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# FUNCTION PSEUDOMETRIC VARIANTS OF THE CARISTI-KIRK THEOREM

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Dedicated to Professor Ioan A. Rus on the occasion of his 70th birthday

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**Abstract.** A functional extension is given for the fixed point result obtained by Kada, Suzuki and Takahashi [Math. Japonica, 44 (1996), 381-191]. This, among others, solves an open problem raised by Petrusel [St. Univ. "Babes-Bolyai" (Math.), 48 (2003), 115-123].
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