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NONTRIVIAL SOLUTIONS TO INTEGRAL AND DIFFERENTIAL EQUATIONS

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Abstract. This survey collects some recent results concerning the existence and multiplicity of nontrivial solutions for nonlinear problems by using variational methods.
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1. Introduction

The existence and multiplicity of nonzero solutions for nonlinear problems when zero is a trivial solution is a widely studied topic of the nonlinear analysis.

In this survey we present some recent results related to this topic and in particular we study:

- a nonlinear equation in L^p-spaces of the type

$$u = K\mathbf{f}(u) \tag{1}$$

where K is a completely continuous linear operator and $\mathbf{f}$ is a superposition operator;

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- a Dirichlet problem of the type

$$\begin{cases} -\Delta u = g(x,u) + \lambda f(x,u) & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega \end{cases} \tag{P_λ}$$

where $\Omega \subset \mathbb{R}^N$ is an open bounded set with sufficiently regular boundary $\partial\Omega$, $f, g : \Omega \times \mathbb{R} \to \mathbb{R}$ are two real functions and λ is a real parameter.

The abstract tools to establish our results are the variational methods. As we will see, if, on one hand, the application of these methods requires further assumptions on the involved linearities and nonlinearities than other methods (such as the fixed point methods), on the other hand, using variational methods, it is easy to find natural condition in order to avoid that the solutions be trivial when a trivial solution exists.

A typical assumption when variational methods are used to study equation (1) is that the operator K be positive definite (see [6, 7, 13] and the references therein). While, when the same methods are used to study problem (P_λ), usually one assumes that f, g be Carathèodory functions with some growth condition with respect to the second variable.

The main features of our results consists in the fact that, as concerns equation (1), we do not assume the operator K positive definite but having at most a finite number of negative eigenvalues. Whereas, as concerns problem (P_λ), we do not assume any growth condition with respect to the second variable on the nonlinearity f.

All the proofs of our main results are based on the application of a general variational theorem established by B. Ricceri in [14]. We state, for our convenience, a particular version of this result which will be useful when we deal with equation (1).

Theorem 1. (Theorem 2.1 of [14]) *Let X be a reflexive real Banach space. Let*

$$\Psi(x) = \frac{1}{2}\|x\|^2 - J(x), \quad x \in X,$$

where $J : X \to \mathbb{R}$ is a sequentially weakly continuous functional. Assume that

$$\inf_{r>0} \inf_{\|x\|<r} \frac{\sup_{\|y\|=r} J(y) - J(x)}{r^2 - \|x\|^2} < \frac{1}{2}. \tag{2}$$

Then, Ψ admits a local minimum in X.

2. Nonlinear Equations in L^p-Spaces

Let $\Omega \subset \mathbb{R}^N$ be a bounded Lebesgue-measurable set, and let p, p_0, q, q_0 four positive real numbers such that

$$p_0 > p > 2 \quad \text{and} \quad \frac{1}{p_0} + \frac{1}{q_0} = \frac{1}{p} + \frac{1}{q} = 1.$$

We consider equation (1) assuming that

- $K : L^{q_0}(\Omega) \to L^{p_0}(\Omega)$ is a completely continuous linear operator;
- $\mathbf{f} : L^p(\Omega) \to L^q(\Omega)$ is a superposition operator defined by

$$\mathbf{f}(u) = f(\cdot, u(\cdot)),$$

  where, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a given Carathèodory function.

Very many articles in literature has been devoted to the study of equation (1). This has been motivated from the fact that linear and nonlinear differential and integral equations can be reduced just to an equation of the type (1).

For instance, the well known Hammerstein integral equation

$$u(x) = \int_\Omega k(x, y) f(y, u(y)) dy$$

with $u \in L^{q_0}(\Omega)$, $x \in \Omega$, $k \in L^{p_0}(\Omega \times \Omega)$, reduces to (1) by putting

$$K(u)(x) = \int_\Omega k(x, y) u(y) dy.$$

Certainly, the fixed point methods have been the most widely used tools to deal with equation (1). Nevertheless, even if considerable less articles use a variational approach to study (1), this latter approach, as mentioned in the introduction, becomes more advantageous when we look for nontrivial solutions of (1).

