\le \lambda_n^2 k^2\|y_n - x_n\|^2 + \beta_n\{\|P_C(x_n - \lambda_n Ax_n) - P_C x_n\| + \lambda_n\|Ay_n\|\}^2 \\
&\le \lambda_n^2 k^2\|y_n - x_n\|^2 + \beta_n(\lambda_n\|Ax_n\| + \lambda_n\|Ay_n\|)^2 \\
&= \lambda_n^2 k^2\|y_n - x_n\|^2 + \beta_n\lambda_n^2(\|Ax_n\| + \|Ay_n\|)^2 \\
&\le \frac{\lambda_n^2 k^2}{(1-\alpha_n)(1-\lambda_n^2 k^2)}(\|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2) \\
&\quad + \frac{\lambda_n^2 k^2}{1-\lambda_n^2 k^2}\cdot 4\beta_n^2\lambda_n^2\|Ax_n\|^2 + \beta_n\lambda_n^2(\|Ax_n\| + \|Ay_n\|)^2.
\end{aligned}
$$

Since $\sum_{n=0}^{\infty}\beta_n^2 < \infty$, we have $\lim_{n\to\infty}\beta_n = 0$. Hence, we get

$$y_n - t_n \to 0 \quad \text{as } n \to \infty.$$

From

$$\|x_n - t_n\| \le \|x_n - y_n\| + \|y_n - t_n\|,$$

we have also

$$x_n - t_n \to 0 \quad \text{as } n \to \infty.$$

Since A is Lipschitz continuous, we have

$$Ay_n - At_n \to 0 \quad \text{as } n \to \infty.$$

As $\{x_n\}$ is bounded, there is a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ that converges weakly to some z. We claim that $z \in F(S)\cap\Omega$. Indeed, first, we show that $z \in \Omega$. Since $x_n - t_n \to 0$ and $y_n - t_n \to 0$, we have $t_{n_i} \rightharpoonup z$ and $y_{n_i} \rightharpoonup z$. Let

$$Tv = \begin{cases} Av + N_C v, & \text{if } v \in C, \\ \emptyset & \text{if } v \notin C. \end{cases}$$

Then, T is maximal monotone and $0 \in Tv$ if and only if $v \in \Omega$; see Ref. 7. Let $(v, w) \in G(T)$. Then, we have

$$w \in Tv = Av + N_C v$$

and hence $w - Av \in N_C v$. So, we have

$$\langle v - u, w - Av\rangle \ge 0, \quad \forall u \in C.$$

On the other hand, from

$$t_n = P_C(x_n - \lambda_n A y_n) \text{ and } v \in C,$$

we have

$$\langle x_n - \lambda_n A y_n - t_n, t_n - v\rangle \geq 0,$$

and hence

$$\langle v - t_n, \frac{t_n - x_n}{\lambda_n} + A y_n\rangle \geq 0.$$

Therefore, from

$$w - Av \in N_C v \text{ and } t_i \in C,$$

we have

$$\begin{aligned}\langle v - t_{n_i}, w\rangle &\geq \langle v - t_{n_i}, Av\rangle \\ &\geq \langle v - t_{n_i}, Av\rangle - \langle v - t_{n_i}, \tfrac{t_{n_i} - x_{n_i}}{\lambda_{n_i}} + A y_{n_i}\rangle \\ &= \langle v - t_{n_i}, Av - At_{n_i}\rangle + \langle v - t_{n_i}, At_{n_i} - Ay_{n_i}\rangle \\ &\ \ -\langle v - t_{n_i}, \tfrac{t_{n_i} - x_{n_i}}{\lambda_{n_i}}\rangle \\ &\geq \langle v - t_{n_i}, At_{n_i} - Ay_{n_i}\rangle - \langle v - t_{n_i}, \tfrac{t_{n_i} - x_{n_i}}{\lambda_{n_i}}\rangle.\end{aligned}$$

