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FIXED POINTS AND CONTINUITY OF ALMOST CONTRACTIONS

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Abstract. Almost contractions form a class of generalized contractions that includes several contractive type mappings like usual contractions, Kannan mappings, Zamfirescu mappings etc. Since any usual contraction is continuous, while a Kannan mapping is not generally continuous but is continuous *at* the fixed point, the main aim of this paper is to study the continuity of both single and multi-valued almost contractions. The main results state that any almost contraction is continuous at its fixed point(s). This answers an open question raised in [Berinde, V., *On the approximation of fixed points of weak contractive mappings* Carpathian J. Math. **19** (2003), No. 1, 7-22].
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1. Introduction

The classical Banach's contraction principle is one of the most useful results in nonlinear analysis. In a metric space setting its full statement is given by

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the next theorem.

Theorem B. *Let (X, d) be a complete metric space and $T : X \to X$ a map satisfying*

$$d(Tx, Ty) \leq a\, d(x, y)\,, \quad \text{for all}\, x, y \in X\,, \tag{1.1}$$

where $0 \leq a < 1$ is constant. Then:

($p1$) *T has a unique fixed point p in X;*

($p2$) *The Picard iteration $\{x_n\}_{n=0}^{\infty}$ defined by*

$$x_{n+1} = Tx_n\,, \quad n = 0, 1, 2, \dots \tag{1.2}$$

converges to p, for any $x_0 \in X$.

($p3$) *The following a priori and a posteriori error estimates hold*:

$$d(x_n, x^*) \leq \frac{a^n}{1-a}\, d(x_0, x_1)\,, \quad n = 0, 1, 2, \dots \tag{1.3}$$

$$d(x_n, x^*) \leq \frac{a}{1-a}\, d(x_{n-1}, x_n)\,, \quad n = 1, 2, \dots \tag{1.4}$$

($p4$) *The rate of convergence of Picard iteration is given by*

$$d(x_n, x^*) \leq a\, d(x_{n-1}, x^*)\,, \quad n = 1, 2, \dots \tag{1.5}$$

A map satisfying ($p1$) and ($p2$) in Theorem B is said to be a *Picard operator*, see Rus [26], [28].

A mapping satisfying (1.1) is usually called *strict contraction* or *a-contraction* or simply *contraction*. Hence, Theorem B essentially shows that any contraction is a Picard operator.

Theorem B has many applications in solving nonlinear equations. Its merit is not only to state the existence and uniqueness of the fixed point of the strict contraction T but also to show that the fixed point can be approximated by means of Picard iteration (1.2). Moreover, for this iterative method both *a priori* (1.3) and *a posteriori* (1.4) error estimates are available. Additionally, inequality (1.5) shows that the rate of convergence of Picard iteration is linear in the class of strict contractions.

Despite these important features, Thoeorem B suffers from one serious drawback - the contractive condition (1.1) forces T to be continuous on the entire space X. It was then naturally to ask if there exist contractive conditions which do not imply the continuity of T. This was answered in the affirmative by R. Kannan [16] in 1968, who proved a fixed point theorem which extends

Theorem B to mappings that need not be continuous, by considering instead of (1.1) the next condition: there exists $0 \le b < 1/2$ such that

$$d(Tx,Ty) \le b\big[d(x,Tx)+d(y,Ty)\big], \quad \text{for all } x,y \in X\,. \tag{1.6}$$

Following Kannan's theorem, a lot of papers were devoted to obtaining fixed point theorems for various classes of contractive type conditions that do not require the continuity of T on X, see for example, Rus [26], [28], Taskovic [32], and references therein.

One of them, actually a sort of dual of Kannan fixed point theorem, due to Chatterjea [11], is based on a condition similar to (1.6): there exists $0 \le c < 1/2$ such that

$$d(Tx,Ty) \le c\big[d(x,Ty)+d(y,Tx)\big], \quad \text{for all } x,y \in X. \tag{1.7}$$

It is well known, see Rhoades [21], that the contractive conditions (1.1) and (1.6), as well as (1.1) and (1.7), respectively, are independent.

In 1972, Zamfirescu [33] obtained a very interesting fixed point theorem, by combining (1.1), (1.6) and (1.7) in a rather unexpected way.

