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STRONG CONVERGENCE THEOREMS FOR A FAMILY OF RELATIVELY NONEXPANSIVE MAPPINGS IN BANACH SPACES

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Abstract. In this paper, we deal with the problem of finding a common fixed point of a family of relatively nonexpansive mappings. We, first of all, discuss the properties of strongly relatively nonexpansive mappings and show a strong convergence theorem for a sequence of relatively nonexpansive mappings under some conditions. Using this result, we obtain a strong convergence theorem for a finite family of relatively nonexpansive mappings. Furthermore, we apply our result to the problem of finding a zero of a maximal monotone operator.

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1. Introduction

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers. Let E be a real smooth Banach space and let J be the duality mapping on E. Let $\phi\colon E \times E \to \mathbb{R}$ be a function defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2 \text{ for all } x, y \in E.$$

Let C be a nonempty closed convex subset of E and let T be a mapping of C into E. We denote by $F(T)$ and $\hat{F}(T)$ the sets of fixed points of T and asymptotic fixed points of T, respectively. A mapping $T\colon C \to E$ is said to be relatively nonexpansive [3, 10] if $F(T) = \hat{F}(T)$ and $\phi(p, Tx) \leq \phi(p, x)$ for all $p \in F(T)$ and $x \in C$. A mapping $T\colon C \to E$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. Nakajo and Takahashi [12] proved the following strong convergence theorem for finding a fixed point of a nonexpansive mapping in a Hilbert space by using the hybrid method in mathematical programming.

Theorem 1.1 ([12]). *Let C be a nonempty closed convex subset of a real Hilbert space H and let T be a nonexpansive mapping of C into itself such that $F(T)$ is nonempty. Let $\{x_n\}$ be a sequence in C defined by*

$$\begin{cases} x_1 = x \in C; \\ y_n = \alpha_n x_n + (1 - \alpha_n) T x_n; \\ C_n = \{z \in C : \|z - y_n\| \leq \|z - x_n\|\}; \\ Q_n = \{z \in C : \langle x_n - z, x - x_n \rangle \geq 0\}; \\ x_{n+1} = P_{C_n \cap Q_n}(x) \end{cases}$$

for $n \in \mathbb{N}$, where $0 \leq \alpha_n \leq a < 1$ for all $n \in \mathbb{N}$ and $P_{C_n \cap Q_n}$ are the metric projections of H onto $C_n \cap Q_n$ for all $n \in \mathbb{N}$. Then $\{x_n\}$ converges strongly to $z = P_{F(T)}(x)$, where $P_{F(T)}$ is the metric projection of H onto $F(T)$.

Later, Matsushita and Takahashi [11] extended this theorem to that of a Banach space in the case when T is a relatively nonexpansive mapping. On the other hand, Reich [15] and Kohsaka and Takahashi [8] proved weak convergence theorems for finding common fixed points of finite families of relatively nonexpansive mappings in Banach spaces.

In this paper, motivated by Matsushita and Takahashi [11], Reich [15], and Kohsaka and Takahashi [8], we establish strong convergence theorems by hybrid methods for finding common fixed points of families of relatively nonexpansive mappings in Banach spaces. For proving them, we obtain some important properties of relatively nonexpansive mappings in Banach spaces. Further, using the methods developed in [5, 11–13], we prove a strong convergence theorem for a sequence of relatively nonexpansive mappings satisfying some conditions. Using this result, we obtain a strong convergence theorem for a finite family of relatively nonexpansive mappings. Furthermore, we apply the result to the problem of finding a zero of a maximal monotone operator.

2. Preliminaries

Throughout this paper, $\mathbb{N}$ denotes the set of positive integers, $\mathbb{R}$ the set of real numbers, E a real Banach space with norm $\|\cdot\|$, E^* the dual of E, and $\langle x, f\rangle$ the value of $f \in E^*$ at $x \in E$. For convenience, the norm of E^* is also denoted by $\|\cdot\|$. Let $\{x_n\}$ be a sequence in E. Strong convergence of $\{x_n\}$ to $x \in E$ is denoted by $x_n \to x$ and weak convergence by $x_n \rightharpoonup x$. The (normalized) duality mapping J of E into 2^{E^*} is defined by

$$Jx = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\}$$

for $x \in E$.

A Banach space E is said to be strictly convex if $\|x\| = \|y\| = 1$ and $x \neq y$ imply $\|(x+y)/2\| < 1$. A Banach space E is said to be uniformly convex if for any $\epsilon > 0$, there exists $\delta > 0$ such that $\|x\| = \|y\| = 1$ and $\|x - y\| \geq \epsilon$ imply $\|(x+y)/2\| \leq 1 - \delta$. We know that a uniformly convex Banach space is reflexive and strictly convex. Let $\{x_n\}$ and $\{y_n\}$ be two bounded sequences in a uniformly convex Banach space E. It is known that $\lim_{n\to 0} \|x_n - y_n\| = 0$ if $\lambda \in (0,1)$ and $\lambda \|x_n\|^2 + (1-\lambda) \|y_n\|^2 - \|\lambda x_n + (1-\lambda)y_n\|^2 \to 0$ as $n \to \infty$. It is also known that if E is a uniformly convex Banach space, then $x_n \to x$ whenever $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$, where $\{x_n\}$ is a sequence in E.

Let $U = \{x \in E : \|x\| = 1\}$. The norm $\|\cdot\|$ of E is said to be Gâteaux differentiable if the limit

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists for all $x, y \in U$. In this case a Banach space E is said to be smooth. The norm of E is said to be uniformly Fréchet differentiable if the limit (2.1) is attained uniformly for $x, y \in U$. In this case a Banach space E is said to be uniformly smooth. It is known that the duality mapping J of E has the following properties (see [19]):

- It is single-valued if E is smooth;
- it is surjective if E is reflexive;
- it is injective if E is strictly convex, i.e., $Jx \cap Jy = \emptyset$ for $x, y \in E$ with $x \neq y$;
- it is uniformly norm-to-norm continuous on every bounded set if E is uniformly smooth.

