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FIXED POINT THEORY FOR ADMISSIBLE PAIRS AND MAPS IN FRÉCHET SPACES VIA DEGREE THEORY

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Abstract. New fixed point results are presented for admissible pairs and maps (admissible in the sense of Górniewicz) defined on subsets of a Fréchet space E. The proof relies on the notion of a pseudo open set, degree theory, and on viewing E as the projective limit of a sequence of Banach spaces.

Key Words and Phrases: fixed point, admissible pair, degree theory.
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1. Introduction

This paper presents applicable fixed point theorems for multivalued admissible maps defined between Fréchet spaces. Our results in particular will apply to R_δ and more generally acyclic maps. Our theory is based on degree theory in Banach spaces and on viewing a Fréchet space as a projective limit of a sequence of Banach spaces $\{E_n\}_{n\in N}$ (here $N=\{1,2,\dots\}$). The usual results in the literature in the non-normable situation are rarely of interest from an application viewpoint (this point seems to be overlooked by many authors) since the set constructed using degree is usually open and bounded and so has empty interior.

For the remainder of this section we present some definitions and known results. Let X and Y be metric spaces. A continuous single valued map

$p : Y \to X$ is called a Vietoris map [4] if the following two conditions are satisfied:
(i). for each $x \in X$, the set $p^{-1}(x)$ is acyclic
(ii). p is a proper map i.e. for every compact $A \subseteq X$ we have that $p^{-1}(A)$ is compact.

Let $D(X,Y)$ be the set of all pairs $X \overset{p}{\leftarrow} Z \overset{q}{\rightarrow} Y$ where p is a Vietoris map, q is continuous and Z is a metric space.

Definition 1.1. A multifunction $\phi : X \to C(Y)$ is admissible, and we write $\phi \in Ad(X,Y)$, if $\phi : X \to C(Y)$ is upper semicontinuous, and if there exists a metric space Z and two continuous maps $p : Z \to X$ and $q : Z \to Y$ such that
(i). p is a Vietoris map
and
(ii). $\phi(x) = q(p^{-1}(x))$ for any $x \in X$;
here $C(Y)$ denotes the family of nonempty, compact subsets of Y.

Remark 1.1. (i). It should be noted that ϕ upper semicontinuous is redundant in Definition 1.1.
(ii). (p,q) is called a selected pair of ϕ and we write $(p,q) \subset \phi$.

Let (X,d) be a metric space and Ω_X the bounded subsets of X. The Kuratowski measure of noncompactness is the map $\alpha : \Omega_X \to [0,\infty]$ defined by (here $A \in \Omega_X$)

$$\alpha(A) = \inf\{r > 0 : \ A \subseteq \cup_{i=1}^{n} A_i \ \text{ and } \ diam\,(A_i) \le r\}.$$

Let S be a nonempty subset of X. For each $x \in X$, define $d(x,S) = \inf_{y\in S} d(x,y)$. We say a set is countably bounded if it is countable and bounded. Now suppose $G : S \to 2^X$; here 2^X denotes the family of nonempty subsets of X. Then $G : S \to 2^X$ is

(i). countably condensing if $\alpha(G(W)) < \alpha(W)$ for all countably bounded sets W of S with $\alpha(W) \neq 0$,
(ii). hemicompact if each sequence $\{x_n\}_{n\in N}$ in S has a convergent subsequence whenever $d(x_n, G(x_n)) \to 0$ as $n \to \infty$.

We now recall a result from the literature [1].

Theorem 1.1. *Let (Y, d) be a metric space, D a nonempty, complete subset of Y, and $G : D \to 2^Y$ a countably condensing map with either D or $G(D)$ bounded. Then G is hemicompact.*

Let E be a normed space. Let A and C be two subsets of E. A pair $A \stackrel{p}{\leftarrow} Z \stackrel{q}{\rightarrow} C$ is called a countably condensing pair from A to C if $\alpha\left(q\left(p^{-1}(\Omega)\right)\right) < \alpha(\Omega)$ for all countably bounded subsets Ω of A with $\alpha(\Omega) \neq 0$.

