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ON THE SOLUTION SET OF A NONCONVEX NONCLOSED SECOND ORDER DIFFERENTIAL INCLUSION

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Abstract. We consider a nonconvex and nonclosed second order differential inclusion with mixed boundary conditions and we prove the arcwise connectedness of the solution set.

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1. Introduction

In this paper we study the second order differential inclusion

$$x'' \in F(t, x, H(t, x)), \quad a.e.\ (I) \tag{1.1}$$

with boundary conditions of the form

$$\begin{aligned} x(0) - k_1 x'(0) &= c_1, \\ x(0) + k_2 x'(0) &= c_2, \end{aligned} \tag{1.2}$$

where $I = [0, 1]$, $F(., ., .) : I \times \mathbf{R}^2 \to \mathcal{P}(\mathbf{R})$, $H(., .) : I \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ and $k_i \in \mathbf{R}_+$, $c_i \in \mathbf{R}$, $i = 1, 2$.

When F does not depend on the last variable (1.1) reduces to

$$x'' \in F(t, x), \quad a.e.\ (I). \tag{1.3}$$

In the theory of ordinary differential equations (i.e., when F is a single valued map) the problem (1.3)-(1.2) is well known as a bilocal problem with mixed boundary conditions.

Qualitative properties of the set of solutions of problem (1.3)-(1.2) may be found in [1,4,7] etc. In all these papers the set-valued map F is assumed to be at least closed-valued. Such an assumption is quite natural in order to obtain good properties of the solution set, but it is interesting to investigate the problem when the right-hand side of the multivalued equation may have nonclosed values.

Following the approach in [6] we consider the problem (1.1)-(1.2), where F and H are closed-valued multifunctions Lipschitzian with respect to the second variable and F is contractive in the third variable. Obviously, the right-hand side of the differential inclusion in (1.1) is in general neither convex nor closed. We prove the arcwise connectedness of the solution set to (1.1)-(1.2). The main tool is a result ([5]) concerning the arcwise connectedness of the fixed point set of a class of nonconvex nonclosed set-valued contractions. This idea was already used for similar results for other classes of differential inclusions ([2,3,6]).

The paper is organized as follows: in Section 2 we recall some preliminary results that we use in the sequel and in Section 3 we prove our main result.

2. Preliminaries

Let Z be a metric space with the distance d_Z and let 2^Z be the family of all nonempty closed subsets of Z. For $a \in Z$ and $A, B \in 2^Z$ set $d_Z(a,B) = \inf_{b\in B} d_Z(a,b)$ and $d_Z^*(A,B) = \sup_{a\in A} d_Z(a,B)$. Denote by D_Z the Pompeiu-Hausdorff generalized metric on 2^Z defined by

$$D_Z(A,B) = \max\{d_Z^*(A,B), d_Z^*(B,A)\}, \quad A, B \in 2^Z.$$

In what follows, when the product $Z = Z_1 \times Z_2$ of metric spaces $Z_i, i = 1,2$, is considered, it is assumed that Z is equipped with the distance $d_Z((z_1,z_2),(z_1',z_2')) = \sum_{i=1}^2 d_{Z_i}(z_i,z_i')$.

Let X be a nonempty set and let $F : X \to 2^Z$ be a set-valued map from X to Z. The range of F is the set $F(X) = \cup_{x\in X} F(x)$. Let $(X, \mathcal{F})$ be a measurable space. The multifunction $F : X \to 2^Z$ is called measurable if $F^{-1}(\Omega) \in \mathcal{F}$ for any open set $\Omega \subset Z$, where $F^{-1}(\Omega) = \{x \in X; F(x) \cap \Omega \neq \emptyset\}$. Let (X, d_X) be a metric space. The multifunction F is called Hausdorff continuous if for any $x_0 \in X$ and every $\epsilon > 0$ there exists $\delta > 0$ such that $x \in X, d_X(x,x_0) < \delta$ implies $D_Z(F(x), F(x_0)) < \epsilon$.

Let $(T, \mathcal{F}, \mu)$ be a finite, positive, nonatomic measure space and let $(X, |.|_X)$ be a Banach space. We denote by $L^1(T, X)$ the Banach space of all (equivalence classes of) Bochner integrable functions $u : T \to X$ endowed with the norm

$$|u|_{L^1(T,X)} = \int_T |u(t)|_X d\mu$$

A nonempty set $K \subset L^1(T, X)$ is called decomposable if, for every $u, v \in K$ and every $A \in \mathcal{F}$, one has

$$\chi_A.u + \chi_{T \setminus A}.v \in K$$

where $\chi_B, B \in \mathcal{F}$ indicates the characteristic function of B.