From these facts, it is obvious that the duality mapping J^{-1} of E^* is single-valued and bijective if E is smooth, strictly convex, and reflexive. Furthermore, we know the following: Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences of a uniformly smooth Banach space E. Then

$$\|x_n - y_n\| \to 0 \text{ implies } \|Jx_n - Jy_n\| \to 0 \tag{2.2}$$

as $n \to \infty$. We know that a Banach space E is uniformly convex if and only if E^* is uniformly smooth. Thus we also obtain the following: Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences of a uniformly convex and smooth Banach space E. Then the duality mapping J^{-1} of E^* is uniformly norm-to-norm continuous on every bounded set and thus

$$\|Jx_n - Jy_n\| \to 0 \text{ implies } \|x_n - y_n\| = \left\|J^{-1}Jx_n - J^{-1}Jy_n\right\| \to 0 \tag{2.3}$$

as $n \to \infty$; see [19] for more details.

Let E be a smooth Banach space. We use the following function $\phi\colon E \times E \to \mathbb{R}$ defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$$

for $x, y \in E$; see [1]. By the definition of ϕ, we immediately obtain

$$(\|x\| - \|y\|)^2 \le \phi(x, y) \le (\|x\| + \|y\|)^2$$

for all $x, y \in E$. Let E be a strictly convex, smooth, and reflexive Banach space and C a nonempty closed convex subset of E. It is known that, for each $x \in E$, there is a unique point $x_0 \in C$ such that

$$\phi(x_0, x) = \min\{\phi(y, x) : y \in C\}.$$

Such a point x_0 is denoted by $\Pi_C x$ and Π_C is said to be the generalized projection of E onto C; see [1] and [5]. We know some lemmas, which are used for the proofs of our main results.

Lemma 2.1 ([1] and [5]). *Let E be a strictly convex, smooth, and reflexive Banach space and C a nonempty closed convex subset of E. Let Π_C be the generalized projection of E onto C, $x \in E$ and $x_0 \in C$. Then $x_0 = \Pi_C x$ if and only if*

$$\langle x_0 - y, Jx - Jx_0\rangle \ge 0$$

for all $y \in C$.

Lemma 2.2 ([1] and [5]). *Let E be a strictly convex, smooth, and reflexive Banach space and C a nonempty closed convex subset of E. Let Π_C be the generalized projection of E onto C. Then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x) \tag{2.4}$$

for all $x \in E$ and $y \in C$.

Lemma 2.3 (Kamimura-Takahashi [5]). *Let E be a smooth and uniformly convex Banach space. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in E with $\lim_{n\to\infty} \phi(x_n, y_n) = 0$. If $\{x_n\}$ or $\{y_n\}$ is bounded, then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Let $\{x_n\}$ and $\{y_n\}$ be two bounded sequences of a smooth Banach space E. We have

$$\begin{aligned} 0 \leq \phi(x_n, y_n) &= \|x_n\|^2 - 2\langle x_n, Jy_n\rangle + \|y_n\|^2 \\ &= \|x_n\|^2 - \|y_n\|^2 - 2\langle x_n - y_n, Jy_n\rangle . \end{aligned}$$

Thus $\phi(x_n, y_n) \to 0$ whenever $\|x_n - y_n\| \to 0$. From this fact combined with (2.2), (2.3) and Lemma 2.3, we conclude the following: Let $\{x_n\}$ and $\{y_n\}$ be two bounded sequences of a uniformly convex and uniformly smooth Banach space E. Then

$$\|x_n - y_n\| \to 0 \Leftrightarrow \|Jx_n - Jy_n\| \to 0 \Leftrightarrow \phi(x_n, y_n) \to 0. \tag{2.5}$$

Let E be a Banach space. A multi-valued mapping A of E into E^* is said to be a monotone operator if $\langle x - y, x^* - y^*\rangle \geq 0$ for all $x, y \in D(A)$, $x^* \in Ax$, and $y^* \in Ay$, where $D(A) = \{x \in E : Ax \neq \emptyset\}$, which is called the effective domain of A. A monotone operator $A \subset E \times E^*$ is said to be maximal if its graph is not properly contained in the graph of any other monotone operators of $E \times E^*$. The following result is well-known.

Lemma 2.4 (Rockafellar [16]). *Let E be a strictly convex, smooth, and reflexive Banach space and $A \subset E \times E^*$ a monotone operator. Then A is maximal if and only if $R(J + rA) = E^*$ for all $r > 0$, where $R(J + rA)$ denotes the range of $J + rA$.*

Let E be a strictly convex, smooth, and reflexive Banach space and $A \subset E \times E^*$ a maximal monotone operator. Let $r > 0$ and $x \in E$ be given. Using

Lemma 2.4, we know that there exists a unique $x_r \in D(A)$ such that

$$Jx \in Jx_r + rAx_r.$$

Thus we may define a single-valued mapping $J_r\colon E \to D(A)$ by $J_r x = x_r$, that is, $J_r = (J + rA)^{-1}J$. Such J_r is said to be the resolvent of A for r. The set of all zeros of A is denoted by $A^{-1}0$, that is, $A^{-1}0 = \{x \in E : Ax \ni 0\}$. It is known that $A^{-1}0$ is a closed convex subset of E and $A^{-1}0 = F(J_r)$, where $F(J_r)$ is the fixed point set of J_r. It is also known that

$$\frac{1}{r}(J - JJ_r)x \in AJ_r x \tag{2.6}$$

for all $r > 0$ and $x \in E$; see [5–7].

3. Strongly Relatively Nonexpansive Mappings

Let E be a smooth Banach space, C a nonempty closed convex subset of E, and T a mapping of C into E. The set of all fixed points of T is denoted by $F(T)$. A point $p \in C$ is said to be an asymptotic fixed point of T [15] if C contains a sequence $\{x_n\}$ which converges weakly to p and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. The set of all asymptotic fixed points of T is denoted by $\hat{F}(T)$.

A mapping T of C into E is said to be relatively nonexpansive [3, 10, 11] if $F(T) = \hat{F}(T)$ and $\phi(p, Tx) \le \phi(p, x)$ for all $x \in C$ and $p \in F(T)$. It is known that $F(T)$ is a closed convex subset of E if T is relatively nonexpansive and E is strictly convex and smooth; see [11, Proposition 2.4]. It is also known that the generalized projection Π_C of E onto C is relatively nonexpansive if E is smooth, strictly convex, and reflexive. A mapping T of C into E is said to be strongly relatively nonexpansive if it is relatively nonexpansive and

$$\lim_{n\to\infty} \phi(Tx_n, x_n) = 0$$

whenever $\{x_n\}$ is a bounded sequence of C and $\lim_{n\to\infty}(\phi(p, x_n) - \phi(p, Tx_n)) = 0$ for some $p \in F(T)$; see [15].

