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FUNCTION PSEUDOMETRIC VARIANTS OF THE CARISTI-KIRK THEOREM

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Abstract. A functional extension is given for the fixed point result obtained by Kada, Suzuki and Takahashi [Math. Japonica, 44 (1996), 381-191]. This, among others, solves an open problem raised by Petrusel [St. Univ. "Babes-Bolyai" (Math.), 48 (2003), 115-123].
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1. Introduction

Let (M, d) be a complete metric space; and $x \vdash \varphi(x)$, some function from M to $R_+ := [0, \infty[$ with

$$\varphi \text{ is } d\text{-lsc over } M\text{: } \liminf_n \varphi(x_n) \geq \varphi(x), \text{ whenever } x_n \to x. \tag{1.1}$$

Let also $T : M \to M$ be a selfmap of M; and put

$$\text{Fix}(T) = \{x \in M; x = Tx\}, \quad \text{Per}(T) = \cup\{\text{Fix}(T^n); n \geq 1\};$$

each point of the former (latter) will be called *fixed* (*periodic*) under T. The following 1975 fixed point result in Caristi and Kirk [9] is basic for us.

Theorem CK. *Suppose that*

$$d(x, Tx) \leq \varphi(x) - \varphi(Tx),\ \forall x \in M \textit{ (T is } (d,\varphi)\textit{-contractive).} \tag{1.2}$$

Then, necessarily,

$$\mathrm{Fix}(T) = \mathrm{Per}(T) \neq \emptyset \quad \textit{(T is strongly fp-admissible);} \tag{1.3a}$$

hence, in particular,

$$\mathrm{Fix}(T) \neq \emptyset \quad \textit{(T is fp-admissible).} \tag{1.3b}$$

The original proof of this result is by transfinite induction; see also Wong [42]. (It works as well for highly specialized versions of Theorem CK; cf. Kirk and Saliga [23]). Note that, in terms of the associated (to φ) order

$$(x, y \in M) \;\; x \leq y \quad \text{iff } d(x, y) \leq \varphi(x) - \varphi(y) \tag{1.4}$$

the contractivity condition (1.2) reads

$$x \leq Tx, \text{ for all } x \in M \;\; \text{(i.e.: } T \text{ is } \textit{progressive}\text{).} \tag{1.5}$$

So, by the Bourbaki meta-theorem [6], this result is logically equivalent with the Zorn maximality principle subsumed to the order (1.4); i.e., with Ekeland's variational principle [12]. Hence, the sequential type argument used in its proof is also working in our framework; see also Pasicki [28]. A proof of Theorem CK involving the chains of the structure $(M, \leq)$ may be found in Turinici [38]; and its sequential translation has been developed in Dancs, Hegedus and Medvegyev [11]. Further aspects involving the general case may be found in Brunner [8] and Manka [25]; see also Taskovic [36], Valyi [41], Nemeth [26] and Isac [17].

Now, the Caristi-Kirk fixed point theorem found (especially via Ekeland's approach) some basic applications to control and optimization, generalized differential calculus, critical point theory and normal solvability; see the above references for details. So, it must be not surprising that, soon after its formulation, many extensions of Theorem CK were proposed. [These refer to its standard version related to (1.3b); and referred to as Theorem CK(st). But, only a few are concerned with the extended version of the same, related to (1.3a); and referred to as Theorem CK(ex)]. For example, in the 1982 paper by Ray and Walker [31], the following "functional" variant of Theorem CK(st) was obtained. Take some function $b : R_+ \to R_+$ with the *normality* properties

$$b \text{ is decreasing and } b(R_+) \subseteq R^0_+ :=]0, \infty[\tag{1.6}$$

$$B(\infty) = \infty \text{ (where } B(t) = \int_0^t b(\tau)d\tau,\ t \geq 0). \tag{1.7}$$

Theorem RW. *Suppose that, for some $a \in M$ one has*

$$b(d(a,x))d(x,Tx) \leq \varphi(x) - \varphi(Tx), \text{ for all } x \in M. \tag{1.8}$$

Then, T has at least one fixed point in M.

Clearly, Theorem 1 includes Theorem CK(st), to which it reduces when $b = 1$. The reciprocal inclusion also holds (cf. Park and Bae [27]). Summing up, Theorem RW is but a logical equivalent of Theorem CK(st). Nevertheless, for technical reasons, it is a preferred tool in many concrete circumstances; see the quoted paper for details. [This result was re-discovered in 1999 by Zhong, Zhu and Zhao [43]; but, no mention has been made about its equivalence with Theorem CK(st)].