We now recall some basic properties of the operator K (see [9]):

Assume K self-adjoint and having a finite number of negative eigenvalues.

Let V be the subspace generated by the eigenvectors corresponding to these eigenvalues. Define

$$K_+(u) = K(u) - 2P_V(u), \quad u \in L^{q_0}(\Omega).$$

Then, K_+ is a self-adjoint, positive definite and completely continuous linear operator from $L^{q_0}(\Omega)$ into $L^{p_0}(\Omega)$.

By the standard theory of the linear operators, K_+ restricted to $L^q(\Omega)$ splits as follows

$$K_+ = H_+ H_+^*$$

where

$$H_+ : L^2(\Omega) \to L^p(\Omega)$$

and

$$H_+^* : L^q(\Omega) \to L^2(\Omega)$$

are completely continuous linear operator satisfying

$$\int_\Omega H_+(u) v dx = \int_\Omega u H_+^*(v) dx,$$

for all $u \in L^2(\Omega)$, $v \in L^q(\Omega)$. We now denote by $\|\cdot\|_m$ the standard L^m-norm $(m \geq 1)$ and put $F(x, \xi) = \int_0^\xi f(x,t)$ for all $(x, \xi) \in \Omega \times \mathbb{R}$. By [9] it is known that the solutions of (1) are exactly the functions of the type $H_+(u)$, where $u \in L^2(\Omega)$ is a critical point of the functional

$$\Psi(v) = \frac{1}{2}\|v\|_2^2 - \left[\|P_V(v)\|_2^2 + \int_\Omega F(x, H_+(v)(x)) dx\right] \tag{3}$$

defined for all $v \in L^2(\Omega)$.

We now can state our existence result for equation (1):

Theorem 2. (Theorem 2.1-2.2 of [4]) *Assume K self-adjoint and denote by E the finite (possibly empty) set of the negative eigenvalues of K. Set $\lambda_{-1} = \max E$ if $E \neq \emptyset$. Suppose that there exists $a \in \mathbb{R}$, with $a > \frac{1}{|\lambda_{-1}|}$ when $E \neq \emptyset$ and $a = 0$ otherwise, such that:*

$$\inf_{r>0} \sup_{\|u\|_2=1} \int_\Omega \left(\frac{F(x, rH_+(u)(x))}{r^2} + a(H_+(u)(x))^2 \right) dx < \frac{\rho}{2}, \tag{4}$$

where $\rho = \frac{a|\lambda_{-1}|-1}{a|\lambda_{-1}|+1}$ if $E \neq \emptyset$ and $\rho = 1$ otherwise. Then, equation (1) has at least a solution in $L^p(\Omega)$.

In addition, suppose that $f(x, 0) = 0$, for a.a. $x \in \Omega$ and assume that one of the following condition holds

i) $E \neq \emptyset$ and there exist $M, N > 0$ with $M < \frac{1}{2|\lambda_{-1}|}$ such that $F(x,t) \geq -Mt^2 - Nt^p$ for a.a. $x \in \Omega$ and $t \in \mathbb{R}$.

ii) $E = \emptyset$ *and there exist* $M, N > 0$ *with* $M > \frac{1}{2\lambda_1}$, *where* λ_1 *is the first eigenvalue of* K, *such that* $F(x,t) \geq Mt^2 - Nt^p$ *for a.a.* $x \in \Omega$ *and* $t \in \mathbb{R}$.

Then, equation (1) *has at least a nonzero solution in* $L^p(\Omega)$.

Outline of the proof. Assume $E \neq \emptyset$ (the proof in the case $E = \emptyset$ is similar). As observed above, we have to prove that the functional Ψ defined by (3) has a critical point. We first observe that, by standard arguments, Ψ turns out to be Gâteaux differentiable. So, to find a critical point, we prove that Ψ admits a local minimum in $L^2(\Omega)$. To this end we want to apply Theorem 1.

At first, we note that, since V is finite dimensional, then the functional

$$J(v) = \|P_V(v)\|_2^2 + \int_\Omega \left(\int_0^{H_+(v)(x)} f(x,t)dt \right) dx$$

is sequentially weakly continuous in $L^2(\Omega)$. Moreover, using Lemma 1.2 at p.308 of [9] we can deduce, after some calculation, that condition (4) implies condition (2).