A metric space Z is called an absolute retract if, for any metric space X and any nonempty closed set $X_0 \subset X$, every continuous function $g : X_0 \to Z$ has a continuous extension $g : X \to Z$ over X. It is obvious that every continuous image of an absolute retract is an arcwise connected space.

In what follows we recall some preliminary results that are the main tools in the proof of our result.

Let $(T, \mathcal{F}, \mu)$ be a finite, positive, nonatomic measure space, S a separable Banach space and let $(X, |.|_X)$ be a real Banach space. To simplify the notation we write E in place of $L^1(T, X)$.

Lemma 2.1 ([6]). *Assume that $\phi : S \times E \to 2^E$ and $\psi : S \times E \times E \to 2^E$ are Hausdorff continuous multifunctions with nonempty, closed, decomposable values, satisfying the following conditions*

a) There exists $L \in [0, 1)$ such that, for every $s \in S$ and every $u, u' \in E$,

$$D_E(\phi(s, u), \phi(s, u')) \leq L|u - u'|_E.$$

b) There exists $M \in [0, 1)$ such that $L + M < 1$ and for every $s \in S$ and every $(u, v), (u', v') \in E \times E$,

$$D_E(\psi(s, u, v), \psi(s, u', v')) \leq M(|u - u'|_E + |v - v'|_E).$$

Set $Fix(\Gamma(s, .)) = \{u \in E; u \in \Gamma(s, u)\}$, where $\Gamma(s, u) = \psi(s, u, \phi(s, u))$, $(s, u) \in S \times E$. Then

1) For every $s \in S$ the set $Fix(\Gamma(s, .))$ is nonempty and arcwise connected.

2) For any $s_i \in S$, and any $u_i \in Fix(\Gamma(s,.)), i = 1,...,p$ there exists a continuous function $\gamma : S \to E$ such that $\gamma(s) \in Fix(\Gamma(s,.))$ for all $s \in S$ and $\gamma(s_i) = u_i, i = 1,...,p$.

Lemma 2.2 ([6]). *Let $U : T \to 2^X$ and $V : T \times X \to 2^X$ be two nonempty closed-valued multifunctions satisfying the following conditions*

a) U is measurable and there exists $r \in L^1(T)$ such that $D_X(U(t), \{0\}) \leq r(t)$ for almost all $t \in T$.

b) The multifunction $t \to V(t,x)$ is measurable for every $x \in X$.

c) The multifunction $x \to V(t,x)$ is Hausdorff continuous for all $t \in T$.

Let $v : T \to X$ be a measurable selection from $t \to V(t, U(t))$.

Then there exists a selection $u \in L^1(T,X)$ such that $v(t) \in V(t,u(t)), t \in T$.

Let $I = [0,1]$, by $C(I)$ we denote the Banach space of all continuous functions from I to $\mathbf{R}$ with the norm $||x(.)||_C = \sup_{t\in I} |x(t)|$, by AC^1 we denote the space of differentiable functions $x(.) : (0,1) \to \mathbf{R}$ whose first derivative $x'(.)$ is absolutely continuous and by L^1 we denote the Banach space of Lebesgue integrable functions $x(.) : [0,1] \to \mathbf{R}$ endowed with the norm $||u(.)||_1 = \int_0^1 |u(t)|dt$.

A function $x(.) \in AC^1$ is said to be a solution of (1.3)-(1.2) if there exists a function $v(.) \in L^1$ with $v(t) \in F(t,x(t))$, *a.e.* (I) such that $x''(t) = v(t)$, *a.e.* (I) and $x(.)$ satisfies conditions (1.2).

The next statement is well known (e.g. [1]).