Hence, letting $n_i \to \infty$, we obtain

$$\langle v - z, w\rangle \geq 0.$$

Since T is maximal monotone, we have $z \in T^{-1}0$ and hence $z \in \Omega$. We show that $z \in F(S)$. Indeed, let $x^* \in F(S) \cap \Omega$. Since it follows from (5) that for all $n \geq N_0$

$$\|St_n - x^*\|^2 \leq \|t_n - x^*\|^2 \leq \|x_n - x^*\|^2 + 4\beta_n^2\lambda_n^2\|Ax_n\|^2,$$

we have

$$\limsup_{n\to\infty} \|St_n - x^*\| \leq c.$$

Further, we have

$$\lim_{n\to\infty} \|\alpha_n(x_n - x^*) + (1-\alpha_n)(St_n - x^*)\| = \lim_{n\to\infty} \|x_{n+1} - x^*\| = c.$$

By Lemma 2.1 we obtain

$$\lim_{n\to\infty} \|St_n - x_n\| = 0.$$

Since

$$\|Sx_n - x_n\| \leq \|Sx_n - St_n\| + \|St_n - x_n\| \leq \|x_n - t_n\| + \|St_n - x_n\|,$$

we have

$$\lim_{n\to\infty} \|Sx_n - x_n\| = 0.$$

By Lemma 2.3, from the demiclosedness of $I - S$, we know that $x_{n_i} \rightharpoonup z$ and $\lim_{n\to\infty} \|Sx_n - x_n\| = 0$ imply $z \in F(S)$.

Let $\{x_{n_j}\}$ be another subsequence of $\{x_n\}$ such that $x_{n_j} \rightharpoonup z'$. Then, $z' \in F(S) \cap \Omega$. Let us show that $z = z'$. Assume that $z \neq z'$. From the Opial condition (Ref. 6) we have

$$\begin{aligned}\lim_{n\to\infty} \|x_n - z\| &= \liminf_{n\to\infty} \|x_{n_i} - z\| < \liminf_{i\to\infty} \|x_{n_i} - z'\| \\ &= \lim_{n\to\infty} \|x_n - z'\| = \liminf_{j\to\infty} \|x_{n_j} - z'\| \\ &< \liminf_{j\to\infty} \|x_{n_j} - z\| = \lim_{n\to\infty} \|x_n - z\|.\end{aligned}$$

This is a contradiction. Consequently, we have $z = z'$. This implies that

$$x_n \rightharpoonup z \in F(S) \cap \Omega.$$

Since $x_n - y_n \to 0$ as $n \to \infty$, we have also

$$y_n \rightharpoonup z \in F(S) \cap \Omega.$$

Now, put $u_n = P_{F(S)\cap\Omega} x_n$. Then we claim that $z = \lim_{n\to\infty} u_n$. Indeed, since

$$u_n = P_{F(S)\cap\Omega} x_n \text{ and } z \in F(S) \cap \Omega,$$

we have

$$\langle z - u_n, u_n - x_n \rangle \geq 0.$$

By Lemma 2.2, $\{u_n\}$ converges strongly to some $z_0 \in F(S) \cap \Omega$. Then, we get $\langle z - z_0, z_0 - z \rangle \geq 0$, and hence $z = z_0$. This completes the proof of Theorem 3.1. □

Remark 3.1. In the proof of Theorem 3.1, whenever $\beta_n = 0$ for all $n \geq 0$, from (5) it follows that

$$\|x_{n+1} - x^*\|^2 \leq \|x_n - x^*\|^2, \quad \forall n \geq N_0.$$

Thus the limit $\lim_{n\to\infty} \|x_n - x^*\|$ exists. Hence $\{x_n\}$ is bounded and so is $\{Ax_n\}$. In this case, we can remove the boundedness restriction of $\{Ax_n\}$. Consequently, Nadezhkina and Takahashi's Theorem 3.1 (Ref. 10) follows immediately from our Theorem 3.1.