Now let I be a directed set with order $\leq$ and let $\{E_\alpha\}_{\alpha \in I}$ be a family of locally convex spaces. For each $\alpha \in I$, $\beta \in I$ for which $\alpha \leq \beta$ let $\pi_{\alpha,\beta} : E_\beta \to E_\alpha$ be a continuous map. Then the set

$$\left\{x = (x_\alpha) \in \prod_{\alpha \in I} E_\alpha : \; x_\alpha = \pi_{\alpha,\beta}(x_\beta) \; \forall \, \alpha, \, \beta \in I, \, \alpha \leq \beta \right\}$$

is a closed subset of $\prod_{\alpha \in I} E_\alpha$ and is called the projective limit of $\{E_\alpha\}_{\alpha \in I}$ and is denoted by $\lim_{\leftarrow} E_\alpha$ (or $\lim_{\leftarrow} \{E_\alpha, \pi_{\alpha,\beta}\}$ or the generalized intersection [5 pp. 439] $\cap_{\alpha \in I} E_\alpha$.)

2. Coincidence degree

A pair (p, q) is called compact if q is compact. Let U be an open subset of a normed space E. By $\mathcal{K}(\overline{U}, E)$ we mean the family of compact pairs (p, q) from $\overline{U}$ to E for which $Fix\,(p, q) \cap \partial\, U = \emptyset$ (recall that a pair (p, q) is from $\overline{U}$ to E if there exists a metric space Z for which $\overline{U} \stackrel{p}{\leftarrow} Z \stackrel{q}{\rightarrow} E$); here $Fix\,(p, q) = \{x \in \overline{U} : \; x \in q\,(p^{-1}(x))\}$. In 1976 Kucharski [6], using the coincidence index in $\mathbf{R}^n$ and Schauder projections, defined the coincidence index on $\mathcal{K}(\overline{U}, E)$ and established the following result.

Theorem 2.1. *There exists a map $I : \mathcal{K}(\overline{U}, E) \to \mathbf{Q}$ (called the coincidence index (degree)) which satisfies the following properties:*

(I). if $I(p, q) \neq 0$ then $Fix\,(p, q) \neq \emptyset$;

and

(II). if $h : Z \times [0, 1] \to E$ is a compact map such that $Fix\,(p, h) \cap \partial U = \emptyset$, Then $I(p, h_0) = I(p, h_1)$; here $h_0(y) = h(y, 0)$, $h_1(y) = h(y, 1)$ and $Fix\,(p, h) = \{x \in \overline{U} : \; x \in h\,(p^{-1}(x) \times \{t\})$ for some $t \in [0, 1]\}$.

We now define the coincidence index for countably condensing pairs as in [3]. Let E be a Banach space and U an open subset of E.

Definition 2.1. A pair $\overline{U} \overset{p}{\leftarrow} Z \overset{q}{\rightarrow} E$ is called a countably condensing pair from $\overline{U}$ to E if $\alpha\left(q\left(p^{-1}(\Omega)\right)\right) < \alpha(\Omega)$ for all countably bounded subsets Ω of $\overline{U}$ with $\alpha(\Omega) \neq 0$.

Definition 2.2. $(p,q) \in \mathcal{M}(\overline{U}, E)$ if $\overline{U}$ is bounded and (p,q) is a countably condensing pair from $\overline{U}$ to E with no fixed points on ∂U (i.e. $Fix\,(p,q) \cap \partial U = \emptyset$).