Lemma 2.3. *If $v(.) : [0,1] \to \mathbf{R}$ is an integrable function then the problem*

$$\begin{gathered} x''(t) = v(t) \quad a.e.\ (I) \\ x(0) - k_1 x'(0) = c_1, \\ x(0) + k_2 x'(0) = c_2, \end{gathered}$$

has a unique solution $x(.) \in AC^1$ given by

$$x(t) = P_c(t) + \int_0^1 G(t,s)v(s)ds,$$

where if $c = (c_1, c_2) \in \mathbf{R}^2$ we denote

$$P_c(t) = \frac{(1 - t + k_2)c_1 + (k_1 + t)c_2}{1 + k_1 + k_2} \tag{2.1}$$

and

$$G(t,s) = \frac{-1}{1+k_1+k_2}\begin{cases} (k_1+t)(1-s+k_2) & \text{if} \quad 0 \le t < s \le 1 \\ (k_1+s)(1-t+k_2) & \text{if} \quad 0 \le s < t \le 1 \end{cases} \tag{2.2}$$

is the Green function of the problem.

Let us note that if $a = (a_1, a_2), b = (b_1, b_2) \in \mathbf{R}^2$ then

$$|P_a(t) - P_b(t)| \le ||a - b||.$$

On the other hand, it is well known that $\sup_{t,s\in I} |G(t,s)| = \frac{1+k_1+k_2}{4}$.

In order to study problem (1.1)-(1.2) we introduce the following hypothesis.

Hypothesis 2.4. *Let $F : I \times \mathbf{R}^2 \to 2^{\mathbf{R}}$ and $H : I \times \mathbf{R} \to 2^{\mathbf{R}}$ be two set-valued maps with nonempty closed values, satisfying the following assumptions:*

i) The set-valued maps $t \to F(t,u,v)$ and $t \to H(t,u)$ are measurable for all $u, v \in \mathbf{R}$.

ii) There exists $l \in L^1$ such that, for every $u, u' \in \mathbf{R}$,

$$D(H(t,u), H(t,u')) \le l(t)|u - u'| \quad a.e.(I).$$

iii) There exist $m \in L^1$ and $\theta \in [0,1)$ such that, for every $u, v, u', v' \in \mathbf{R}$,

$$D(F(t,u,v), F(t,u',v')) \le m(t)|u - u'| + \theta|v - v'| \quad a.e.(I).$$

iv) There exist $f, g \in L^1$ such that

$$d(0, F(t,0,0)) \le f(t), \quad d(0, H(t,0)) \le g(t) \quad a.e.(I).$$

For $c = (c_1, c_2) \in \mathbf{R}^2$ we denote by $S(c)$ the solution set of (1.1)-(1.2).

In what follows $N(t) := \max\{l(t), m(t)\}$.

3. The main result

Even if the multifunction from the right-hand side of (1.1) has, in general, nonclosed nonconvex values, the solution set $S(c)$ has some meaningful properties, stated in Theorem 3.1 below.

Theorem 3.1. *Assume that Hypothesis 2.4 is satisfied and*

$$2 \sup_{t,s\in I} |G(t,s)| \cdot \int_0^1 N(s)ds + \theta < 1.$$

Then

1) For every $c\in \mathbf{R}^2$, the solution set $S(c)$ of (1.1)-(1.2) is nonempty and arcwise connected in the space $C(I)$.

2) For any $c_i\in \mathbf{R}^2$ and any $u_i\in S(c_i)$, $i=1,...,p$, there exists a continuous function $s:\mathbf{R}^2\to C(I)$ such that $s(c)\in S(c)$ for any $c\in \mathbf{R}^2$ and $s(c_i)=u_i, i=1,...,p$.

3) The set $S=\cup_{c\in \mathbf{R}^2}S(c)$ is arcwise connected in $C(I)$.

Proof. 1) For $c\in \mathbf{R}^2$ and $u\in L^1$, set

$$u_c(t)=P_c(t)+\int_0^1 G(t,s)u(s)ds,\quad t\in I,$$

where $P_c(.)$ and $G(.,.)$ are defined in (2.1) and (2.2), respectively.

We prove that the multifunctions $\phi:\mathbf{R}^2\times L^1\to 2^{L^1}$ and $\psi:\mathbf{R}^2\times L^1\times L^1\to 2^{L^1}$ given by

$$\phi(c,u)=\{v\in L^1;\quad v(t)\in H(t,u_c(t))\quad a.e.\ (I)\},$$

$$\psi(c,u,v)=\{w\in L^1; w(t)\in F(t,u_c(t),v(t))\quad a.e.\ (I)\},$$

$c\in \mathbf{R}^2, u,v\in L^1$ satisfy the hypotheses of Lemma 2.1.