Therefore, applying Theorem 1 we infer that Ψ admits a local minimum $u \in L^2(\Omega)$.

If the additional assumptions of the theorem are satisfied, the critical value $\Psi(u)$ turns out to be negative and this implies, in turn, that the solution $H_+(u)$ must be nonzero. $\square$

The key assumption of Theorem 1 is, of course, condition (4). The proposition below shows that (4) holds under an appropriate growth condition on F.

Proposition 1. *Let* $m \in]2,p]$, $b \in L^{\frac{p}{p-m}}(\Omega)$ *with* $b \in L^\infty(\Omega)$ *if* $m = p$, *and* $c \in L^1(\Omega)$. *Put* $b_0 = \sup_{\|u\|_2=1} \int_\Omega b(x)|H_+(u)(x)|^m dx$, $c_0 = \int_\Omega |c(x)|dx$ *and suppose that* $m(b_0)^{\frac{2}{m}} \left(\frac{m-2}{2c_0}\right)^{\frac{2}{m}-1} < \rho$.

Then, if

$$F(x,\xi) \leq -a\xi^2 + b(x)|\xi|^m + c(x)$$

for a.a $x \in \Omega$ *and all* $\xi \in \mathbb{R}$, *where* a, ρ *are as in Theorem 2, we have that condition* (4) *of Theorem 2 is satisfied.*

In view of the above Proposition, it is useful to have an upper estimation of the constant b_0. Denoting by λ_p the norm of the operator K_+ acting from $L^q(\Omega)$ into $L^p(\Omega)$, we have the following estimation

$$b_0 \leq \|b\|_{\frac{p}{p-m}} \lambda_p^{\frac{m}{2}}. \tag{5}$$

The above inequality corrects the estimation given in Remark 2.1 of [4]. To deduce (5), let $\varphi \in L^2(\Omega)$. Using the Hölder-Schwartz inequality and the splitting representation of K_+, we have

$$\begin{aligned}
&\|H(\varphi)\|_p^p = \int_\Omega H(\varphi)H(\varphi)^{p-1}dx = \int_\Omega \varphi H^*(H(\varphi)^{p-1})dx \\
&\leq \|\varphi\|_2 \|H^*(H(\varphi)^{p-1})\|_2 = \\
&\|\varphi\|_2 \sqrt{\int_\Omega H(\varphi)^{p-1} H(H^*(H(\varphi)^{p-1}))dx} = \\
&\|\varphi\|_2 \sqrt{\int_\Omega H(\varphi)^{p-1} K_+(H(\varphi)^{p-1}))dx} \leq \\
&\|\varphi\|_2 \sqrt{\|H(\varphi)^{p-1}\|_q \|K_+(H(\varphi)^{p-1})\|_p} \leq \sqrt{\lambda_p}\|\varphi\|_2 \|H(\varphi)^{p-1}\|_q = \\
&\|\varphi\|_2 \sqrt{\lambda_p} \|H(\varphi)\|_p^{p-1}
\end{aligned}$$

from which

$$\|H(\varphi)\|_p \leq \sqrt{\lambda_p}\|\varphi\|_2.$$

Then, (5) follows applying the Schwartz inequality.

3. The Dirichlet Problem

Let $\Omega \subset \mathbb{R}^N$ be an open bounded set with sufficiently regular boundary $\partial\Omega$. Let $f, g : \Omega \times \mathbb{R} \to \mathbb{R}$ be two Carathèodory functions. Finally, let $\lambda > 0$.

Consider the following Dirichlet problem

$$\begin{cases} -\Delta u = g(x,u) + \lambda f(x,u) & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega \end{cases} \tag{P_λ}$$

In recent years some authors have investigated the existence and multiplicity of solutions to problem (P_λ) for λ small enough, imposing no growth condition on f.

Among the most interesting papers , we cite, for instance, [10, 11, 12].

A common feature in the above articles is that g is supposed symmetric with respect to the second variable and/or satisfying an Ambrosetti - Rabinovitz superlinear type condition. Instead, the nonlinearity f is only assumed continuous in $\overline{\Omega} \times \mathbb{R}$.

Here we want to present some new contributions on this topic where completely different assumptions are imposed on the nonlinearity g and no growth condition is imposed on f (in fact, we assume on f even weaker conditions than the solely continuity, as in the aforementioned articles.)