Now let $(p,q) \in \mathcal{M}(\overline{U}, E)$. We <u>claim</u> that we can associate with each pair (p,q) a compact pair $(p,q^\star)$ with

$$Fix\,(p,q) = Fix\,(p,q^\star); \tag{2.1}$$

here of course $Fix\,(p,q) = \{x \in \overline{U} : \; x \in q\,(p^{-1}\,(x))\}$. To see this let

$$K_1 = \overline{co}\,(q\,(p^{-1}\,(\overline{U}))) \quad \text{and} \quad K_n = \overline{co}\,(q\,(p^{-1}\,(\overline{U} \cap K_{n-1}))) \quad \text{for} \quad n = 2, 3, \dots .$$

In [2] we showed

$$q\,(p^{-1}\,(\overline{U} \cap K_n)) \subseteq K_{n+1} \quad \text{and} \quad Fix\,(p,q) \subseteq K_n \quad \text{for each} \quad n. \tag{2.2}$$

Now there are two possibilities that can occur, namely

$$K_n \neq \emptyset \quad \text{for each} \quad n \tag{2.3}$$

or

$$K_i \neq \emptyset \quad \text{for} \quad i = 1, \dots, m \quad \text{and} \quad K_{m+j} = \emptyset \quad \text{for each} \quad j \in \{1, 2, \dots\}. \tag{2.4}$$

If (2.4) holds, then we can choose $x_0 \in K_m$ and let

$$q^\star : Z \to E \quad \text{be} \quad q^\star(z) = x_0. \tag{2.5}$$

Clearly $(p,q^\star)$ is a compact pair and (2.2) guarantees that $Fix\,(p,q) = \emptyset$. Also if $x \in \overline{U}$ with $x \in q^\star\,(p^{-1}\,(x))$ then it is immediate that $x = x_0$ so $K_{m+1} \neq \emptyset$, a contradiction. Thus $Fix\,(p,q^\star) = \emptyset$ and so $Fix\,(p,q) = Fix\,(p,q^\star) = \emptyset$. In this case we define the coincidence index (degree) $I(p,q)$ as

$$I(p,q) = I(p,q^\star); \tag{2.6}$$

of course one could define $I(p,q) = 0$ immediately (if (2.4) occurs) since $Fix\,(p,q) = \emptyset$. Next suppose (2.3) holds. It is well known (see [7, Theorem 2.2] if $(p,q) \in \mathcal{M}(\overline{U},E)$) that

$$K_\infty = \cap_{n=1}^{\infty} K_n$$

is compact. Also from (2.2) we have that $Fix\,(p,q) \subseteq K_\infty$. Now define a compact pair $(p,\tilde{q})$ as follows:

$$\overline{U} \cap K_\infty \xleftarrow{p} p^{-1}(\overline{U} \cap K_\infty) \xrightarrow{\tilde{q}} K_\infty$$

with $\tilde{q} = q(u)$ for all u. Also in [2] we showed

$$Fix\,(p,\tilde{q}) = \left\{x \in \overline{U} \cap K_\infty :\ x \in \tilde{q}\,(p^{-1}\,(x))\right\} = Fix\,(p,q).$$

Now Dugundji's extension theorem guarantees that we can extend $\tilde{q}$ to a compact map $q^\star : Z \to K_\infty$. As a result $(p,q^\star)$ is a compact pair with

$$Fix\,(p,q) = Fix\,(p,q^\star) = \left\{x \in \overline{U} :\ x \in q^\star\,(p^{-1}\,(x))\right\}. \tag{2.7}$$

To see (2.7), we will show $Fix\,(p,q^\star) = Fix\,(p,\tilde{q})$. Certainly $Fix\,(p,\tilde{q}) \subseteq Fix\,(p,q^\star)$ since $q^\star(y) = \tilde{q}(y)$ for $y \in p^{-1}(\overline{U} \cap K_\infty)$. Next suppose $x \in Fix\,(p,q^\star)$, so $x \in \overline{U}$ and $x \in q^\star\,(p^{-1}\,(x))$. Now since $q^\star\,(p^{-1}\,(x)) \subseteq K_\infty$ we have $x \in \overline{U} \cap K_\infty$. Thus $q^\star\,(p^{-1}\,(x)) = \tilde{q}\,(p^{-1}\,(x))$ since $q^\star(y) = \tilde{q}(y)$ for $y \in p^{-1}(\overline{U} \cap K_\infty)$. As a result $x \in \tilde{q}\,(p^{-1}\,(x))$ with $x \in \overline{U} \cap K_\infty$ i.e. $x \in Fix\,(p,\tilde{q})$. Thus (2.7) holds. In this case we define the coincidence index (degree) $I(p,q)$ as

