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CONVERGENCE OF KRASNOSELSKII ITERATIONS IN THE REAL LINE BASED ON ASYMPTOTICALLY NON-EXPANSIVE MAPPING

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Abstract. In this paper, we will discuss the convergence of Krasnoselskii iterations in the real line based on asymptotically non-expansive mapping.
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1. Introduction

Following Krasnoselskii's [3] fixed point theorem, namely " Let K be a bounded closed convex subset of a uniformly convex Banach space X. Let F be a non-expansive mapping of K into a compact subset of K. Let x_0 be an arbitrary point of K. Then the sequence defined by

$$x_{n+1} = \frac{x_n + Fx_n}{2}, \ (n = 0, 1, 2, \ldots)$$

converges to a fixed point of F in K" and Beardon's [2] theorem in relation to the contractions on the real line, namely " Suppose that $f : \Re \to \Re$ satisfies $|f(x) - f(y)| < |x - y|$ whenever $x \neq y$. Then there is some ζ in $[-\infty, +\infty]$ such that for any real x, $f^n(x) \to \zeta$ as $n \to \infty$", we have already considered the convergence of Krasnoselskii iterations in the real line based on a non-expansive mapping in [1] as in the following:

"Suppose that T is a non-expansive map defined on the real line $\Re$. Then for any real x, the sequence defined by

$$x_{n+1} = \frac{x_n + Tx_n}{2},\ (n = 0, 1, 2, \ldots)$$

converges either to a fixed point in $\Re$ or to an element $\zeta \in \{-\infty, +\infty\}$".

Therefore, having discussed the convergence of Krasnoselskii iterations in the real line for a non - expansive mapping already in [1], we are now going to deal with the same theorem for an asymptotically non - expansive mapping and so we are going to consider the convergence based on asymptotically non-expansive mapping as in the following: "Suppose that $T : \Re \longrightarrow \Re$ is an asymptotically non-expansive mapping. Moreover assume that

$$\prod_{i=N}^{n} \left(\frac{1 + k_n}{2}\right)$$

is convergent. Then for any real x, the sequence defined by

$$x_{n+1} = \frac{x_n + T^n x}{2},\ (n = 0, 1, 2, \ldots)$$

converges either to a fixed point in $\Re$ or to an element $\zeta \in \{-\infty, +\infty\}$".

2. Basic definitions

Definition 2.1. *A space X is said to be uniformly convex if and only if given $\epsilon > 0$, there exists a $\delta(\epsilon) > 0$ such that*

$$\left\| \frac{x + y}{2} \right\| \leq 1 - \delta(\epsilon) \quad \textit{whenever} \quad \|x - y\| \geq \epsilon \ \textit{ and } \ \|x\| = \|y\| = 1.$$

The above inequality is equivalent to the following:
If $x, y, p \in X$, $R > 0$ and $r \in [0, 2R]$, then

$$\left.\begin{array}{l} ||x - p|| \leq R \\ ||y - p|| \leq R \\ ||x - y|| \geq r \end{array}\right\} \Rightarrow \left\| \frac{x + y}{2} \right\| \leq \left(1 - \delta\left(\frac{r}{R}\right)\right) R.$$

Definition 2.2. *Let K be a nonempty subset of a Banach space X. A mapping $F : K \longrightarrow K$ is said to be asymptotically non-expansive if*

$$\|F^n x - F^n y\| \leq k_n \|x - y\| \ \textit{for all } x, y \in K \ \textit{and} \lim_{n \to \infty} k_n = 1.$$

3. Main Results

Having prepared the ground, let us now first look at the fixed point theorem of Krasnoselskii [3] in the following:

Theorem 3.1. *Let K be a bounded closed convex subset of a uniformly convex Banach space X. Let F be a non-expansive mapping of K into a compact subset of K. Let x_0 be an arbitrary point of K. Then the sequence defined by*

$$x_{n+1} = \frac{x_n + Fx_n}{2}, \ (n = 0, 1, 2, \ldots)$$

converges to a fixed point of F in K.

Keeping the above theorem at the back of our mind, let us now see what Beardon [2] says in his theorem:

Theorem 3.2. *Suppose that $f : \Re \to \Re$ satisfies $|f(x) - f(y)| < |x - y|$ whenever $x \neq y$. Then there is some ζ in $[-\infty, +\infty]$ such that for any real x, $f^n(x) \to \zeta$ as $n \to \infty$.*

Following Theorems 3.1 and 3.2, we have already discussed the convergence of Krasnoselskii iterations in the real line for non-expansive mapping in [1] and so we give the statement of the same as in the following:

Theorem 3.3. *Suppose that $T : \Re \to \Re$ is a mapping which satisfies $|Tx - Ty| \leq |x - y|$. Then for any real x, the sequence defined by*

$$x_{n+1} = \frac{x_n + Tx_n}{2}, \quad (n = 0, 1, 2, \ldots)$$

converges either to a fixed point in $\Re$ or to an element $\zeta \in \{-\infty, +\infty\}$.

Now, in this paper, we are going to discuss the convergence of Krasnoselskii iterations in the real line for asymptotically non-expansive mapping and so let us prove the theorem in the following:

Theorem 3.4. *Suppose that $T : \Re \longrightarrow \Re$ is an asymptotically non-expansive mapping. Moreover assume that*

$$\prod_{i=N}^{n} \left(\frac{1 + k_n}{2} \right)$$

is convergent. Then for any real x, the sequence defined by

$$x_{n+1} = \frac{x_n + T^n x}{2}, \quad (n = 0, 1, 2, \ldots)$$

converges either to a fixed point in $\Re$ or to an element $\zeta \in \{-\infty, +\infty\}$.