Example 3.1. Let E be a smooth, strictly convex, and reflexive Banach space and C a nonempty closed convex subset of E. Let $T\colon C \to E$ be a relatively nonexpansive mapping that satisfies the following:

$$\phi(p, Tx) + \phi(Tx, x) \le \phi(p, x)$$

or equivalently:

$$\langle p - Tx, Jx - JTx \rangle \leq 0$$

for all $p \in F(T)$ and $x \in C$. Then T is strongly relatively nonexpansive. Thus it follows from (2.4) that the generalized projection Π_C of E onto C is an example of strongly relatively nonexpansive mappings. Let J_r be the resolvent of a maximal monotone operator $A \subset E \times E^*$ for $r > 0$. We know that

$$\phi(u, J_r x) + \phi(J_r x, x) \leq \phi(u, x)$$

for all $u \in A^{-1}0$ and $x \in E$; see [7, Lemma 3.1]. Moreover, suppose that E is uniformly smooth. In this case we also know that J_r is relatively nonexpansive with respect to $A^{-1}0$; see [11, Theorem 4.3] for more details. From these facts, we see that resolvents of a maximal monotone operator are also strongly relatively nonexpansive.

Before proving the first theorem, we need the following:

Lemma 3.2. *Let E be a smooth Banach space and C a nonempty closed convex subset of E. Let S be a strongly relatively nonexpansive mapping of C into E, T a relatively nonexpansive mapping of C into E, and U a mapping of C into E defined by $Ux = J^{-1}(\lambda JSx + (1-\lambda)JTx)$ for $x \in C$, where $\lambda \in (0,1)$ is a constant. Suppose $F(S) \cap F(T) \neq \emptyset$. Let $\{x_n\}$ be a bounded sequence in C and $w \in F(S) \cap F(T)$. If $\phi(w, x_n) - \phi(w, Ux_n) \to 0$, then $\phi(w, x_n) - \phi(w, Sx_n) \to 0$, $\phi(w, x_n) - \phi(w, Tx_n) \to 0$, and $\phi(Sx_n, x_n) \to 0$ as $n \to 0$.*

Proof. Let $x \in C$ be given. Since $\|\cdot\|^2$ is a convex function on E and both T and S are relatively nonexpansive, we have

$$\begin{aligned}
\phi(w, Ux) &= \|w\|^2 - 2\langle w, JUx \rangle + \|Ux\|^2 \\
&\leq \|w\|^2 - 2\langle w, \lambda JSx + (1-\lambda)JTx \rangle + \lambda\|Sx\|^2 + (1-\lambda)\|Tx\|^2 \\
&= \lambda\phi(w, Sx) + (1-\lambda)\phi(w, Tx) \\
&\leq \lambda\phi(w, Sx) + (1-\lambda)\phi(w, x) \leq \phi(w, x).
\end{aligned}$$

This shows that

$$0 \leq \lambda(\phi(w, x_n) - \phi(w, Sx_n)) \leq \phi(w, x_n) - \phi(w, Ux_n)$$

for every $n \in \mathbb{N}$. Thus we conclude that $\phi(w, x_n) - \phi(w, Sx_n) \to 0$. Since S is strongly relatively nonexpansive, we also have $\phi(Sx_n, x_n) \to 0$. Similarly, we obtain $\phi(w, x_n) - \phi(w, Tx_n) \to 0$. □

Theorem 3.3. *Let E be a uniformly convex and uniformly smooth Banach space and C a nonempty closed convex subset of E. Let S be a strongly relatively nonexpansive mapping of C into E, T a relatively nonexpansive mapping of C into E, and U a mapping of C into E defined by $Ux = J^{-1}(\lambda JSx + (1-\lambda)JTx)$ for $x \in C$, where $\lambda \in (0,1)$ is a constant. If $F(S) \cap F(T)$ is nonempty, then U is strongly relatively nonexpansive and $F(U) = \hat{F}(U) = F(S) \cap F(T)$.*

Proof. We first prove $\hat{F}(U) \subset \hat{F}(S) \cap \hat{F}(T)$. Let $z \in \hat{F}(U)$. Then there exists a sequence $\{z_n\}$ in C such that $z_n \rightharpoonup z$ and

$$\|z_n - Uz_n\| \to 0 \tag{3.1}$$

as $n \to \infty$. Note that $\{z_n\}$ is bounded and hence $\{Sz_n\}$, $\{Tz_n\}$, and $\{Uz_n\}$ are also bounded. Choose $w \in F(S) \cap F(T)$ arbitrarily. Since E is uniformly smooth and

$$\phi(w, Uz_n) - \phi(w, z_n) = \|Uz_n\|^2 - \|z_n\|^2 - 2\langle w, JUz_n - Jz_n\rangle,$$

it follows that $\phi(w, Uz_n) - \phi(w, z_n) \to 0$ as $n \to \infty$. Thus Lemmas 3.2 and 2.3 imply that

$$\|Sz_n - z_n\| \to 0 \tag{3.2}$$

as $n \to \infty$. Since S is relatively nonexpansive, we conclude that $z \in \hat{F}(S)$. On the other hand, we have

$$\begin{aligned}(1-\lambda)\|Jz_n - JTz_n\| &= \|Jz_n - (\lambda JSz_n + (1-\lambda)JTz_n) - \lambda(Jz_n - JSz_n)\| \\ &\leq \|Jz_n - JUz_n\| + \lambda\|Jz_n - JSz_n\|.\end{aligned}$$

From (3.2), (3.1) and (2.2), we know that both $\|Jz_n - JUz_n\|$ and $\|Jz_n - JSz_n\|$ converge to 0 as $n \to \infty$. Therefore we have $\|Jz_n - JTz_n\| \to 0$ and hence $\|z_n - Tz_n\| \to 0$ because of (2.3). Thus, we conclude that $z \in \hat{F}(T)$. From all observations above, we have

$$\hat{F}(S) \cap \hat{F}(T) = F(S) \cap F(T) \subset F(U) \subset \hat{F}(U) \subset \hat{F}(S) \cap \hat{F}(T)$$

and hence $F(S) \cap F(T) = F(U) = \hat{F}(U)$. From this fact, it is easy to check that U is relatively nonexpansive.