$$I(p,q) = I(p,q^\star). \tag{2.8}$$

Of course we now need to check that the definition is independent of the extension $q^\star$. Let the compact map $\overline{q} : Z \to K_\infty$ be another extension of $\tilde{q}$ with $Fix(p,q) = Fix\,(p,\overline{q})$. We must show $I(p,q^\star) = I(p,\overline{q})$. Let $R : E \to K_\infty$ be a retraction (guaranteed from Dugundji's extension theorem) and consider the compact map $h : Z \times [0,1] \to K_\infty$ given by

$$h(x,t) = \begin{cases} (1-2\,t)\,q^\star(x) + 2\,t\,R\,q(x), & t \in \left[0,\frac{1}{2}\right] \text{ and } x \in Z \\ (2-2\,t)\,R\,q(x) + (2\,t-1)\,\overline{q}(x), & t \in \left(\frac{1}{2},1\right] \text{ and } x \in Z. \end{cases}$$

Notice $h_0(x) = q^\star(x)$ and $h_1(x) = \overline{q}(x)$. If we show

$$Fix\,(p,h) \cap \partial U = \emptyset, \tag{2.9}$$

then Theorem 2.1 guarantees that $I(p,h_0) = I(p,h_1)$. That is $I(p,q^\star) = I(p,\overline{q})$, and we are finished. It remains to check (2.9). Suppose there exists $t \in [0,1]$ (without loss of generality assume $t \in \left[0,\frac{1}{2}\right]$) and $x \in \partial U$ with $x \in h\left(p^{-1}(x) \times \{t\}\right)$. Then $x = h(y,t)$ for some y with $p(y) = x$. As a result

$$p(y) = (1-2\,t)\,q^\star(y) + 2\,t\,R\,q(y). \tag{2.10}$$

We know $q^\star(y) \in K_\infty$ and $R\,q(y) \in K_\infty$ so these together with (2.10) yield $p(y) \in K_\infty$. Also $p(y) = x \in \partial U$ and so $y \in p^{-1}(\overline{U} \cap K_\infty)$. Thus

$$q^\star(y) = \tilde{q}(y) = q(y) \quad \text{and} \quad R\,q(y) = q(y) \tag{2.11}$$

since $q(y) \in K_\infty$ (note $q\left(p^{-1}(\overline{U} \cap K_\infty)\right) \subseteq K_\infty$ from (2.2)). As a result

$$p(y) = (1-2\,t)\,q(y) + 2\,t\,q(y) = q(y),$$

i.e. $x \in q\left(p^{-1}(x)\right)$ with $x \in \partial U$. This is a contradiction since $Fix\,(p,q) \cap \partial U = \emptyset$.

Theorem 2.2. *If* $(p,q) \in \mathcal{M}(\overline{U},E)$ *and* $I(p,q) \neq 0$ *then* $Fix\,(p,q) \neq \emptyset$.

Proof. Now from above there exists a compact pair $(p,q^\star)$ with $I(p,q) = I(p,q^\star)$. Theorem 2.1 implies $Fix\,(p,q^\star) \neq \emptyset$ so (2.1) guarantees that $Fix\,(p,q) \neq \emptyset$. $\square$

Remark 2.1. If the pair in Definition 2.2 was condensing (see [3]) instead of countably condensing then $\overline{U}$ bounded is not needed in the above argument (see [3]).