At first, we recall that if $h : \Omega \times \mathbb{R} \to \mathbb{R}$ is Carathèodory function satisfying, if $N \geq 2$,

$$\sup_{(x,t) \in \Omega \times \mathbb{R}} \frac{h(x,t)}{1 + |t|^m} < +\infty$$

for some $m > 0$ with $m < \frac{N+2}{N-2}$ if $N \geq 3$, then by standard argument we have that the functional

$$u \in W_0^{1,2}(\Omega) \to \int_\Omega \left(\int_0^{u(x)} h(x,t)dt \right) dx$$

is (strongly) continuous, sequentially weakly semicontinuous and Gâteuaux differentiable on $W_0^{1,2}(\Omega)$. This fact will be implicitly used in all the successive results. We recall that a weak solution of problem (P_λ) is exactly a critical point of the corresponding energy functional

$$u \in W_0^{1,2}(\Omega) \to \int_\Omega \left(\frac{1}{2}|\nabla u|^2 - \int_0^{u(x)} (g(x,t) + \lambda f(x,t))dt \right) dx$$

while a strong solution of problem (P_λ) is any $u \in W^{2,1}(\Omega) \cap W_0^{1,2}(\Omega) \cap C(\overline{\Omega})$ satisfying the equation $-\Delta u = g(x,u) + \lambda f(x,u)$ a.e. in Ω and the boundary condition pointwise. Hereafter, we denote by $\|u\| := \sqrt{\int_\Omega |\nabla u|^2 dx}$ the standard norm of $W_0^{1,2}(\Omega)$. Moreover, we will make use of the following definition: given a function $h : \Omega \times \mathbb{R} \to \mathbb{R}$ we say that a function $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is the truncation of h with respect to the interval $]a,b[\subseteq \mathbb{R}$ if

$$g(x,t) = \begin{cases} h(x,a) & \text{if } (x,t) \in \Omega \times]-\infty, a] \\ h(x,t) & \text{if } (x,t) \in \Omega \times]a,b[\\ h(x,b) & \text{if } (x,t) \in \Omega \times [b, +\infty[\end{cases}.$$

We start with the following theorem where a sublinear growth condition is imposed on the nonlinearity g

Theorem 3. (Theorem 2.1 of [3]) *Let* $s \in]1,2[$, $q > \frac{N}{2}$ *and* $a > 0$. *Let* $D \subseteq \Omega$ *be a non empty open set. Assume* f, g *satisfying the following conditions*

i) $\sup\limits_{0 \le \xi \le r} |f(\cdot, \xi)| \in L^q(\Omega)$ *for all* $r > 0$ *;*

ii) $f(x, 0) = 0$ *for a.e.* $x \in \Omega$*;*

iii) $|g(x,t)| \le at^{s-1}$ *for all* $t \ge 0$ *and a.e* $x \in \Omega$.

iv) $\liminf\limits_{\xi \to 0^+} \dfrac{\inf_{x \in D} \int_0^\xi g(x,t)dt}{\xi^2} = +\infty$

Then, there exist $\sigma, \overline{\lambda} > 0$ *such that, for every* $\lambda \in [-\overline{\lambda}, \overline{\lambda}]$, *there exists a nonzero nonnegative strong solution* $u_\lambda \in W_0^{1,2}(\Omega) \cap W^{2,q}(\Omega)$ *of problem* (P_λ) *with* $\|u_\lambda\|_{W^{2,q}(\Omega)} \le \sigma$.

Outline of the proof. Let g_0 be the truncation of g with respect to $]0, +\infty[$ and put

$$\Psi(u) = \frac{1}{2}\|u\|^2 - \int_\Omega \left(\int_0^{u(x)} g_0(x,t)dt \right) dx.$$

for all $u \in W_0^{1,2}(\Omega)$. By condition $iii)$ it is easy to prove that Ψ is coercive.

Moreover, condition $iv)$ assures that $\inf_{W_0^{1,2}(\Omega)} \Psi < 0$. Thus, we can fix

$$t \in \left] \inf_{W_0^{1,2}(\Omega)} \Psi, 0 \right[.$$

Now, by Theorem 8.16 of [8], there exists a constant $C_0 = C_0(N, q, \Omega)$ such that, for each $h \in L^q(\Omega)$ and for each weak solution $u \in W_0^{1,2}(\Omega)$ of the equation $-\Delta u = h$ on Ω, one has $\|u\|_\infty \le C_0 \|h\|_q$.