Now, it is our aim in this exposition to show that this reduction scheme goes beyond the metrical setting. Precisely, (cf. Section 5) we shall establish that the "functional" versions of the fixed point results in Suzuki [34], Lin and Du [24] or Turinici [40] are logical equivalents of these. This is also true for the related statement in Kada, Suzuki and Takahashi [20]; note that the corresponding statement solves an open problem in Petrusel [29]. The basic tool for our developments is a fixed point result (in Section 3) obtained via transitive versions of the Brezis-Browder ordering principle [7] (exposed in Section 2). And the specific tool of these is the (already specified) concept of *normal* function (discussed in Section 4). Further aspects will be delineated elsewhere.

2. Transitive BB principles

Let M be some nonempty set. Take a *quasi-order* (i.e.: reflexive and transitive relation) $(\leq)$ over M; as well as a function $x \vdash \varphi(x)$ from M to R_+. Call the point $z \in M$, $(\leq, \varphi)$-*maximal* when

$$w \in M \text{ and } z \leq w \text{ imply } \varphi(z) = \varphi(w). \tag{2.1}$$

A basic result about the existence of such points is the 1976 Brezis-Browder ordering principle [7]:

Proposition 1. *Assume that*

$$(M,\leq) \text{ is sequentially inductive: each ascending sequence in } M \text{ has an upper bound (modulo } (\leq)) \tag{2.2}$$

$$\varphi \text{ is } (\leq)\text{-decreasing } (x \leq y \Longrightarrow \varphi(x) \geq \varphi(y)). \tag{2.3}$$

Then **a)** *for each* $u \in M$ *there exists a* $(\leq,\varphi)$*-maximal* $v \in M$ *with* $u \leq v$, **aa)** *if* $T : M \to M$ *is* $(\leq)$*-progressive (cf. (1.5)) we have (in addition)* $\varphi(v) = \varphi(Tv)$.

In particular, when (2.3) is taken in the stronger sense

$$\varphi \text{ is strictly } (\leq)\text{-decreasing } (x \leq y, x \neq y \Longrightarrow \varphi(x) > \varphi(y)) \tag{2.4}$$

the concept (2.1) means

$$w \in M, z \leq w \Longrightarrow z = w \quad [z \text{ is } (\leq)\text{-}maximal]; \tag{2.5}$$

and the Brezis-Browder ordering principle includes directly Caristi-Kirk's [9] (Theorem CK(ex)). Note that Codom$(\varphi) \subseteq R_+$ is not essential for the conclusion above; cf. Carja and Ursescu [10]. Further aspects may be found in Altman [1], Anisiu [2] and Turinici [39]; see also Kang and Park [21].

Now, Proposition 1 is applicable to all questions handled by Theorem CK. In addition, it may solve problems not attainable by the quoted statement (cf. Section 5). These, as a rule, require *transitive* variants of Proposition 1. To describe them, we need some conventions. Let (∇) be some *transitive* (over M) relation ($x\nabla y$ and $y\nabla z$ imply $x\nabla z$). The relation

$$(x, y \in M) \;\; x \leq y \text{ iff either } x = y \text{ or } x\nabla y \tag{2.6}$$

is a *quasi-order* on M; which, in addition, fulfills

$$[(x\nabla y, y \leq z) \text{ or } (x \leq y, y\nabla z)] \Longrightarrow x\nabla z. \tag{2.7}$$

Denote, for simplicity reasons

$$M(x,\nabla) = \{y \in M; x\nabla y\} \quad (\text{the } x\text{-}section \text{ of } (\nabla)),\ x \in M.$$

Further, take a function $\varphi : M \to R_+$. The (∇)-decreasing property for it is that of (2.3) (with (∇) in place of $(\leq)$). Note that, by (2.6) above,

$$\varphi \text{ is } (\nabla)\text{-decreasing} \Longleftrightarrow \varphi \text{ is } (\leq)\text{-decreasing}.$$

Further, call the point $z \in M$, (∇, φ)-*maximal*, provided

$$w \in M \text{ and } z\nabla w \text{ imply } \varphi(z) = \varphi(w). \tag{2.8}$$

For a non-trivial concept, we must take z as (∇)-*starting* (in the sense: $M(z, \nabla) \neq \emptyset$); for, otherwise, z is vacuously (∇, φ)-maximal. Note that such a requirement holds whenever z is (∇)-*reflexive* (i.e.: $z\nabla z$). Again by (2.6), the generic property holds

$$\text{(for each } z \in M) \ (\nabla, \varphi)\text{-maximal} \Longleftrightarrow (\leq, \varphi)\text{-maximal.}$$

As a consequence, maximality results involving our transitive relation (∇) are deductible from the Brezis-Browder principle (written for its associated quasi-order $(\leq)$). The key moment of this approach is that of (2.2) being assured. It would be useful to have this condition expressed in terms of (∇). Call the sequence (x_n), *ascending* (modulo (∇)) when

$$x_n \nabla x_{n+1}, \forall n \text{ (or, equivalently: } x_n \nabla x_m \text{ if } n < m).$$