Let E be a Banach space, U an open bounded subset of E and let $\phi \in Ad(\overline{U},E)$ be countably condensing with $Fix\,\phi \cap \partial U = \emptyset$; here $Fix\,\phi = \{x \in U :\ x \in \phi(x)\}$ and ϕ is called countably condensing if there exists a selected pair (p,q) of ϕ which is countably condensing. We define the coincidence index (degree) $I(\phi,U)$ by putting

$$I(\phi,U) = \{I(p,q) :\ (p,q) \subset \phi \text{ such that } (p,q) \text{ is } P\text{-concentrative}\}\,;$$

note $Fix\,\phi = Fix\,(p,q)$.

If $I(\phi,U) \neq \{0\}$ then $Fix\,\phi \neq \emptyset$. To see this note if $I(\phi,U) \neq \{0\}$ then there exists a selected pair (p,q) of ϕ which is countably condensing with $I(p,q) \neq 0$. Then Theorem 2.2 guarantees that $Fix\,(p,q) \neq \emptyset$ and so $Fix\,\phi \neq \emptyset$.

3. Fixed point theory in Fréchet spaces

Let $E = (E, \{|\cdot|_n\}_{n\in N})$ be a Fréchet space with the topology generated by a family of seminorms $\{|\cdot|_n : n \in N\}$. We assume that the family of seminorms satisfies

$$|x|_1 \le |x|_2 \le |x|_3 \le \dots \quad \text{for every } x \in E. \tag{3.1}$$

A subset X of E is bounded if for every $n \in N$ there exists $r_n > 0$ such that $|x|_n \le r_n$ for all $x \in X$. To E we associate a sequence of Banach spaces $\{(\mathbf{E}_n, |\cdot|_n)\}$ described as follows. For every $n \in N$ we consider the equivalence relation $\sim_n$ defined by

$$x \sim_n y \quad \text{iff} \quad |x-y|_n = 0. \tag{3.2}$$

We denote by $\mathbf{E}^n = (E/\sim_n, |\cdot|_n)$ the quotient space, and by $(\mathbf{E}_n, |\cdot|_n)$ the completion of $\mathbf{E}^n$ with respect to $|\cdot|_n$ (the norm on $\mathbf{E}^n$ induced by $|\cdot|_n$ and its extension to $\mathbf{E}_n$ are still denoted by $|\cdot|_n$). This construction defines a continuous map $\mu_n : E \to \mathbf{E}_n$. Now since (3.1) is satisfied the seminorm $|\cdot|_n$ induces a seminorm on $\mathbf{E}_m$ for every $m \ge n$ (again this seminorm is denoted by $|\cdot|_n$). Also (3.2) defines an equivalence relation on $\mathbf{E}_m$ from which we obtain a continuous map $\mu_{n,m} : \mathbf{E}_m \to \mathbf{E}_n$ since $\mathbf{E}_m/\sim_n$ can be regarded as a subset of $\mathbf{E}_n$. We now assume the following condition holds:

$$\begin{cases} \text{for each } n \in N, \quad \text{there exists a Banach space } (E_n, |\cdot|_n) \\ \text{and an isomorphism (between normed spaces) } j_n : \mathbf{E}_n \to E_n. \end{cases} \tag{3.3}$$

Remark 3.1. (i). For convenience the norm on E_n is denoted by $|\cdot|_n$.
(ii). Usually in applications $\mathbf{E}_n = \mathbf{E}^n$ for each $n \in N$.
(iii). Note if $x \in \mathbf{E}_n$ (or $\mathbf{E}^n$) then $x \in E$. However if $x \in E_n$ then x is not necessaily in E and in fact E_n is easier to use in applications (even though E_n is isomorphic to $\mathbf{E}_n$). For example if $E = C[0,\infty)$, then $\mathbf{E}^n$ consists of the class of functions in E which coincide on the interval $[0,n]$ and $E_n = C[0,n]$.