At this point, fix a constant $C > (aC_0)^{\frac{1}{2-s}} m(\Omega)^{\frac{1}{q(2-s)}}$, where $m(\Omega)$ is the Lebesgue measure of Ω, and let f_0 be the truncation of f with respect to $]0, C[$. Applying Theorem 2.1 of [14] to the functional Ψ jointly to the functionals

$$\Phi_\pm(u) := \pm \int_\Omega \left(\int_0^{u(x)} f_0(x,t)dt \right) dx, \quad u \in W_0^{1,2}(\Omega),$$

we find $\overline{\lambda} > 0$ such that for all $\lambda \in [-\overline{\lambda}, \overline{\lambda}]$ there exists a critical point u_λ of the functional $\Psi + \lambda \Phi_-$ satisfying

$$\Psi(u_\lambda) < t < 0. \tag{6}$$

So, in particular u_λ is non-zero.

Moreover, using a standard argument, it follows that u_λ is nonnegative. Finally, using Schauder estimates and taking the choice of C into account, we also have, choosing a smaller $\overline{\lambda}$ if necessary, that $u_\lambda(x) \leq C$ for a.a. $x \in \Omega$. This implies that u_λ is a weak solution of problem (P_λ). To finish the prove, we observe that, by standard regularity results ([1]), one has $u_\lambda \in W^{2,q}(\Omega)$ (hence $u_\lambda \in C(\overline{\Omega})$) and u_λ is a strong solution of (P_λ). Finally, the existence of an upper estimate of $\|u_\lambda\|_{W^{2,q}(\Omega)}$ which is independent of λ follows again by the Schauder estimates. $\square$

From Theorem 3, with a clear change of the assumptions, we obtain the following multiplicity result

Theorem 4. (Theorem 2.3 of [3]) *Let $s \in]1,2[$ $q > \frac{N}{2}$ and $a > 0$. Let $D \subseteq \Omega$ be a non empty open set. Assume f, g satisfying the following conditions*

i) $\sup\limits_{|\xi| \leq r} |f(\cdot, \xi)| \in L^q(\Omega)$ *for all $r > 0$;*

ii) $f(x, 0) = 0$ *for a.e. $x \in \Omega$;*

iii) $|g(x,t)| \leq a|t|^{s-1}$ *for all $t \in \mathbb{R}$ and a.e $x \in \Omega$.*

iv) $\liminf\limits_{\xi \to 0} \dfrac{\operatorname*{ess\,inf}\limits_{x \in D} \int_0^\xi g(x,t)dt}{\xi^2} = +\infty$

Then, there exist $\sigma, \overline{\lambda} > 0$ such that, for every $\lambda \in [-\overline{\lambda}, \overline{\lambda}]$, there exist a strong nonzero nonnegative solution $u_\lambda \in W_0^{1,2}(\Omega) \cap W^{2,q}(\Omega)$ and a strong nonzero nonpositive solution $v_\lambda \in W_0^{1,2}(\Omega) \cap W^{2,q}(\Omega)$ of problem (P_λ) with $\max\{\|u_\lambda\|_{W^{2,q}(\Omega)}, \|v_\lambda\|_{W^{2,q}(\Omega)}\} \leq \sigma$.

We now pass to state another existence result for problem (P_λ) where different conditions are imposed on the nonlinearity g. However, also in this result, as the previous one, we impose no growth condition on f.

Theorem 5. (Theorem 1 of [2]) *Let q, λ_0, t_1, t_2 be four positive constants with $t_2 > t_1$ and $q > \frac{N}{2}$. Let $D \subseteq \Omega$ be a non empty open set. Assume f, g satisfying the following conditions:*

i) $g(x, 0) + \lambda f(x, 0) \geq 0$ *for almost all $x \in \Omega$ and all $\lambda \in [-\lambda_0, \lambda_0]$;*

ii) $\operatorname{ess\,sup}_{x \in \Omega} \sup_{t\in[t_1,t_2]} g(x,t) < 0$;

iii) $\sup_{t\in[0,t_2]} (|g(\cdot,t)| + |f(\cdot,t)|) \in L^q(\Omega)$;

iv) $\liminf_{\xi\to 0^+} \dfrac{\operatorname{ess\,inf}_{x\in D} \int_0^\xi g(x,t)dt}{\xi^2} = +\infty$;