Let us prove that U is strongly relatively nonexpansive. Let $\{x_n\}$ be a bounded sequence in C and $p \in F(U)$. Suppose that $\phi(p, x_n) - \phi(p, Ux_n) \to 0$. We only have to show that $\phi(Ux_n, x_n) \to 0$. Lemma 3.2 implies that

$$\phi(p, x_n) - \phi(p, Sx_n) \to 0 \text{ and } \phi(p, x_n) - \phi(p, Tx_n) \to 0.$$

This shows that

$$\begin{aligned}
&\lambda \|JSx_n\|^2 + (1-\lambda) \|JTx_n\|^2 - \|\lambda JSx_n + (1-\lambda) JTx_n\|^2 \\
&\quad = \lambda \|Sx_n\|^2 + (1-\lambda) \|Tx_n\|^2 - \|Ux_n\|^2 \\
&\quad = \lambda(\phi(p, Sx_n) - \phi(p, x_n)) + (1-\lambda)(\phi(p, Tx_n) - \phi(p, x_n)) \\
&\qquad - (\phi(p, Ux_n) - \phi(p, x_n)) \to 0
\end{aligned}$$

as $n \to \infty$. Since E^* is uniformly convex and both $\{JSx_n\}$ and $\{JTx_n\}$ are bounded, we obtain $\|JSx_n - JTx_n\| \to 0$. On the other hand, it follows from Lemmas 3.2 and 2.3 that $\|Sx_n - x_n\| \to 0$. Using (2.2), we conclude $\|JSx_n - Jx_n\| \to 0$. These facts imply that

$$\begin{aligned}
\|Jx_n - JUx_n\| &= \|(1-\lambda)(JSx_n - JTx_n) + Jx_n - JSx_n\| \\
&\le (1-\lambda) \|JSx_n - JTx_n\| + \|Jx_n - JSx_n\| \to 0.
\end{aligned}$$

So, we have that $\|Jx_n - JUx_n\| \to 0$ and therefore $\phi(Ux_n, x_n)$ converges to 0 by (2.5). This completes our proof. $\square$

It is clear that the identity mapping I on C is strongly relatively nonexpansive. Putting $S = I$ in Theorem 3.3, we immediately obtain the following:

Corollary 3.4. *Let E be a uniformly convex and uniformly smooth Banach space and C a nonempty closed convex subset of E. Let T be a relatively nonexpansive mapping of C into E, and U a mapping of C into E defined by $Ux = J^{-1}(\lambda Jx + (1-\lambda)JTx)$ for $x \in C$, where $\lambda \in (0,1)$ is a constant. If $F(T)$ is nonempty, then U is strongly relatively nonexpansive and $F(U) = \hat{F}(U) = F(T)$.*

By induction and Theorem 3.3, we get the following result:

Corollary 3.5. *Let E be a uniformly convex and uniformly smooth Banach space and C a nonempty closed convex subset of E. Let $\{S_k\}_{k=1}^N$ be a finite family of relatively nonexpansive mappings of C into E, where N is some positive integer. Let $\{\lambda^k\}_{k=0}^N$ be a finite sequence in $(0,1)$ with $\sum_{k=0}^N \lambda^k = 1$.*

Let U be a mapping of C into E defined by $Ux = J^{-1}\sum_{k=0}^{N}\lambda^k JS_k x$ for $x \in C$, where S_0 is the identity mapping on C. If $\bigcap_{k=1}^{N} F(S_k)$ is nonempty, then U is strongly relatively nonexpansive and $F(U) = \hat{F}(U) = \bigcap_{k=1}^{N} F(S_k)$.

4. Strong Convergence Theorems

Using an iterative method developed in [5, 11–13], we first prove the following theorem for a sequence of relatively nonexpansive mappings.

Theorem 4.1. *Let E be a uniformly convex and uniformly smooth Banach space and C a nonempty closed convex subset of E. Let $\{T_n\}$ be a sequence of relatively nonexpansive mappings of C into E with $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Suppose that for any nonempty bounded closed convex subset B of C and any subsequence $\{T_{n_i}\}$ of $\{T_n\}$, there exist a subsequence $\{T_{n_{i_j}}\}$ of $\{T_{n_i}\}$ and a relatively nonexpansive mapping U of C into E such that*

$$F(U) = \bigcap_{n=1}^{\infty} F(T_n) \text{ and } \lim_{j\to\infty}\sup_{y\in B}\left\|Uy - T_{n_{i_j}}y\right\| = 0.$$

Let $\{x_n\}$ and $\{y_n\}$ be two sequences of E defined by the following:

$$\begin{cases} x_1 = x \in C; \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JT_n x_n); \\ H_n = \{z \in C : \phi(z, y_n) \le \phi(z, x_n)\}; \\ W_n = \{z \in C : \langle x_n - z, Jx - Jx_n\rangle \ge 0\}; \\ x_{n+1} = \Pi_{H_n\cap W_n}(x) \end{cases}$$

for each $n \in \mathbb{N}$, where $\{\alpha_n\}$ is a sequence in $[0,1]$ with $\limsup_{n\to\infty}\alpha_n < 1$ and $\Pi_{H_n\cap W_n}$ is the generalized projection of E onto $H_n \cap W_n$. Then $\{x_n\}$ converges strongly to $\Pi_{F(U)}(x)$, where $\Pi_{F(U)}$ is the generalized projection of E onto $F(U) = \bigcap_{n=1}^{\infty} F(T_n)$.