Since $u_c(.)$ is measurable and H satisfies Hypothesis 2.4 i) and ii), the multifunction $t\to H(t,u_c(t))$ is measurable and nonempty closed-valued, it has a measurable selection. Therefore due to Hypothesis 2.4 iv), the set $\phi(c,u)$ is nonempty. The fact that the set $\phi(c,u)$ is closed and decomposable follows by simple computation. In the same way we obtain that $\psi(c,u,v)$ is a nonempty closed decomposable set.

Pick $(c,u),(c_1,u_1)\in \mathbf{R}^2\times L^1$ and choose $v\in\phi(c,u)$. For each $\varepsilon>0$ there exists $v_1\in\phi(c_1,u_1)$ such that, for every $t\in I$, one has

$$|v(t)-v_1(t)|\leq D(H(t,u_c(t)),H(t,u_{c_1}(t)))+\varepsilon\leq N(t)[|P_c(t)-P_{c_1}(t)|+$$

$$\int_0^1|G(t,s)|.|u(s)-u_1(s)|ds]+\varepsilon\leq N(t)[||c-c_1||+\sup_{t,s\in I}|G(t,s)|.||u-u_1||_1]+\varepsilon.$$

Hence

$$||v-v_1||_1\leq ||c-c_1||.\int_0^1 N(t)dt+\sup_{t,s\in I}|G(t,s)|\int_0^1 N(t)dt||u-u_1||_1+\varepsilon$$

for any $\varepsilon>0$.

This implies

$$d_{L^1}(v,\phi(c_1,u_1)) \leq ||c-c_1||.\int_0^1 N(t)dt + \sup_{t,s\in I}|G(t,s)|\int_0^1 N(t)dt||u-u_1||_1$$

for all $v\in\phi(c,u)$. Therefore,

$$d^*_{L^1}(\phi(c,u),\phi(c_1,u_1)) \leq ||c-c_1||.\int_0^1 N(t)dt + \sup_{t,s\in I}|G(t,s)|\int_0^1 N(t)dt||u-u_1||_1.$$

Consequently,

$$D_{L^1}(\phi(c,u),\phi(c_1,u_1)) \leq ||c-c_1||.\int_0^1 N(t)dt + \sup_{t,s\in I}|G(t,s)|\int_0^1 N(t)dt||u-u_1||_1$$

which shows that ϕ is Hausdorff continuous and satisfies the assumptions of Lemma 2.1.

Pick $(c,u,v),(c_1,u_1,v_1)\in\mathbf{R}^2\times L^1\times L^1$ and choose $w\in\psi(c,u,v)$. Then, as before, for each $\varepsilon>0$ there exists $w_1\in\psi(c_1,u_1,v_1)$ such that for every $t\in I$

$$\begin{aligned}|w(t)-w_1(t)| &\leq D(F(t,u_c(t),v(t)),F(t,u_{c_1}(t),v_1(t)))+\varepsilon\\ \leq N(t)|u_c(t)-u_{c_1}(t)| &+ \theta|v(t)-v_1(t)|+\varepsilon \leq N(t)[|P_c(t)-P_{c_1}(t)|\\ +\int_0^1 |G(t,s)|.|u(s)-u_1(s)|ds] &+ \theta|v(t)-v_1(t)|+\varepsilon \leq N(t)[||c-c_1||\\ &+\sup_{t,s\in I}|G(t,s)|.||u-u_1||_1]+\theta|v(t)-v_1(t)|+\varepsilon.\end{aligned}$$

Hence

$$\begin{aligned}||w-w_1||_1 \leq ||c-c_1||.\int_0^1 N(t)dt &+ \sup_{t,s\in I}|G(t,s)|\int_0^1 N(t)dt||u-u_1||_1\\ &+\theta||v-v_1||_1+\varepsilon\end{aligned}$$

$$\leq ||c-c_1||.\int_0^1 N(t)dt + (\sup_{t,s\in I}|G(t,s)|\int_0^1 N(t)dt+\theta)d_{L^1\times L^1}((u,v),(u_1,v_1))+\varepsilon.$$

As above, we deduce that

$$D_{L^1}(\psi(c,u,v),\psi(c_1,u_1,v_1)) \leq$$

$$\leq ||c-c_1||.\int_0^1 N(t)dt + (\sup_{t,s\in I}|G(t,s)|\int_0^1 N(t)dt+\theta)d_{L^1\times L^1}((u,v),(u_1,v_1)),$$

namely, the multifunction ψ is Hausdorff continuous and satisfies the hypothesis of Lemma 2.1.