Then, there exist $\overline{\lambda}, \sigma > 0$ *such that, for all* $\lambda \in [-\overline{\lambda}, \overline{\lambda}]$ *there exists a strong nonzero solution* $u_\lambda \in W_0^{1,2}(\Omega) \cap W^{2,q}(\Omega)$ *of problem* (P_λ) *satisfying* $0 \le u_\lambda(x) \le t_2$ *for all* $x \in \Omega$, *with* $\|u_\lambda\|_{W^{2,q}(\Omega)} \le \sigma$.

Outline of the proof. Let $g_0, f_0 : \Omega\times\mathbb{R} \to \mathbb{R}$ be respectively the truncations of g, f with respect to $]0, t_2[$. Put

$$\Psi(u) = \frac{1}{2}\|u\|^2 - \int_\Omega \left(\int_0^{u(x)} g_0(x,t)dt \right) d$$

and

$$\Phi(u) = -\int_\Omega \left(\int_0^{u(x)} f_0(x,t)dt \right) dx.$$

for all $u \in W_0^{1,2}(\Omega)$. Observe that the functional Ψ is coercive. Moreover, denoting by K the set of its global minima, it is possible to shows that $\inf_{\partial K_\varepsilon} \Psi > \inf_{W_0^{1,2}(\Omega)} \Psi$ for all $\varepsilon > 0$, where K_ε is the set of $u \in W_0^{1,2}(\Omega)$ whose distance from K is less than or equal to ε. Since by condition *iv)* one has $\inf_{W_0^{1,2}(\Omega)} \Psi < 0$ we can consider a decreasing sequence ε_k of positive real numbers converging to zero and such that the zero-function does not belong to K_{ε_k}. Also, choose $r_k \in]\inf_{W_0^{1,2}(\Omega)} \Psi, \inf_{\partial K_{\varepsilon_k}} \Psi[$ for all $k \in \mathbb{N}$. At this point, since K_{ε_k} turns out to be weakly compact, we can apply Theorem 2.1 of [14] to the restriction to K_{ε_k} of the functionals Ψ and Φ. Then, we find $\lambda_k \in]0, \lambda_0[$ such that, for all $\lambda \in [0, \lambda_k]$, there exists a non-zero critical point $u_{\lambda,k}$ of $\Psi + \lambda\Phi_{|K_{\varepsilon_k}}$ with $\Psi(u_{\lambda_k}) < r_k$. Using standard arguments, we deduce that $u_{\lambda,k}$ is nonnegative and $u_{\lambda,k} \in W^{2,q}(\Omega)$ (and so $u_{\lambda,k} \in C^0(\overline{\Omega})$). Moreover, by the Maximum Principle it is easy to see that every point $u \in K$ satisfies $u(x) \in [0, t_1]$ for all $x \in \Omega$. Hence, assuming λ_k converging to 0, by regularity results and the Schauder estimates ([1]), it is possible to prove for that, for k sufficiently large, one has $u_{\lambda,k}(x) \le t_2$ and $\sup_{\lambda\in[0,\lambda_k]} \|u_\lambda\|_{W^{2,q}(\Omega)} < +\infty$. As a consequence, for such k's, $u_{\lambda,k}$ is actually a strong solution of problem (P_λ).

To finish the prove, it is now enough to repeat the above argument with $-f$ in place of f. $\square$

From Theorem 5, with a clear change of the assumptions, we obtain the following multiplicity result