Proof. From the definition of H_n and W_n, it is clear that H_n and W_n are closed convex subsets of C for every $n \in \mathbb{N}$. First we show that $\bigcap_{n=1}^{\infty} F(T_n) \subset H_n$. Since T_n is relatively nonexpansive, $u \in \bigcap_{n=1}^{\infty} F(T_n)$ implies

$$\begin{aligned}\phi(u, y_n) &= \|u\|^2 - 2\langle u, \alpha_n Jx_n + (1-\alpha_n)JT_n x_n\rangle + \|\alpha_n Jx_n + (1-\alpha_n)JT_n x_n\|^2 \\ &\le \alpha_n(\|u\|^2 - 2\langle u, Jx_n\rangle + \|Jx_n\|^2) + (1-\alpha_n)(\|u\|^2 - 2\langle u, JT_n x_n\rangle + \|JT_n x_n\|^2) \\ &= \alpha_n\phi(u, x_n) + (1-\alpha_n)\phi(u, T_n x_n)\end{aligned}$$

$$\le \alpha_n \phi(u, x_n) + (1 - \alpha_n)\phi(u, x_n) = \phi(u, x_n).$$

Thus we know that $\bigcap_{n=1}^{\infty} F(T_n) \subset H_n$ for every $n \in \mathbb{N}$. We next show that $\bigcap_{n=1}^{\infty} F(T_n) \subset W_n$. It is obvious that $\bigcap_{n=1}^{\infty} F(T_n) \subset C = W_1$. Suppose that $u \in \bigcap_{n=1}^{\infty} F(T_n) \subset W_k$ for some $k \in \mathbb{N}$. Since $x_{k+1} = \Pi_{H_k \cap W_k}(x)$ and $u \in H_k \cap W_k$, we obtain $\langle x_{k+1} - u, Jx - Jx_{k+1}\rangle \ge 0$ from Lemma 2.1. This means that $u \in W_{k+1}$. Thus, by induction on k, we conclude that $u \in W_n$ for every $n \in \mathbb{N}$. Therefore we see that $\bigcap_{n=1}^{\infty} F(T_n) \subset H_n \cap W_n$ and $H_n \cap W_n$ is a nonempty closed convex subset of E for every $n \in \mathbb{N}$.

We verify that $\{x_n\}$ is bounded and $\lim_{n\to\infty} \|x_n - T_n x_n\| = 0$. Again, let $u \in \bigcap_{n=1}^{\infty} F(T_n)$ be fixed. Using Lemma 2.2 combined with the fact that $x_n = \Pi_{W_n}(x)$ and $u \in W_n$, we have

$$\phi(u, x_n) + \phi(x_n, x) \le \phi(u, x).$$

Hence, for every $n \in \mathbb{N}$,

$$(\|x_n\| - \|x\|)^2 \le \phi(x_n, x) \le \phi(u, x) \le (\|u\| + \|x\|)^2.$$

This means that both $\{\phi(x_n, x)\}$ and $\{x_n\}$ are bounded. Then we may assume, without loss of generality, that C is bounded. Using Lemma 2.2 combined with the fact that $x_{n+1} \in W_n$ and $x_n = \Pi_{W_n}(x)$, we have

$$\phi(x_{n+1}, x_n) + \phi(x_n, x) \le \phi(x_{n+1}, x) \tag{4.1}$$

for every $n \in \mathbb{N}$. This shows that $\{\phi(x_n, x)\}$ is nondecreasing and hence it is convergent. From (4.1), we have $\phi(x_{n+1}, x_n) \le \phi(x_{n+1}, x) - \phi(x_n, x)$. This yields

$$\lim_{n\to\infty} \phi(x_{n+1}, x_n) = 0. \tag{4.2}$$

By the definition of H_n, we see that $\phi(x_{n+1}, y_n) \le \phi(x_{n+1}, x_n)$. Therefore we also obtain

$$\lim_{n\to\infty} \phi(x_{n+1}, y_n) = 0. \tag{4.3}$$

From the definition of y_n, (4.2), (4.3), and (2.5), it follows that

$$(1 - \alpha_n)(Jx_{n+1} - JT_n x_n) = Jx_{n+1} - Jy_n - \alpha_n(Jx_{n+1} - Jx_n) \to 0.$$

Therefore we have $\|Jx_{n+1} - JT_n x_n\| \to 0$ because of the assumption on $\{\alpha_n\}$. This fact combined with (4.2) and (2.3) shows that

$$\|JT_n x_n - Jx_n\| \le \|JT_n x_n - Jx_{n+1}\| + \|Jx_{n+1} - Jx_n\| \to 0$$

and hence

$$\lim_{n\to\infty} \|T_n x_n - x_n\| = 0. \tag{4.4}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup v$. Let us show that $v \in \bigcap_{n=1}^{\infty} F(T_n)$. Let $\{T_{n_i}\}$ be the subsequence of $\{T_n\}$ corresponding to $\{x_{n_i}\}$. By assumption, for C and $\{T_{n_i}\}$, there exist a subsequence $\{T_{n_{i_j}}\}$ of $\{T_{n_i}\}$ and a relatively nonexpansive mapping U of C into E such that

$$F(U) = \bigcap_{n=1}^{\infty} F(T_n) \tag{4.5}$$

and

$$\lim_{j\to\infty} \sup_{y\in C} \left\| Uy - T_{n_{i_j}} y \right\| = 0. \tag{4.6}$$

It is clear that

$$\begin{aligned} \|Ux_n - x_n\| &\leq \|Ux_n - T_n x_n\| + \|T_n x_n - x_n\| \\ &\leq \sup_{y\in C} \|Uy - T_n y\| + \|T_n x_n - x_n\| \end{aligned}$$

for every $n \in \mathbb{N}$. From (4.4) and (4.6), we obtain

$$\lim_{j\to\infty} \left\| Ux_{n_{i_j}} - x_{n_{i_j}} \right\| = 0.$$

By (4.5) and the relative nonexpansiveness of U, we conclude that $v \in F(U) = \bigcap_{n=1}^{\infty} F(T_n)$.