Note the generic (sequential) relation

$$\text{ascending (modulo } (\nabla)) \Longrightarrow \text{ascending (modulo } (\leq)).$$

The reciprocal is not in general true. For example, the constant sequence $(x_n = a; n \in N)$ is ascending (modulo $(\leq)$); but not ascending (modulo (∇)), whenever $a\nabla a$ is false. Further, given the sequence (x_n) in M, let us say that $u \in M$ is an *upper bound* (modulo (∇)) of it provided

$$x_n \nabla u, \text{ for all } n \text{ (written as: } (x_n)\nabla u).$$

If u is generic in this convention, we say that (x_n) is *bounded above* (modulo (∇)). As before, the relation below is clear

$$\text{bounded above (modulo } (\nabla)) \Longrightarrow \text{bounded above (modulo } (\leq)).$$

(The converse is not in general valid). Finally, let the concept of *sequential inductivity* for (M, ∇) be that of (2.2) [with (∇) in place of $(\leq)$].

Lemma 1. *Under the specified setting,*

$$(M, \nabla) \text{ sequentially inductive} \Longleftrightarrow (M, \leq) \text{ sequentially inductive.} \tag{2.9}$$

Proof. The right to left implication is clear, via (2.7); so, it remains the opposite one. Assume (M, ∇) is sequentially inductive; and let (x_n) be an ascending (modulo $(\leq)$) sequence in M:

$$x_n \leq x_{n+1}, \forall n \text{ (or, equivalently: } x_n \leq x_m, \quad \text{whenever } n \leq m).$$

If this sequence is stationary beyond a certain rank

$$\exists k \text{ such that: } \forall n > k \text{ one has } x_n = x_k$$

we are done; because $(x_n) \leq u (= x_k)$. Otherwise,

$$\forall p, \exists q > p, \text{ such that } x_p \neq x_q \quad (\text{hence } x_p \nabla x_q).$$

Consequently, a subsequence $(y_n = x_{i(n)})$ of (x_n) may be constructed with the property of being ascending (modulo (∇)); wherefrom $(y_n)\nabla t$ for some $t \in M$. But then, $(x_n) \leq t$; hence the conclusion. □

We are now in position to give an appropriate answer to the posed question. Call the selfmap $T : M \to M$, (∇)-*progressive* if (1.5) holds with (∇) in place of $(\leq)$.

Proposition 2. *Assume that* (M, ∇) *is sequentially inductive and* φ *is* (∇)*-decreasing. Then*

b) *for each* (∇)*-starting* $u \in M$ *there exists a* (∇, φ)*-maximal* $v \in M$ *with* $u \nabla v$

bb) *if* $T : M \to M$ *is* (∇)*-progressive we have (in addition)* $\varphi(v) = \varphi(Tv)$.

Proof. Let $(\leq)$ stand for the quasi-order (2.6). By the remarks above (and Lemma 1), Proposition 1 is applicable to $(M, \leq)$ and φ.

b) Let $u \in M$ be (∇)-starting. For the arbitrary fixed $u_1 \in M(u, \nabla)$ there exists $v \in M$ with

$$u_1 \leq v (\text{ i.e.: either } u_1 = v \text{ or } u_1 \nabla v); \text{ and } v \text{ is } (\leq, \varphi)\text{-maximal.}$$

This, along with (2.7), yields $u \nabla v$; and proves the first part.

bb) Each point of M is (∇)-starting; so, by the previous argument, we are done. □

Clearly, the Brezis-Browder principle [7] (Proposition 1) follows from Proposition 2. The reciprocal inclusion also holds, by the argument above; hence these results are logically equivalent. Nevertheless, a direct use of Proposition 2 is more profitable in all concrete situations involving explicitly (∇); cf. Section 5.

An interesting completion of this statement is to be given under the remarks following Proposition 1. Precisely, after the model of (2.5), we may introduce the concept of (∇)-*maximal* element (with (∇) in place of $(\leq)$); this is a stronger version of the concept (2.8). As before, it is effective only if z is (∇)-starting; for, otherwise, z is vacuously (∇)-maximal. To get a result involving such points, we need an extra condition upon our data:

$$(\nabla) \text{ is } \varphi\text{-}\textit{sufficient}\text{: } z\nabla x\nabla y,\ \varphi(z)=\varphi(x)=\varphi(y) \Longrightarrow x=y. \tag{2.10}$$