Theorem Z. *Let (X,d) be a complete metric space and $T : X \to X$ be a map for which there exist the real numbers a, b and c satisfying $0 \le a < 1$, $0 \le b$, $c < 1/2$ such that for each pair x, y in X, at least one of the following is true*:

(z_1) $d(Tx,Ty) \le a\, d(x,y)$;

(z_2) $d(Tx,Ty) \le b\big[d(x,Tx)+d(y,Ty)\big]$;

(z_3) $d(Tx,Ty) \le c\big[d(x,Ty)+d(y,Tx)\big]$.

Then T is a Picard operator.

One of the most general contraction conditions for which the map satisfying it is still a Picard operator, has been obtained by Ciric [13] in 1974: there exists $0 \le h < 1$ such that

$$d(Tx,Ty) \le h\cdot\max\big\{d(x,y),d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx)\big\}\,, \quad \text{for all } x,y \in X\,. \tag{1.8}$$

Remark. A mapping satisfying (1.8) is commonly called *quasi contraction.* It is obvious that each of the conditions (1.1), (1.6), (1.7) and (z_1)-(z_3) implies (1.8).

There exist many other fixed point theorems based on contractive conditions of the type considered in this section: see [11]-[18], [19], [21], [22] and also the monographs [3], [26], [29] and [32].

A more general class of contractive type mappings called *weak contractions* were introduced and studied in [2], [4]-[8], [10], [20], see also [9]. This class includes mappings satisfying the previous contractive conditions (except for quasi contractions, which are known to be only in part included in the class of weak contractions). The next section survey the most significant results obtained in [4], [5] and [6].

2. Almost contractions

The following concept has been introduced in [6], where we used the term of *weak contraction* instead of the present *almost contraction*.

Definition 1. Let (X, d) be a metric space. A map $T : X \to X$ is called *almost contraction* or (δ, L)*-contraction* if there exist a constant $\delta \in (0, 1)$ and some $L \geq 0$ such that

$$d(Tx, Ty) \leq \delta \cdot d(x, y) + Ld(y, Tx)\,, \quad \textit{for all}\ \ x, y \in X\,. \tag{2.1}$$

Remark 1. Due to the symmetry of the distance, the almost contraction condition (2.1) implicitly includes the following dual one

$$d(Tx, Ty) \leq \delta \cdot d(x, y) + L \cdot d(x, Ty)\,, \quad \text{for all}\ \ x, y \in X\,, \tag{2.2}$$

obtained from (2.1) by formally replacing $d(Tx, Ty)$ and $d(x, y)$ by $d(Ty, Tx)$ and $d(y, x)$, respectively, and then interchanging x and y.

Consequently, in order to check the almost contractiveness of T, it is necessary to check both (2.1) and (2.2).

Obviously, any strict contraction satisfies (2.1), with $\delta = a$ and $L = 0$, and hence is an almost contraction (that possesses a unique fixed point).

Other examples of almost contractions are given in [5]-[8]. There are many other examples of contractive conditions which implies the almost contractiveness condition, see for example Taskovic [32], Rus [29] and Berinde [3].

The main results in [6] are given below as Theorem 1 (an existence theorem) and Theorem 2 (an existence and uniqueness theorem). Their main merit is that extend Theorems B and Z and many other related results in literature to the larger class of almost contractions, in the spirit of Theorem B, that is, in such a way that they offer a method for approximating the fixed point, for which both *a priori* and *a posteriori* error estimates are available.

Theorem 1. *Let (X, d) be a complete metric space and $T : X \longrightarrow X$ a weak contraction. Then*

1) *$Fix\,(T) = \{x \in X : Tx = x\} \neq \phi$;*

2) *For any $x_0 \in X$, the Picard iteration $\{x_n\}_{n=0}^{\infty}$ given by (1.2) converges to some $x^* \in Fix\,(T)$;*

3) *The following estimates*

$$d(x_n, x^*) \leq \frac{\delta^n}{1-\delta}\, d(x_0, x_1)\,, \quad n = 0, 1, 2, \dots$$

$$d(x_n, x^*) \leq \frac{\delta}{1-\delta}\, d(x_{n-1}, x_n)\,, \quad n = 1, 2, \dots$$

hold, where δ is the constant appearing in (2.1).

Remark 2. 1) Note that, although the fixed point theorems B and Z actually forces the uniqueness of the fixed point, the almost (weak) contractions need not have a unique fixed point, as shown by Example 1 in [6].