Define $\Gamma(c,u)=\psi(c,u,\phi(c,u)),(c,u)\in\mathbf{R}^2\times L^1$. According to Lemma 2.1, the set $Fix(\Gamma(c,.))=\{u\in L^1; u\in\Gamma(c,u)\}$ is nonempty and arcwise connected in L^1. Moreover, for fixed $c_i \in \mathbf{R}^2$ and $v_i \in Fix(\Gamma(c_i,.)), i = 1, ..., p$, there exists a continuous function $\gamma : \mathbf{R}^2 \to L^1$ such that

$$\gamma(c) \in Fix(\Gamma(c,.)), \quad \forall c \in \mathbf{R}^2, \tag{3.1}$$

$$\gamma(c_i) = v_i, \quad i = 1, ..., p. \tag{3.2}$$

We shall prove that

$$Fix(\Gamma(c,.)) = \{u \in L^1; \quad u(t) \in F(t, u_c(t), H(t, u_c(t))) \quad a.e.\ (I)\}. \tag{3.3}$$

Denote by $A(c)$ the right-hand side of (3.3). If $u \in Fix(\Gamma(c,.))$ then there is $v \in \phi(c, v)$ such that $u \in \psi(c, u, v)$. Therefore, $v(t) \in H(t, u_c(t))$ and

$$u(t) \in F(t, u_c(t), v(t)) \subset F(t, u_c(t), H(t, u_c(t))) \quad a.e.\ (I),$$

so that $Fix(\Gamma(c,.)) \subset A(c)$.

Let now $u \in A(c)$. By Lemma 2.2, there exists a selection $v \in L^1$ of the multifunction $t \to H(t, u_c(t)))$ satisfying

$$u(t) \in F(t, u_c(t), v(t)) \quad a.e.\ (I).$$

Hence, $v \in \phi(c, v)$, $u \in \psi(c, u, v)$ and thus $u \in \Gamma(c, u)$, which completes the proof of (3.3).

We next note that the function $T : L^1 \to C(I)$,

$$T(u)(t) := \int_0^1 G(t,s)u(s)ds, \quad t \in I$$

is continuous and one has

$$S(c) = P_c(.) + T(Fix(\Gamma(c,.))), \quad c \in \mathbf{R}^2. \tag{3.4}$$

Since $Fix(\Gamma(c,.))$ is nonempty and arcwise connected in L^1, the set $S(c)$ has the same properties in $C(I)$.

2) Let $c_i \in \mathbf{R}^2$ and let $u_i \in S(c_i), i = 1, ..., p$ be fixed. By (3.4) there exists $v_i \in Fix(\Gamma(c_i,.))$ such that

$$u_i = P_{c_i}(.) + T(v_i), \quad i = 1, ..., p.$$

If $\gamma : \mathbf{R}^2 \to L^1$ is a continuous function satisfying (3.1) and (3.2) we define, for every $c \in \mathbf{R}^2$,

$$s(c) = P_c(.) + T(\gamma(c)).$$

Obviously, the function $s : \mathbf{R}^2 \to C(I)$ is continuous, $s(c) \in S(c)$ for all $c \in \mathbf{R}^2$, and

$$s(c_i) = P_{c_i}(.) + T(\gamma(c_i)) = P_{c_i}(.) + T(v_i) = u_i, \quad i = 1, ..., p.$$

3) Let $u_1, u_2 \in S = \cup_{c\in\mathbf{R}^2} S(c)$ and choose $c_i \in \mathbf{R}^2$, $i = 1, 2$ such that $u_i \in S(c_i)$, $i = 1, 2$. From the conclusion of 2) we deduce the existence of a continuous function $s : \mathbf{R}^2 \to C(I)$ satisfying $s(c_i) = u_i$, $i = 1, 2$ and $s(c) \in S(c)$, $c \in \mathbf{R}^2$. Let $h : [0, 1] \to \mathbf{R}^2$ be a continuous mapping such that $h(0) = c_1$ and $h(1) = c_2$. Then the function $s \circ h : [0, 1] \to C(I)$ is continuous and verifies

$$s \circ h(0) = u_1, \quad s \circ h(1) = u_2,$$
$$s \circ h(c) \in S(h(c)) \subset S, \quad c \in [0, 1].$$

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