Finally, let us prove that $x_{n_i} \rightharpoonup v$ implies $v = z$, where $z = \Pi_{F(U)}(x)$. Since $z \in F(U) = \bigcap_{n=1}^{\infty} F(T_n) \subset H_n \cap W_n$, it follows that

$$\phi(x_{n+1}, x) = \min\{\phi(y, x) : y \in H_n \cap W_n\} \leq \phi(z, x)$$

for every $n \in \mathbb{N}$. From the fact that $\|\cdot\|^2$ is weakly lower semicontinuous and $\lim_{n\to\infty} \phi(x_n, x)$ exists, we get

$$\begin{aligned} \phi(v, x) &= \|v\|^2 - 2\langle v, Jx\rangle + \|x\|^2 \\ &\leq \liminf_{i\to\infty} (\|x_{n_i}\|^2 - 2\langle x_{n_i}, Jx\rangle + \|x\|^2) \\ &= \liminf_{i\to\infty} \phi(x_{n_i}, x) \\ &= \lim_{n\to\infty} \phi(x_n, x) \\ &\leq \lim_{n\to\infty} \phi(z, x) = \phi(z, x). \end{aligned}$$

Since $v \in F(U)$ and $\{z\} = \operatorname{argmin}\{\phi(y,x) : y \in F(U)\}$, we see that $z = v$ and $\lim_{n\to\infty} \phi(x_n, x) = \phi(z, x)$. Therefore $x_n \rightharpoonup z$ and hence

$$\|x_n\|^2 - \|z\|^2 = \phi(x_n, x) - \phi(z, x) + 2\langle x_n - z, Jx\rangle \to 0$$

as $n \to \infty$. This shows $\lim_{n\to\infty} \|x_n\| = \|z\|$. Since E is uniformly convex, we conclude that $\lim_{n\to\infty} x_n = z$. This completes our proof. □

Using Theorem 4.1 and Corollary 3.5, we obtain the following theorem.

Theorem 4.2. *Let E be a uniformly convex and uniformly smooth Banach space and C a nonempty closed convex subset of E. Let $\{S_k\}_{k=1}^N$ be a finite family of relatively nonexpansive mappings of C into E with $\bigcap_{k=1}^N F(S_k) \neq \emptyset$, where N is some positive integer. Let $\{\lambda_n^k\}$ is a sequence in $(0,1)$ with two indices $n \in \mathbb{N}$ and $k = 0, \ldots, N$. Suppose that $\sum_{k=0}^N \lambda_n^k = 1$ for every $n \in \mathbb{N}$ and $\inf\{\lambda_n^k : n \in \mathbb{N}\} > 0$ for every $k = 0, \ldots, N$. Let $\{x_n\}$ and $\{y_n\}$ be two sequences of E defined by the following:*

$$\begin{cases} x_1 = x \in C; \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)\sum_{k=0}^N \lambda_n^k JS_k x_n); \\ H_n = \{z \in C : \phi(z, y_n) \leq \phi(z, x_n)\}; \\ W_n = \{z \in C : \langle x_n - z, Jx - Jx_n\rangle \geq 0\}; \\ x_{n+1} = \Pi_{H_n \cap W_n}(x) \end{cases}$$

for each $n \in \mathbb{N}$, where $\{\alpha_n\}$ is a sequence in $[0,1]$ with $\limsup_{n\to\infty} \alpha_n < 1$, $\Pi_{H_n \cap W_n}$ is the generalized projection of E onto $H_n \cap W_n$, and S_0 is the identity mapping on C. Then $\{x_n\}$ converges strongly to $\Pi_F(x)$, where Π_F is the generalized projection of E onto $F = \bigcap_{k=1}^N F(S_k)$.

Proof. Put $T_n = J^{-1}\sum_{k=0}^N \lambda_n^k JS_k$. Then Corollary 3.5 implies that each T_n is (strongly) relatively nonexpansive and $F(T_n) = \bigcap_{k=1}^N F(S_k)$. Let B be a nonempty bounded closed convex subset of C and $\{T_{n_i}\}$ a subsequence of $\{T_n\}$. Let $\{\lambda_{n_i}^k\}$ be the subsequence of $\{\lambda_n^k\}$ corresponding to $\{T_{n_i}\}$. By assumption, for each $k = 0, \ldots, N$, there exist $\lambda^k \in (0,1)$ and a subsequence $\{\lambda_{n_{i_j}}^k\}$ of $\{\lambda_{n_i}^k\}$ such that $\lim_{j\to\infty} \lambda_{n_{i_j}}^k = \lambda^k$. Define a mapping U of C into E by

$$U = J^{-1}\sum_{k=0}^N \lambda^k JS_k.$$

Then Corollary 3.5 also implies that U is relatively nonexpansive and $F(U) = \bigcap_{k=1}^{N} F(S_k)$. Hence $F(U) = \bigcap_{n=1}^{\infty} F(T_n)$. Note that $\sup\{\|S_k y\| : y \in B,\ k = 0, \dots, N\} < \infty$. Further we have

$$\begin{aligned}\|JUy - JT_n y\| &= \left\| \sum_{k=0}^{N} (\lambda^k - \lambda_n^k) JS_k y \right\| \\ &\le \sum_{k=0}^{N} \left| \lambda^k - \lambda_n^k \right| \|JS_k y\| \\ &\le \sum_{k=0}^{N} \left| \lambda^k - \lambda_n^k \right| M\end{aligned}$$

for all $y \in B$, where $M = \sup\{\|S_k y\| : y \in B,\ k = 0, \dots, N\}$. Therefore

$$\lim_{j\to\infty} \sup_{y \in B} \left\| JUy - JT_{n_{i_j}} y \right\| \le \lim_{j\to\infty} \sum_{k=0}^{N} \left| \lambda^k - \lambda_{n_{i_j}}^k \right| M = 0.$$

This shows that

$$\lim_{j\to\infty} \sup_{y \in B} \left\| Uy - T_{n_{i_j}} y \right\| = 0.$$

Using Theorem 4.1, we conclude that $\{x_n\}$ converges strongly to $\Pi_{F(U)}(x) = \Pi_F(x)$. $\square$

Applying Theorem 4.1, we can also obtain the following result:

Theorem 4.3 (Matsushita-Takahashi [11]). *Let E be a uniformly convex and uniformly smooth Banach space and C a nonempty closed convex subset of E. Let T be a relatively nonexpansive mapping of C into E with $F(T) \neq \emptyset$ and $\{\alpha_n\}$ a sequence of real numbers such that $\alpha_n \in [0, 1)$ for every $n \in \mathbb{N}$ and $\limsup_{n\to\infty} \alpha_n < 1$. Let $\{x_n\}$ and $\{y_n\}$ be two sequences of E defined by the following:*

$$\begin{cases} x_1 = x \in C; \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n) JTx_n); \\ H_n = \{z \in C : \phi(z, y_n) \le \phi(z, x_n)\}; \\ W_n = \{z \in C : \langle x_n - z, Jx - Jx_n \rangle \ge 0\}; \\ x_{n+1} = \Pi_{H_n \cap W_n}(x) \end{cases}$$

for each $n \in \mathbb{N}$, *where* $\Pi_{H_n \cap W_n}$ *is the generalized projection of* E *onto* $H_n \cap W_n$. *Then* $\{x_n\}$ *converges strongly to* $\Pi_{F(T)}(x)$, *where* $\Pi_{F(T)}$ *is the generalized projection of* E *onto* $F(T)$.