Recall, see Rus [28], [29], [31], that an operator $T : X \to X$ is said to be a *weakly Picard operator* if the sequence $\{T^n x_0\}_{n=0}^{\infty}$ converges for all $x_0 \in X$ and the limits are fixed points of T. Therefore, Theorem 1 provides a large class of weakly Picard operators.

2) It is easy to see that condition (2.1) implies the so called Banach orbital condition

$$d(Tx, T^2x) \leq a\, d(x, Tx)\,, \quad \textit{for all} \;\; x \in X\,,$$

studied by various authors in the context of fixed point theorems, see the references in [6].

It is possible to force the uniqueness of the fixed point of an almost contraction, see [5] and [6], by imposing an additional contractive condition, quite similar to (2.1), as shown by the next theorem.

Theorem 2. *Let (X, d) be a complete metric space and $T : X \to X$ an almost contraction for which there exist $\theta \in (0, 1)$ and some $L_1 \geq 0$ such that*

$$d(Tx, Ty) \leq \theta \cdot d(x, y) + L_1 \cdot d(x, Tx)\,, \quad \textit{for all} \;\; x, y \in X\,. \tag{2.3}$$

Then

1) *T has a unique fixed point, i.e., $Fix\,(T) = \{x^*\}$;*

2) *The Picard iteration $\{x_n\}_{n=0}^{\infty}$ given by (1.2) converges to x^*, for any $x_0 \in X$;*

3) *The a priori and a posteriori error estimates*

$$\begin{aligned} d(x_n, x^*) &\leq \frac{\delta^n}{1-\delta}\, d(x_0, x_1)\,, \quad n = 0, 1, 2, \ldots \\ d(x_n, x^*) &\leq \frac{\delta}{1-\delta}\, d(x_{n-1}, x_n)\,, \quad n = 1, 2, \ldots \end{aligned}$$

hold.

4) *The rate of convergence of the Picard iteration is given by*

$$d(x_n, x^*) \leq \theta\, d(x_{n-1}, x^*)\,, \quad n = 1, 2, \ldots$$

Remark 3. Note that, by the symmetry of the distance, (2.3) is satisfied for all $x, y \in X$ if and only if

$$d(Tx, Ty) \leq \theta\, d(x, y) + L_1 d(y, Ty)\,, \tag{2.4}$$

also holds, for all $x, y \in X$.

So, similarly to the case of the dual conditions (2.1) and (2.2), in concrete applications it is necessary to check that both conditions (2.3) and (2.4) are satisfied.

As shown in [6], an operator T satisfying one of the conditions (1.1), (1.6), (1.7), or the conditions in Theorem Z, also satisfies the uniqueness conditions (2.1) and (2.4). Therefore, in view of Example 1 in [6], Theorem 2 (and also Theorem 1) in this section properly generalizes Theorem Z. The results obtained for single-valued mappings have been extended to the multi-valued case in [2], [10] and [20].

3. Continuity of single-valued almost contractions

The next Example shows that an almost contraction (and a quasi-contraction, too) need not be continuous.

Example 1. Let $[0, 1]$ be the unit interval with the usual norm and let $T : [0, 1] \to [0, 1]$ be given by $Tx = \dfrac{2}{3}$, for $x \in [0, 1)$ and $T1 = 0$. Then: 1) T

satisfies (1.5) with $h \in \left[\frac{2}{3}, 1\right)$; 2) T satisfies (2.1) with $1 > \delta \geq \frac{2}{3}$ and $L \geq \delta$; 3) T has a unique fixed point, $x^* = \frac{2}{3}$; 4) T does not satisfy (2.3); 5) T is not continuous.

In [23], Rhoades found a large class of contractive type mappings which are continuous at their *unique* fixed point, but are not continuous on the whole space X.

It was then naturally to ask in [6] a similar question for the almost contractions, whose set of fixed points $Fix\,(T)$ is not a singleton: is a almost contraction continuous at any fixed point of it ? The positive answer is given by the next Theorem.

Theorem 3. *Let (X, d) be a complete metric space and $T : X \to X$ be an almost contraction. Then T is continuous at p, for any $p \in Fix\,(T)$.*

Proof. Since T is an almost contraction, there exist a constant $\delta \in (0, 1)$ and some $L \geq 0$ such that (2.1) is satisfied. We know by Theorem 1 that for any $x_0 \in X$ the Picard iteration $\{x_n\}_{n=0}^{\infty}$ defined by $x_{n+1} = Tx_n$, $n = 0, 1, 2, \ldots$ converges to some $p \in Fix\,(T)$.