Next we give two applications of Theorem 3.1.

Corollary 3.1. *Let H be a real Hilbert space. Let $A : H \to H$ be a monotone k-Lipschitz continuous mapping and let $S : H \to H$ be a nonexpansive mapping such that $F(S) \cap A^{-1}0 \neq \emptyset$. Let $\{x_n\}, \{y_n\}$ be the sequences generated by*

$$\begin{cases} x_0 = x \in H, \\ y_n = x_n - \lambda_n A x_n, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n) S(x_n - \lambda_n A y_n), \ \forall n \geq 0. \end{cases}$$

Suppose $\{\alpha_n\}$ is a sequence in $[0,1]$ and $\{\lambda_n\}$ is a sequence in $(0, 1/k)$ such that

$$\limsup_{n\to\infty} \alpha_n < 1 \text{ and } 0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < 1/k.$$

Then, the sequences $\{x_n\}, \{y_n\}$ converge weakly to the same point $z \in F(S) \cap A^{-1}$, where $z = \lim_{n\to\infty} P_{F(S)\cap A^{-1}} x_n$.

Proof. We have $A^{-1}0 = \Omega$ and $P_H = I$. Then

$$\begin{aligned} y_n &= P_H[(1-\beta_n)(x_n - \lambda_n A x_n) + \beta_n P_H(x_n - \lambda_n A x_n)] \\ &= (1-\beta_n)(x_n - \lambda_n A x_n) + \beta_n (x_n - \lambda_n A x_n) \\ &= x_n - \lambda_n A x_n, \end{aligned}$$

and

$$\begin{aligned} x_{n+1} &= \alpha_n x_n + (1-\alpha_n) S P_H (x_n - \lambda_n A y_n) \\ &= \alpha_n x_n + (1-\alpha_n) S(x_n - \lambda_n A y_n). \end{aligned}$$

Note that inequality (5) yields

$$\|x_{n+1} - x^*\|^2 \leq \|x_n - x^*\|^2.$$

This implies that $\{x_n\}$ is bounded and so is $\{Ax_n\}$. Hence by Theorem 3.1 we obtain the desired result. □

Remark 3.1. Notice that $F(S) \cap A^{-1}0$ is contained in the set of solutions of the variational inequality problem VI$(F(S), A)$. See also Yamada (Ref. 9) for the case when $A : H \to H$ is strongly monotone and Lipschitz continuous and $S : H \to H$ is a nonexpansive mapping.

Theorem 3.2. *Let H be a real Hilbert space. Let $A : H \to H$ be a monotone k-Lipschitz continuous mapping and $B : H \to 2^H$ be a maximal monotone mapping such that $A^{-1}0 \cap B^{-1}0 \neq \emptyset$. Let J_r^B be the resolvent of B for each*

$r > 0$. Let $\{x_n\}, \{y_n\}$ be the sequences generated by

$$\begin{cases} x_0 = x \in H, \\ y_n = x_n - \lambda_n A x_n, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n) J_r^B (x_n - \lambda_n A y_n), \ \forall n \geq 0. \end{cases}$$

Suppose $\{\alpha_n\}, \{\beta_n\}$ are sequences in $[0,1]$ and $\{\lambda_n\}$ is a sequence in $(0, 1/k)$ such that

$$\limsup_{n\to\infty} \alpha_n < 1 \text{ and } 0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < 1/k.$$

Then, the sequences $\{x_n\}, \{y_n\}$ converge weakly to the same point $z \in A^{-1}0 \cap B^{-1}0$, where $z = \lim_{n\to\infty} P_{A^{-1}0\cap B^{-1}0} x_n$.

Proof. We have $F(J_r^B) = B^{-1}0$. By Corollary 3.1 we obtain the desired result. □

Remark 3.2. Corollary 3.1 and Corollary 3.2 are, essentially, Nadezhkina and Takahashi's Theorems 3.1 and 3.2 (Ref. 10), respectively.

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