Proof. Putting $T_n = T$ for all $n \in \mathbb{N}$, we have that $\{T_n\}$ satisfies the condition in Theorem 4.1. So, we obtain the desired result by using Theorem 4.1. $\square$

Finally, we apply Theorem 4.1 to the problem of finding a zero of a maximal monotone operator in a Banach space. This problem has been studied by many researchers; see, for example, [9], [17], [14], [15], [18], [4, 5], [13], [6], [7], and [10, 11].

Before solving the problem, we begin with the following lemma:

Lemma 4.4. *Let* E *be a strictly convex, smooth, and reflexive Banach space and* J_r *the resolvent of a maximal monotone operator* $A \subset E \times E^*$ *for* $r > 0$. *Then*

$$\lambda\phi(J_\lambda x, J_\mu x) + \mu\phi(J_\mu x, J_\lambda x) \leq (\lambda - \mu)(\phi(J_\lambda x, x) - \phi(J_\mu x, x))$$

for all $x \in E$ *and* $\lambda, \mu > 0$. *In particular,*

$$\phi(J_\lambda x, J_\mu x) \leq \frac{|\lambda - \mu|}{\lambda} |\phi(J_\lambda x, x) - \phi(J_\mu x, x)| \tag{4.7}$$

for all $x \in E$ *and* $\lambda, \mu > 0$.

Proof. Let $x \in E$ and $\lambda, \mu > 0$ be fixed. From (2.6) and the monotonicity of A, we have

$$\begin{aligned} 0 &\leq \left\langle J_\lambda x - J_\mu x, \frac{1}{\lambda}(J - JJ_\lambda)x - \frac{1}{\mu}(J - JJ_\mu)x \right\rangle \\ &= \frac{1}{\lambda\mu} \left\langle J_\lambda x - J_\mu x, \mu(Jx - JJ_\lambda x) - \lambda(Jx - JJ_\mu x) \right\rangle. \end{aligned}$$

So, we have

$$0 \leq \mu \left\langle J_\lambda x - J_\mu x, Jx - JJ_\lambda x \right\rangle - \lambda \left\langle J_\lambda x - J_\mu x, Jx - JJ_\mu x \right\rangle$$

and hence

$$0 \leq (\mu - \lambda) \left\langle J_\lambda x - J_\mu x, Jx \right\rangle - \mu \left\langle J_\lambda x - J_\mu x, JJ_\lambda x \right\rangle + \lambda \left\langle J_\lambda x - J_\mu x, JJ_\mu x \right\rangle.$$

Therefore we obtain

$$
\begin{aligned}
&-\left(\lambda \langle J_\lambda x, JJ_\mu x\rangle + \mu \langle J_\mu x, JJ_\lambda x\rangle\right) \\
&\qquad\qquad \leq (\mu-\lambda)\langle J_\lambda x - J_\mu x, Jx\rangle - \mu \|J_\lambda x\|^2 - \lambda \|J_\mu x\|^2 .
\end{aligned}
$$

Applying this inequality, we have

$$
\begin{aligned}
&\lambda\phi(J_\lambda x, J_\mu x) \\
&\quad \leq \lambda\phi(J_\lambda x, J_\mu x) + \mu\phi(J_\mu x, J_\lambda x) \\
&\quad = \lambda(\|J_\lambda x\|^2 - 2\langle J_\lambda x, JJ_\mu x\rangle + \|J_\mu x\|^2) \\
&\qquad + \mu(\|J_\mu x\|^2 - 2\langle J_\mu x, JJ_\lambda x\rangle + \|J_\lambda x\|^2) \\
&\quad = (\lambda+\mu)(\|J_\lambda x\|^2 + \|J_\mu x\|^2) - 2(\lambda\langle J_\lambda x, JJ_\mu x\rangle + \mu\langle J_\mu x, JJ_\lambda x\rangle) \\
&\quad \leq (\lambda+\mu)(\|J_\lambda x\|^2 + \|J_\mu x\|^2) \\
&\qquad + 2((\mu-\lambda)\langle J_\lambda x - J_\mu x, Jx\rangle - \mu\|J_\lambda x\|^2 - \lambda\|J_\mu x\|^2) \\
&\quad = \lambda(\|J_\lambda x\|^2 + \|J_\mu x\|^2 - 2\langle J_\lambda x - J_\mu x, Jx\rangle - 2\|J_\mu x\|^2) \\
&\qquad - \mu(-\|J_\lambda x\|^2 - \|J_\mu x\|^2 - 2\langle J_\lambda x - J_\mu x, Jx\rangle + 2\|J_\lambda x\|^2) \\
&\quad = (\lambda-\mu)(\|J_\lambda x\|^2 - 2\langle J_\lambda x, Jx\rangle + \|x\|^2 - \|J_\mu x\|^2 + 2\langle J_\mu x, Jx\rangle - \|x\|^2) \\
&\quad = (\lambda-\mu)(\phi(J_\lambda x, x) - \phi(J_\mu x, x)) \\
&\quad = |\lambda-\mu|\,|\phi(J_\lambda x, x) - \phi(J_\mu x, x)| .
\end{aligned}
$$

This completes the proof. □

Using Theorem 4.1 and Lemma 4.4, we show the following strong convergence theorem. A similar result was obtained in [11]; compare this theorem with [11, Theorem 4.3]; see also [7, Theorem 3.3].