Let $\{y_n\}_{n=0}^{\infty}$ be any sequence in X converging to p. Then by taking $y := y_n$ and $x := p$ in the almost contraction condition (2.1),

$$d(Tx, Ty) \leq \delta \cdot d(x, y) + Ld(y, Tx)\,,$$

we get

$$d(Tp, Ty_n) \leq \delta d(p, y_n) + Ld(y_n, Tp),\ n = 0, 1, 2, \ldots$$

which, in view of $Tp = p$, is equivalent to

$$d(Ty_n, Tp) \leq (\delta + L) \cdot d(y_n, p),\ n = 0, 1, 2, \ldots. \tag{3.1}$$

Now by letting $n \to \infty$ in (3.1) we get $Ty_n \to Tp$ as $n \to \infty$, which shows that T is continuous at p.

As the fixed point p has been chosen arbitrarily in $Fix\,(T)$, the proof is complete. □

A mapping for which there exist a constant $\delta \in (0, 1)$ and some $L \geq 0$ such that

$$d(Tx, Ty) \leq \delta \cdot d(x, y) + L \min\{d(x, Tx), d(y, Ty), d(x, Ty),$$

$$d(y, Tx)\}, \forall x, y \in X. \tag{3.2}$$

is said to be a *strict almost contraction.*

The strict almost contraction have been introduced and studied in [1] by Babu, Sanddhya and Kameswari, by answering an open question in [6]. As condition (3.2) is obtained by combining the almost contraction condition (2.1) and the uniqueness condition (2.3), they obtained a fixed point theorem that ensure the uniqueness of the fixed point. It is obvious that any strict almost contraction is an almost contraction, i.e., it does satisfy (2.1) and also satisfies the uniqueness condition (2.3) but the reverse is not generally true, see the examples in [1].

By Theorem 3 we get the following result for the class of strict almost contractions, for which we give a direct proof.

Corollary 1. *Let (X, d) be a complete metric space and $T : X \to X$ be a generalized almost contraction, i.e., a mapping satisfying (3.2) and let $Fix\,(T) = \{p\}$. Then T is continuous at p.*

Proof. Since T is a strict almost contraction, there exist a constant $\delta \in (0, 1)$ and some $L \geq 0$ such that (3.2) is satisfied. We know by Theorem 2.9 in [1] that T has a unique fixed point, say p.

Let $\{y_n\}_{n=0}^{\infty}$ be any sequence in X converging to p. Then by taking $y := y_n$ and $x := p$ in the strict almost contraction condition (3.2),

$$d(Tx, Ty) \leq \delta \cdot d(x, y) + L \min\{d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}$$

we get

$$d(Tp, Ty_n) \leq \delta d(p, y_n),\ n = 0, 1, 2, \ldots$$

since

$$\min\{d(p, Tp), d(y_n, Ty_n), d(p, Ty_n), d(y_n, Tp)\} = d(p, Tp) = 0.$$

and therefore, the previous inequality is equivalent to

$$d(Ty_n, Tp) \leq \delta \cdot d(y_n, p),\ n = 0, 1, 2, \ldots. \tag{3.3}$$

Now by letting $n \to \infty$ in (3.3) we get $Ty_n \to Tp$ as $n \to \infty$, which shows that T is continuous at p. $\square$

4. Continuity of multi-valued almost contractions

In order to study the much more complicated case of *multi-valued* almost contractions (mappings that were introduced and studied in [2], [10] and [20]) we need the following concepts and results.

Let (X, d) be a metric space and let $\mathcal{P}(X)$ ($\mathcal{C}(X)$ and $\mathcal{CB}(X)$) denote the family of all nonempty subsets of X (nonempty closed, nonempty closed and bounded, respectively).

For $A, B \subset X$ and $a \in X$, we consider the following functionals:

$$d(a, B) = \inf\{d(a, b) : b \in B\}\,,\ \text{the distance between } a \text{ and } B,$$

$$D(A, B) = \inf\{d(a, b) : a \in A, b \in B\}\,,\ \text{the distance between } A \text{ and } B,$$

$$\delta(A, B) = \sup\{d(a, b) : a \in A, b \in B\}\,,\ \text{the diameter of } A \text{ and } B$$

and

$$H(A, B) = \max\{\sup\{d(a, B) : a \in A\}, \sup\{d(b, A) : b \in B\}\},$$

the *Hausdorff-Pompeiu metric* on $\mathcal{CB}(X)$ induced by d.