Proof. Suppose that T has a fixed point in $\Re$, say z. Let us first show that $|x_{n+1} - z| \leq |x_n - z|$, $n = 0, 1, 2, \ldots$. First of all, for any arbitrary real x_n, the sequence can be defined (by the hypothesis) by

$$x_{n+1} = \frac{x_n + T^n x_n}{2}, (n = 0, 1, 2, \ldots)$$

Since $Tz = z$, we have $T^n z = z$. Therefore, we have

$$\begin{aligned}
|x_{n+1} - z| &= \left|\frac{1}{2}(x_n + T^n x_n) - \frac{1}{2}(z + z)\right| \\
&= \left|\frac{1}{2}(x_n + T^n x_n) - \frac{1}{2}(z + T^n z)\right| \\
&= \left|\frac{1}{2}(x_n - z) + \frac{1}{2}\,(T^n x_n - T^n z)\right| \\
&\leq \frac{1}{2}\,|x_n - z| + \frac{1}{2}\,|T^n x_n - T^n z| \\
&\leq \frac{1}{2}\,|x_n - z| + \frac{1}{2}\,k_n\,|x_n - z| \\
&\qquad (Since\ T\ is\ asymptotically\ non-expansive) \\
&= \frac{1}{2}(1 + k_n)\,|x_n - z|.
\end{aligned}$$

Therefore,

$$|x_{n+1} - z| \ \leq\ \frac{1}{2}\,(1 + k_n)\,|x_n - z|$$

That is,

$$\frac{|x_{n+1} - z|}{\left(\dfrac{1 + k_n}{2}\right)} \ \leq\ |x_n - z|.$$

Let M be the greatest lower bound of the sequence $\{x_n - z\}$. Then there exists N such that $|x_N - z| \ \leq\ M + \epsilon$ for all $n > N$. Therefore,

$$\frac{|x_n - z|}{\prod_{i=N}^{n}(1 + k_i)} \ \leq\ |x_N - z| \ \leq\ M + \epsilon.$$

This implies that

$$M \leq |x_n - z| \leq \prod_{i=N}^{n} (1 + k_i)(M + \epsilon).$$

And therefore this implies that $|x_n - z|$ converges.
Suppose there exists an $\epsilon > 0$ and N, such that

$$|x_n - T^n x_n| \geq \epsilon \text{ for all } n \geq N. \tag{3.1}$$

Then $|x_n - z - (T^n x_n - T^n z)| \geq \epsilon \ for \ all \ n \geq N.$
Since T is asymptotically non-expansive, we have

$$|T^n x_n - T^n z| \leq k_n \ |x_n - z|$$

Since the space $\Re$ is uniformly convex, there exists a constant δ, $0 < \delta < 1$ such that

$$\begin{aligned} |x_{n+1} - z| &= \left| \frac{1}{2} (x_n - z) + \frac{1}{2} (T^n x_n - T^n z) \right| \\ &\leq \delta \max \{ |x_n - z| , |T^n x_n - T^n z| \} \ (\ by \ definition) \\ &= \max \{ \delta \ |x_n - z| , \delta \ k_n \ |T^n x_n - T^n z| \} \\ &< \delta \ k_n \ |x_n - z| \quad for \ all \ n \geq N. \end{aligned}$$

Therefore $\lim_{n\to\infty} x_n = z$ where $Tz = z$.
If there does not exist an $\epsilon > 0$ for which (3.1) holds, there exists a subsequence $x_{n_k} \in [-\infty, +\infty]$ such that $\lim_{k\to\infty} (x_{n_k} - Tx_{n_k}) = 0$ and such that the sequence $\{Tx_{n_k}\}$ converges. Suppose $\{Tx_{n_k}\}$ converges to a fixed point. This implies that $\lim_{k\to\infty} (x_{n_k}) = \zeta = \lim_{k\to\infty} (Tx_{n_k})$ and hence $T\zeta = \zeta$.
As $\lim_{k\to\infty} |(x_{n_k}) - \zeta| = 0$, we have $\lim_{n\to\infty} |x_n - \zeta| = 0$.
Suppose $Tx_{n_k} \to \infty$. Then $x_{n_k} \to \infty$. Therefore, $x_{n_{k+1}} \to \infty$. Similarly if $Tx_{n_k} \to -\infty$, then $x_{n_{k+1}} \to -\infty$. If $x_n < Tx_n$, then $x_n < x_{n+1} < Tx_n$.
Suppose $Tx_{n+1} < x_{n+1}$. This implies

$$\begin{aligned} |Tx_{n+1} - Tx_n| &= |Tx_{n+1} - x_{n+1}| + |x_{n+1} - Tx_n| \\ &= |Tx_{n+1} - x_{n+1}| + |x_n - x_{n+1}|. \end{aligned}$$

Since T does not have a fixed point, $x_{n+1} < Tx_{n+1}$. This implies that $x_{n+1} < x_{n+2}$. Therefore $\{x_n\}$ is an asymptotically increasing sequence. Therefore, since $x_{n_k} \to \infty$, we have $x_n \to \infty$. If $Tx_n < x_n$, then $x_n > x_{n+1} > Tx_n$. Suppose $Tx_{n+1} > x_{n+1}$. This implies

$$\begin{aligned} |Tx_{n+1} - Tx_n| &= |Tx_{n+1} - x_{n+1}| + |x_{n+1} - Tx_n| \\ &= |Tx_{n+1} - x_{n+1}| + |x_n - x_{n+1}|. \end{aligned}$$

Since T does not have a fixed point, $x_{n+1} > Tx_{n+1}$.
This implies that $x_{n+1} > x_{n+2}$.
Therefore $\{x_n\}$ is an asymptotically decreasing sequence.
Since $x_{n_k} \longrightarrow -\infty$, we have $x_n \longrightarrow -\infty$.
Hence the theorem. □

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