Theorem 4.5. *Let E be a uniformly convex and uniformly smooth Banach space, $A \subset E \times E^*$ a maximal monotone operator with $A^{-1}0 \neq \emptyset$, and J_r the resolvent of A for $r > 0$. Let $\{r_n\}$ be a bounded sequence of positive real numbers which is bounded away from 0. Let $\{x_n\}$ and $\{y_n\}$ be two sequences* *of E defined by the following:*

$$\begin{cases} x_1 = x \in E; \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JJ_{r_n}x_n); \\ H_n = \{z \in E : \phi(z, y_n) \leq \phi(z, x_n)\}; \\ W_n = \{z \in E : \langle x_n - z, Jx - Jx_n\rangle \geq 0\}; \\ x_{n+1} = \Pi_{H_n \cap W_n}(x) \end{cases}$$

for each $n \in \mathbb{N}$, where $\{\alpha_n\}$ is a sequence in $[0,1]$ with $\limsup_{n\to\infty} \alpha_n < 1$ and $\Pi_{H_n\cap W_n}$ is the generalized projection of E onto $H_n \cap W_n$. Then $\{x_n\}$ converges strongly to $\Pi_{A^{-1}0}(x)$, where $\Pi_{A^{-1}0}$ is the generalized projection of E onto $A^{-1}0$.

Proof. Put $T_n = J_{r_n}$. Then each T_n is relatively nonexpansive and $\bigcap_{n=1}^{\infty} F(T_n) = A^{-1}0$. Let B be a nonempty bounded closed convex subset of E and $\{T_{n_i}\}$ a subsequence of $\{T_n\}$. Let $\{r_{n_i}\}$ be the subsequence of $\{r_n\}$ corresponding to $\{T_{n_i}\}$. By assumption, there exist $s > 0$ and a subsequence $\{r_{n_{i_j}}\}$ of $\{r_{n_i}\}$ such that $\lim_{j\to\infty} r_{n_{i_j}} = s > 0$. Let U be a mapping of E into $D(A)$ defined by $U = J_s$. It is clear that U is also relatively nonexpansive and $F(U) = A^{-1}0$. Hence $F(U) = \bigcap_{n=1}^{\infty} F(T_n)$. Further we have, by (4.7),

$$\sup_{y\in B} \phi(J_s y, J_{r_n} y) \leq \frac{|s - r_n|}{s} \sup_{y\in B} |\phi(J_s y, y) - \phi(J_{r_n} y, y)| .$$

It is easy to check that $\{\sup_{y\in B} |\phi(J_s y, y) - \phi(J_{r_n} y, y)|\}$ is bounded.

Thus it follows that

$$\lim_{j\to\infty} \sup_{y\in B} \phi(Uy, T_{n_{i_j}} y) = 0.$$

This shows that

$$\lim_{j\to\infty} \sup_{y\in B} \left\| Uy - T_{n_{i_j}} y \right\| = 0.$$

Using Theorem 4.1, we conclude that $\{x_n\}$ converges strongly to $\Pi_{A^{-1}0}(x)$. $\square$

References

[1] Y. I. Alber, *Metric and generalized projection operators in Banach spaces: properties and applications*, Theory and applications of nonlinear operators of accretive and monotone type, Lecture Notes in Pure and Appl. Math., vol. 178, Dekker, New York, 1996, pp. 15–50.

[2] R. E. Bruck and S. Reich, *Nonexpansive projections and resolvents of accretive operators in Banach spaces*, Houston J. Math. **3** (1977), 459–470.

[3] D. Butnariu, S. Reich, and A. J. Zaslavski, *Asymptotic behavior of relatively nonexpansive operators in Banach spaces*, J. Appl. Anal. **7** (2001), 151–174.

[4] S. Kamimura and W. Takahashi, *Approximating solutions of maximal monotone operators in Hilbert spaces*, J. Approx. Theory **106** (2000), 226–240.

[5] ———, *Strong convergence of a proximal-type algorithm in a Banach space*, SIAM J. Optim. **13** (2002), 938–945 (electronic) (2003).

[6] S. Kamimura, F. Kohsaka, and W. Takahashi, *Weak and strong convergence theorems for maximal monotone operators in a Banach space*, Set-Valued Anal. **12** (2004), 417–429.

[7] F. Kohsaka and W. Takahashi, *Strong convergence of an iterative sequence for maximal monotone operators in a Banach space*, Abstr. Appl. Anal. (2004), 239–249.

[8] ———, *Block iterative methods for a finite family of relatively nonexpansive mappings in Banach spaces*, Fixed Point Theory Appl. (2007), Art. ID 21972, 18.

[9] B. Martinet, *Régularisation d'inéquations variationnelles par approximations successives*, Rev. Française Informat. Recherche Opérationnelle **4** (1970), 154–158 (French).

[10] S. Matsushita and W. Takahashi, *Weak and strong convergence theorems for relatively nonexpansive mappings in Banach spaces*, Fixed Point Theory Appl. (2004), 37–47.

[11] ———, *A strong convergence theorem for relatively nonexpansive mappings in a Banach space*, J. Approx. Theory **134** (2005), 257–266.

[12] K. Nakajo and W. Takahashi, *Strong convergence theorems for nonexpansive mappings and nonexpansive semigroups*, J. Math. Anal. Appl. **279** (2003), 372–379.

[13] S. Ohsawa and W. Takahashi, *Strong convergence theorems for resolvents of maximal monotone operators in Banach spaces*, Arch. Math. (Basel) **81** (2003), 439–445.

[14] G. B. Passty, *Ergodic convergence to a zero of the sum of monotone operators in Hilbert space*, J. Math. Anal. Appl. **72** (1979), 383–390.

[15] S. Reich, *A weak convergence theorem for the alternating method with Bregman distances*, Theory and applications of nonlinear operators of accretive and monotone type, Lecture Notes in Pure and Appl. Math., vol. 178, Dekker, New York, 1996, pp. 313–318.

[16] R. T. Rockafellar, *On the maximality of sums of nonlinear monotone operators*, Trans. Amer. Math. Soc. **149** (1970), 75–88.

[17] ———, *Monotone operators and the proximal point algorithm*, SIAM J. Control Optimization **14** (1976), 877–898.

[18] M. V. Solodov and B. F. Svaiter, *Forcing strong convergence of proximal point iterations in a Hilbert space*, Math. Program. **87** (2000), 189–202.

[19] W. Takahashi, *Nonlinear Functional Analysis-Fixed Point Theory and its Applications*, Yokohama Publishers, Yokohama, 2000.

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