Theorem 6. (Theorem 3 of [2]) *Let $q, \lambda_0, t_1, t_2, \tilde{t_1}, \tilde{t_2}$ be six positive constants with $t_2 > t_1$, $\tilde{t_2} > \tilde{t_1}$ and $q > \frac{N}{2}$. Let $D \subseteq \Omega$ be a non empty open set. Assume f, g satisfying the following conditions:*

i) $g(x, 0) + \lambda f(x, 0) = 0$ *for almost all* $x \in \Omega$ *and all* $\lambda \in [-\lambda_0, \lambda_0]$*;*

ii) $\operatorname*{ess\,inf}_{x\in\Omega} \inf_{t\in[-\tilde{t}_2,-\tilde{t}_1]} g(x, t) > 0$*;*

ii) $\operatorname*{ess\,sup}_{x\in\Omega} \sup_{t\in[t_1,t_2]} g(x, t) < 0$*;*

iii) $\sup_{t\in[-\tilde{t}_2,t_2]} (|g(\cdot, t)| + |f(\cdot, t)|) \in L^q(\Omega)$;

iv) $\liminf_{\xi\to 0} \dfrac{\operatorname*{ess\,inf}_{x\in D} \int_0^\xi g(x, t)dt}{\xi^2} = +\infty$;

Then, there exist $\overline{\lambda}, \sigma > 0$ such that, for all $\lambda \in [-\overline{\lambda}, \overline{\lambda}]$ there exist two strong nonzero solutions $u_\lambda, v_\lambda \in W_0^{1,2}(\Omega) \cap W^{2,q}(\Omega)$ of problem (P_λ) satisfying $0 \le u_\lambda(x) \le t_2$ and $-\tilde{t_2} \le v_\lambda(x) \le 0$ for all $x \in \Omega$, with $\|u_\lambda\|_{W^{2,p}(\Omega)} + \|v_\lambda\|_{W^{2,p}(\Omega)} \le \sigma$.

We now consider the case in which $g(x, \cdot)$ has an oscillating behavior. We will show that, assuming again only a mild summability condition on f, it is possible to establish, for any fixed $k \in \mathbb{N}$, the existence of at least k-weak solutions for problem (P_λ) for λ small enough. We want to stress out that these results are directly comparable with the results of [10] where completely different conditions are imposed on g.

The first theorem we state deals with the case in which $g(x, \cdot)$ has an oscillating behavior in every neighborhood of the origin

Theorem 7. (Theorem 2.1 of [5]) *Let $s_0 > 0$ and let $\{b_n\}, \{c_n\} \subset]0, +\infty[$ and $\{d_n\} \subset] - \infty, 0[$ be three sequences with $\lim_{n\to+\infty} c_n = \lim_{n\to+\infty} d_n = 0$. Let $D \subseteq \Omega$ be a non empty open set. Finally, let $p \ge 1$ with $p > \frac{2N}{N+2}$ if $N \ge 2$. Assume f, g satisfying the following conditions:*

i) $\sup_{|t|\le s_0} |f(\cdot, t)|, \sup_{|t|\le s_0} |g(\cdot, t)| \in L^p(\Omega)$, *ii)* $\operatorname*{ess\,sup}_{x\in\Omega}(g(x,c_n)+\lambda f(x,c_n))<0,\quad \operatorname*{ess\,inf}_{x\in\Omega}(g(x,d_n)+\lambda f(x,d_n))>0,$ *for all* $\lambda\in[-b_n,b_n]$ *and all* $n\in\mathbb{N}$.

iii) $\displaystyle\liminf_{\xi\to 0}\frac{\operatorname*{ess\,inf}_{x\in D}\int_0^\xi g(x,t)dt}{\xi^2}=+\infty;$

Then, for every $k\in\mathbb{N}$ *and* $\sigma>0$ *there exists* $b_{k,\sigma}>0$ *such that, for every* $\lambda\in[-b_{k,\sigma},b_{k,\sigma}]$*, problem* (P_λ) *has at least* k *distinct weak solutions whose norms in* $W_0^{1,2}(\Omega)$ *are less than* σ.

Outline of the proof. Let $n_0\in\mathbb{N}$ such that $c_n,d_n\in[-s_0,s_0]$ for all $n\geq n_0$. For every $n\geq n_0$, let g_n,f_n be respectively the truncations of g,f with respect to the interval $]c_n,d_n[$ and put

$$\Psi_n(u)=\frac{1}{2}\|u\|^2-\int_\Omega\left(\int_0^{u(x)}f_n(x,t)dt\right)dx,$$

$$\Phi_n(u)=-\int_\Omega\left(\int_0^{u(x)}g_n(x,t)dt\right)dx$$

for all $u\in W_0^{1,2}(\Omega)$.