Finally we assume

$$E_1 \supseteq E_2 \supseteq \dots \text{ and for each } n \in N,\ |x|_n \le |x|_{n+1}\ \forall\, x \in E_{n+1}. \tag{3.4}$$

Let $\lim_{\leftarrow} E_n$ (or $\cap_1^\infty E_n$ where $\cap_1^\infty$ is the generalized intersection [5]) denote the projective limit of $\{E_n\}_{n\in N}$ (note $\pi_{n,m} = j_n\,\mu_{n,m}\,j_m^{-1} : E_m \to E_n$ for $m \geq n$) and note $\lim_{\leftarrow} E_n \cong E$, so for convenience we write $E = \lim_{\leftarrow} E_n$.

For each $X \subseteq E$ and each $n \in N$ we set $X_n = j_n\,\mu_n(X)$, and we let $\overline{X_n}$ and ∂X_n denote respectively the closure and the boundary of X_n with respect to $|\cdot|_n$ in E_n. Also the pseudo-interior of X is defined by

$$pseudo - int\,(X) = \{x \in X :\ j_n\,\mu_n(x) \in \overline{X_n} \setminus \partial X_n \ \text{ for every } \ n \in N\}.$$

The set X is pseudo-open if $X = pseudo - int\,(X)$.

If U is a pseudo-open bounded subset of E then for each $n \in N$ we have that U_n is open and bounded.

To see that U_n is open first notice $U_n \subseteq \overline{U_n} \setminus \partial U_n$ since if $y \in U_n$ then there exists $x \in U$ with $y = j_n\mu_n(x)$ and this together with $U = pseudo - int\,U$ yields $j_n\mu_n(x) \in \overline{U_n} \setminus \partial U_n$ i.e. $y \in \overline{U_n} \setminus \partial U_n$. In addition notice

$$\overline{U_n} \setminus \partial U_n = (int\,U_n \cup \partial U_n) \setminus \partial U_n = int\,U_n \setminus \partial U_n = int\,U_n$$

since $int\,U_n \cap \partial U_n = \emptyset$. Consequently

$$U_n \subseteq \overline{U_n} \setminus \partial U_n = int\,U_n, \quad \text{so} \ \ U_n = int\,U_n.$$

As a result U_n is open. Finally U_n is bounded since U is bounded (note if $y \in U_n$ then there exists $x \in U$ with $y = j_n\mu_n(x)$).

We begin with a result for Volterra type operators.

Theorem 3.1. *Let E and E_n be as described above, $F : \Omega \to 2^E$ where Ω is a pseudo-open bounded subset of E. Also assume for each $n \in N$ that $F : \overline{\Omega_n} \to C(E_n)$. Suppose the following conditions are satisfied:*

$$\text{for each } \ n \in N,\ F \in Ad(\overline{\Omega_n}, E_n) \ \text{ is countably condensing} \tag{3.5}$$

$$\text{for each } \ n \in N,\ Fix\,F \cap \partial\Omega_n = \emptyset \tag{3.6}$$

$$\text{for each } \ n \in N,\ I(F, \Omega_n) \neq \{0\} \tag{3.7}$$

and

$$\begin{cases} \text{for each } \ n \in \{2, 3, \dots\} \ \text{ if } \ y \in \overline{\Omega_n} \ \text{ solves } \ y \in F\,y \ \text{ in } \ E_n \\ \text{then } \ y \in \overline{\Omega_k} \ \text{ for } \ k \in \{1, \dots, n-1\}. \end{cases} \tag{3.8}$$

Then F has a fixed point in E.

Proof. Fix $n \in N$. Now there exists $y_n \in \overline{\Omega_n}$ with $y_n \in F\,y_n$. Lets look at $\{y_n\}_{n\in N}$. Notice $y_1 \in \overline{\Omega_1}$ and $y_k \in \overline{\Omega_1}$ for $k \in N\backslash\{1\}$ from (3.8). As a result $y_n \in \overline{\Omega_1}$ for $n \in N$, $y_n \in F\,y_n$ in E_n together with (3.5) implies there is a subsequence $N_1^\star$ of N and a $z_1 \in \overline{\Omega_1}$ with $y_n \to z_1$ in E_1 as $n \to \infty$ in $N_1^\star$. Let $N_1 = N_1^\star \setminus \{1\}$. Now $y_n \in \overline{\Omega_2}$ for $n \in N_1$ together with (3.5) guarantees that there exists a subsequence $N_2^\star$ of N_1 and a $z_2 \in \overline{\Omega_2}$ with $y_n \to z_2$ in E_2 as $n \to \infty$ in $N_2^\star$. Note from (3.4) that $z_2 = z_1$ in E_1 since $N_2^\star \subseteq N_1$. Let $N_2 = N_2^\star \setminus \{2\}$. Proceed inductively to obtain subsequences of integers