It is known that $\mathcal{CB}(X)$ is a metric space equipped with the Hausdorff-Pompeiu distance function H. It is also known, see for example Lemma 8.1.4 in Rus [29], that if (X, d) is a complete metric space then $(\mathcal{CB}(X), H)$ is a complete metric space, too.

A multi-valued map $T : X \to \mathcal{C}(X)$ is said to be continuous at a point p if

$$\lim_{n\to\infty} d(x_n, p) = 0 \ \text{ implies} \lim_{n\to\infty} H(Tx_n, Tp) = 0.$$

(Note that in [24] instead of H is used the functional D).

Let $T : X \to \mathcal{P}(X)$ be a multi-valued mapping. An element $x \in X$ such that $x \in T(x)$ is called a *fixed point* of T. We denote by $Fix\,(T)$ the set of all fixed points of T, i.e.,

$$Fix\,(T) = \{x \in X : x \in T(x)\}.$$

The next theorem shows that any generalized strict almost contraction is continuous at the fixed point.

Theorem 4. *Let (X, d) be a complete metric space and $T : X \to \mathcal{CB}(X)$ be a generalized multi-valued (θ, L)-strict almost contraction, i.e., a mapping satisfying for which there exist $\theta \in (0, 1)$ and some $L \geq 0$ such that $\forall x, y \in X$,*

$$H(Tx, Ty) \leq \theta \cdot d(x, y) + L \min\{d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}$$

$$(4.1)$$

Then $Fix\,(T) \neq \emptyset$ and for any $p \in Fix\,(T)$, T is continuous at p.

Proof. The first part of the conclusion follows by Theorem 7 in [10].

Let $\{y_n\}_{n=0}^{\infty}$ be any sequence in X converging to p. Then by taking $y := y_n$ and $x := p$ in the generalized strict almost contraction condition (4.1), we get

$$d(Tp, Ty_n) \leq \delta d(p, y_n),\ n = 0, 1, 2, \ldots$$

since

$$\min\{d(p, Tp), d(y_n, Ty_n), d(p, Ty_n), d(y_n, Tp)\} = d(p, Tp) = 0.$$

and therefore, the previous inequality is equivalent to

$$d(Ty_n, Tp) \leq \delta \cdot d(y_n, p),\ n = 0, 1, 2, \ldots. \tag{4.2}$$

Now by letting $n \to \infty$ in (4.2) we get $Ty_n \to Tp$ as $n \to \infty$, which shows that T is continuous at p. $\square$

The previous result can be extended without any difficulty to the more general class of generalized almost contractions.

Theorem 5. *Let (X, d) be a complete metric space and $T : X \to \mathcal{CB}(X)$ be a generalized multi-valued (θ, L)-almost contraction, i.e., a mapping satisfying for which there exist $\theta \in (0, 1)$ and some $L \geq 0$ such that $\forall x, y \in X$,*

$$H(Tx, Ty) \leq \theta \cdot d(x, y) + Ld(y, Tx). \tag{4.3}$$

Then $Fix\,(T) \neq \emptyset$ and for any $p \in Fix\,(T)$, T is continuous at p.

Proof. The first part of the conclusion follows by Theorem 3 in [2]. Let $\{y_n\}_{n=0}^{\infty}$ be any sequence in X converging to p. Then by taking $y := y_n$ and $x := p$ in the generalized almost contraction condition (4.3), we get

$$d(Tp, Ty_n) \leq \delta d(p, y_n) + Ld(y_n, Tp),\ n = 0, 1, 2, \ldots$$

which, in view of $Tp = p$, is equivalent to

$$d(Ty_n, Tp) \leq (\delta + L) \cdot d(y_n, p),\ n = 0, 1, 2, \ldots. \tag{4.4}$$

Now by letting $n \to \infty$ in (4.4) we get $Ty_n \to Tp$ as $n \to \infty$, which shows that T is continuous at p. $\square$

For other concepts of continuity of *multi-valued* mappings and other classes of generalized multi-valued almost contractions (see our papers [2], [10] and [20]) the study will be presented in a future work.

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