As a usual, condition *iii)* guarantees that $\inf_{W_0^{1,2}(\Omega)}\Psi_n<0$. Moreover, since Ψ_n is coercive, it attains its global minimum in some $u_n\in W_0^{1,2}(\Omega)$. It is easy to check that $\lim_{n\to+\infty}\Psi_n(u_n)=0$. Thus, up to a subsequence, we can suppose $\Psi_n(u_n)$ strictly increasing. Now, fix $k\in\mathbb{N}$ and $\sigma>0$. For every $i\in\mathbb{N}$ pick

$$r_i\in\left]\inf_{W_0^{1,2}(\Omega)}\Psi_i,\ \inf_{W_0^{1,2}(\Omega)}\Psi_{i+1}\right[.$$

Applying Theorem 2.1 of [14] to the functionals Ψ_i,Φ_i we find $\tilde{b}_i>0$ such that, for every $\lambda\in[0,\tilde{b}_i]$, the functional $\Psi_i+\lambda\Phi_i$ has a local minima $u_{i,\lambda}$ which satisfies $\Psi_i(u_{i,\lambda})<r_i<0$. Moreover, we also deduce that for some $n_\sigma\geq n_0$ sufficiently large one has $\|u_{i,\lambda}\|\leq\sigma$ for every $i\geq n_\sigma$ and $\lambda\in[0,\tilde{b}_i]$. At this point, put

$$b_{k,\sigma}=\min\{b_{n_\sigma},...,b_{n_\sigma+k-1},\tilde{b}_{n_\sigma},...,\tilde{b}_{n_\sigma+k-1}\}.$$

Then, by condition *ii*) and the Maximum Principle, for every $\lambda \in [0, b_{k,\sigma}]$ and $i = n_\sigma, \ldots, n_\sigma + k - 1$, we have

$$d_i \leq u_{i,\lambda}(x) \leq c_i \quad \text{for every } x \in \Omega.$$

Thus, the functions $u_{i,\lambda}$ $(i = n_\sigma, \ldots, n_\sigma + k - 1)$ are k-weak solutions of problem (P_λ). It remains to prove that they are distinct. This is true since we can prove that

$$\Psi_i(u_{j,\lambda}) > \Psi_i(u_{i,\lambda})$$

for every $i, j \in \{n_\sigma, ..., n_\sigma + k - 1\}$ with $i < j$.

To finish the prove, it is enough to repeat the same above arguments with $-f$ in place of f and choosing a smaller $b_{k,\sigma}$ if necessary. □

In the next result we assume $g(x, \cdot)$ having an oscillating behavior in any neighborhood of $+\infty$. Its prove is based on the same arguments of that one of Theorem 7 and so it is omitted.

Theorem 8. (Theorem 2.2 of [5]) *Let $d_0 < 0$ and let $\{b_n\}, \{c_n\} \subset]0, +\infty[$ be two sequences with $\lim_{n\to+\infty} c_n = +\infty$. Let $D \subseteq \Omega$ be a non empty open set. Finally, let $p \geq 1$ with $p > \frac{2N}{N+2}$ if $N \geq 2$. Assume f, g satisfying the following conditions:*

i) $\sup\limits_{d_0 \leq t \leq s} |f(\cdot, t)|, \sup\limits_{d_0 \leq t \leq s} |g(\cdot, t)| \in L^p(\Omega)$ *for all* $s \geq 0$,

ii) $\operatorname*{ess\,sup}\limits_{x \in \Omega} (g(x, c_n) + \lambda f(x, c_n)) < 0, \quad \operatorname*{ess\,inf}\limits_{x \in \Omega} (g(x, d_n) + \lambda f(x, d_0)) > 0,$ *for all* $\lambda \in [-b_n, b_n]$ *and all* $n \in \mathbb{N}$.

iii) $\liminf\limits_{\xi \to +\infty} \dfrac{\operatorname*{ess\,inf}\limits_{x \in D} \int_0^\xi g(x, t)dt}{\xi^2} = +\infty;$

Then, for every $k \in \mathbb{N}$ there exist $\sigma_k > 0$ $b_k^* > 0$ such that, for every $\lambda \in [-b_k^*, b_k^*]$, problem (P_λ) has at least k distinct weak solutions whose norms in $W_0^{1,2}(\Omega)$ are less than σ_k.

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