$$N_1^\star \supseteq N_2^\star \supseteq \dots, \quad N_k^\star \subseteq \{k, k+1, \dots\}$$

and $z_k \in \overline{\Omega_k}$ with $y_n \to z_k$ in E_k as $n \to \infty$ in $N_k^\star$. Note $z_{k+1} = z_k$ in E_k for $k \in \{1, 2, \dots\}$. Also let $N_k = N_k^\star \setminus \{k\}$.

Fix $k \in N$. Let $y = z_k$ in E_k. Notice y is well defined and $y \in \lim_{\leftarrow} E_n = E$. Now $y_n \in F\,y_n$ in E_n for $n \in N_k$ and $y_n \to y$ in E_k as $n \to \infty$ in N_k (since $y = z_k$ in E_k) together with the fact that $F : \overline{\Omega_k} \to 2^{E_k}$ is upper semicontinuous (note $y_n \in \overline{\Omega_k}$ for $n \in N_k$) implies $y \in F\,y$ in E_k. We can do this for each $k \in N$ so $y \in F\,y$ in E. $\square$

Our next result was motivated by Urysohn type operators. In this case the map F_n will be related to F by the closure property (3.14).

Theorem 3.2. *Let E and E_n be as described in the beginning of Section 3, Ω a pseudo-open bounded subset of E and $F : \Omega \to 2^E$. Also assume for each $n \in N$ that $F_n : \overline{\Omega_n} \to C(E_n)$. Suppose the following conditions are satisfied:*

$$\overline{\Omega_1} \supseteq \overline{\Omega_2} \supseteq \dots \tag{3.9}$$

$$\textit{for each } \; n \in N, \; F_n \in Ad(\overline{\Omega_n}, E_n) \tag{3.10}$$

$$\textit{for each } \; n \in N, \; Fix\, F_n \cap \partial\Omega_n = \emptyset \tag{3.11}$$

$$\textit{for each } \; n \in N, \; I(F_n, \Omega_n) \neq \{0\} \tag{3.12}$$

$$\begin{cases} \textit{for each } \; n \in N, \quad \textit{the map } \; \mathcal{K}_n : \overline{\Omega_n} \to 2^{E_n}, \quad \textit{given by} \\ \mathcal{K}_n(y) = \cup_{m=n}^{\infty} F_m(y) \quad \textit{(see Remark 3.2), is} \\ \textit{countably condensing} \end{cases} \tag{3.13}$$

and

$$\text{(3.14)} \qquad \begin{cases} \textit{if there exists a } w \in E \textit{ and a sequence } \{y_n\}_{n\in N} \\ \textit{with } y_n \in \overline{\Omega_n} \textit{ and } y_n \in F_n\, y_n \textit{ in } E_n \textit{ such that} \\ \textit{for every } k \in N \textit{ there exists a subsequence} \\ S \subseteq \{k+1, k+2, \dots\} \textit{ of } N \textit{ with } y_n \to w \textit{ in } E_k \\ \textit{as } n \to \infty \textit{ in } S, \textit{ then } w \in F\, w \textit{ in } E. \end{cases}$$

Then F *has a fixed point in* E.

Remark 3.2. The definition of $\mathcal{K}_n$ in (3.13) is as follows. If $y \in \overline{\Omega_n}$ and $y \notin \overline{\Omega_{n+1}}$ then $\mathcal{K}_n(y) = F_n(y)$, whereas if $y \in \overline{\Omega_{n+1}}$ and $y \notin \overline{\Omega_{n+2}}$ then $\mathcal{K}_n(y) = F_n(y) \cup F_{n+1}(y)$, and so on.

Proof. Fix $n \in N$. Now there exists $y_n \in \overline{\Omega_n}$ with $y_n \in F_n\, y_n$ in E_n. Lets look at $\{y_n\}_{n\in N}$. Now Theorem 1.1 (with $Y = E_1$, $G = \mathcal{K}_1$, $D = \overline{\Omega_1}$ and note $d_1(y_n, \mathcal{K}_1(y_n)) = 0$ for each $n \in N$ since $|x|_1 \le |x|_n$ for all $x \in E_n$ and $y_n \in F_n\, y_n$ in E_n; here $d_1(x, Z) = \inf_{y\in Z} |x - y|_1$ for $Z \subseteq Y$) guarantees that there exists a subsequence $N_1^\star$ of N and a $z_1 \in E_1$ with $y_n \to z_1$ in E_1 as $n \to \infty$ in $N_1^\star$. Let $N_1 = N_1^\star \setminus \{1\}$. Look at $\{y_n\}_{n\in N_1}$. Now Theorem 1.1 (with $Y = E_2$, $G = \mathcal{K}_2$ and $D = \overline{\Omega_2}$) guarantees that there exists a subsequence $N_2^\star$ of N_1 and a $z_2 \in E_2$ with $y_n \to z_2$ in E_2 as $n \to \infty$ in $N_2^\star$. Note $z_2 = z_1$ in E_1 since $N_2^\star \subseteq N_1^\star$. Let $N_2 = N_2^\star \setminus \{2\}$. Proceed inductively to obtain subsequences of integers

$$N_1^\star \supseteq N_2^\star \supseteq \dots, \quad N_k^\star \subseteq \{k, k+1, \dots\}$$

and $z_k \in E_k$ with $y_n \to z_k$ in E_k as $n \to \infty$ in $N_k^\star$. Note $z_{k+1} = z_k$ in E_k for $k \in N$. Also let $N_k = N_k^\star \setminus \{k\}$.

Fix $k \in N$. Let $y = z_k$ in E_k. Notice y is well defined and $y \in \lim_{\leftarrow} E_n = E$. Now $y_n \in F_n\, y_n$ in E_n for $n \in N_k$ and $y_n \to y$ in E_k as $n \to \infty$ in N_k (since $y = z_k$ in E_k) together with (3.14) implies $y \in F\, y$ in E. $\square$

References

[1] R.P. Agarwal, M. Frigon and D. O'Regan, *A survey of recent fixed point theory in Fréchet spaces*, Nonlinear analysis and applications: to V. Lakshmikantham on his 80th birthday, Vol 1, 75-88, Kluwer Acad. Publ., Dordrecht, 2003.

[2] R.P. Agarwal and D. O'Regan, *Countably P-concentrative pairs and the coincidence index*, Applied Math. Letters, **12**(2002), 439-444.

[3] R.P. Agarwal and D. O'Regan, *A topological degree for pairs,* Fixed Point Theory and Applications, Vol 4 (edited by Y.J. Cho, J.K. Kim and S.M. Kang), Nova Science Publishers, New York, 2003, 11-17.

[4] L. Górniewicz, *Topological Fixed Point Theory of Multivalued Mappings,* Kluwer Academic Publishers, Dordrecht, 1999.

[5] L.V. Kantorovich and G.P. Akilov, *Functional Analysis in Normed Spaces,* Pergamon Press, Oxford, 1964.

[6] Z. Kucharski, *A coincidence index,* Bull. Acad. Polon. Sci., **24**(1976), 245-252.

[7] M. Vath, *Fixed point theorems and fixed point index for countably condensing maps*, Topological Methods in Nonlinear Analysis, **13**(1999), 341-363.

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