Proposition 3. *Suppose that conditions of Proposition 2 hold, as well as (2.10). Then*

c) *for each* (∇)*-starting* $u \in M$ *there exists a* (∇)*-maximal* $w \in M$ *with* $u\nabla w$

cc) *if* $T : M \to M$ *is* (∇)*-progressive, we have (in addition)* $w \in \mathrm{Fix}(T)$

ccc) *each* (∇)*-progressive selfmap is strongly fp-admissible.*

Proof. **c)** By Proposition 2, there must be some (∇,φ)-maximal $v \in M$ with $u\nabla v$. If v is (∇)-maximal, we are done (with $w = v$); so, it remains the alternative of v fulfilling the opposite property:

$$v\nabla w \text{ (hence } \varphi(v)=\varphi(w)\text{), for some } w \in M \setminus \{v\}.$$

In this case, w is our desired element. Assume not: $w\nabla y$, for some $y \in M$, $y \neq w$. By the preceding relation we get $v\nabla y$ (hence $\varphi(v)=\varphi(y)$). Summing up, $v\nabla w\nabla y$ and $\varphi(v)=\varphi(w)=\varphi(y)$; wherefrom (by (2.10)) $w=y$, contradiction. Hence, the claim **c)** is proved.

cc) Each point of M is (∇)-starting; wherefrom, all is clear.

ccc) By the previous argument, $\mathrm{Fix}(T) \neq \emptyset$; so, it remains to show that $\mathrm{Per}(T) \subseteq \mathrm{Fix}(T)$. Let $z \in \mathrm{Per}(T)$ be arbitrary fixed; so (by definition) $z = T^k(z)$, for some $k \geq 1$. By the (∇)-progressivity condition (imposed upon T),

$$z\nabla Tz\nabla ...\nabla T^k(z) = z; \text{ wherefrom } z\nabla Tz\nabla z.$$

This (by the (∇)-decreasing property of φ, yields $\varphi(z)=\varphi(Tz)$; so that (by simply taking (2.10) into account) $z = Tz$ (hence, z is fixed under T). The proof is thereby complete. □

The obtained statement is nothing but a "transitive" form of the Bourbaki maximality principle [6] for these structures; cf. Hazen and Morin [16]. It may be also viewed as a counterpart of the "reflexive" type version of Proposition 1 obtained in Bae, Cho and Yeom [3]. Further aspects were delineated in Gajek and Zagrodny [14]; see also Sonntag and Zalinescu [33].

3. Main results

Let M be some nonempty set. By a *pseudometric* over it we shall mean any map $e : M \times M \to R_+$. Suppose that we fixed such an object; which, in addition, is *triangular*

$$e(x,z) \le e(x,y) + e(y,z), \quad \text{for all } x,y,z \in M.$$

Let also $\varphi : M \to R_+$ be some function For an easy reference, we shall formulate the basic regularity condition involving our data. This will necessitate some conventions and auxiliary facts. Call the sequence (x_n) in M, *strongly* e*-asymptotic* when

$$\text{the series } \sum_n e(x_n, x_{n+1}) \text{ converges (in } R).$$

Further, let the e-*Cauchy* property of this object be the usual one

$$\forall \delta > 0, \exists n(\delta), \text{ such that } n(\delta) \le p < q \Longrightarrow e(x_p, x_q) \le \delta.$$

The generic relation below is clear (by the triangular property of e)

$$\text{(for each sequence) strongly } e\text{-asymptotic} \Longrightarrow e\text{-Cauchy}; \tag{3.1}$$

but the converse is not in general true. Nevertheless, in many conditions involving *all* such objects, this is retainable. A concrete example is to be constructed under the lines below. Let us introduce an e-*convergence* structure over M by

$$x_n \xrightarrow{e} x \text{ iff } \quad e(x_n, x) \to 0 \quad \text{as } n \to \infty.$$

We consider the regularity condition

$$\begin{array}{l} (e,\varphi) \text{ is weakly descending complete: for each strongly} \\ e\text{-asymptotic sequence } (x_n) \text{ in } M \text{ with } (\varphi(x_n)) \text{ descending} \\ \text{there exists } x \in M \text{ with } x_n \xrightarrow{e} x \text{ and } \lim_n \varphi(x_n) \ge \varphi(x). \end{array} \tag{3.2}$$

By (3.1) above, it includes its (stronger) counterpart

$$\begin{aligned}&(e,\varphi) \text{ is descending complete: for each } e\text{-Cauchy sequence}\\&(x_n) \text{ in } M \text{ with } (\varphi(x_n)) \text{ descending there exists } x\in M\\&\text{with the properties } x_n \xrightarrow{e} x \text{ and } \lim_n \varphi(x_n)\geq\varphi(x).\end{aligned} \tag{3.3}$$

A remarkable fact to be added is that the reciprocal inclusion also holds:

Lemma 2. *Under the specified conditions,*

$$(3.2) \Longrightarrow (3.3); \quad \text{hence } (3.2) \Longleftrightarrow (3.3).$$

Proof. Assume that (3.2) holds; and let (x_n) be an e-Cauchy sequence in M with $(\varphi(x_n))$ descending. By the very definition of this property, there must be a subsequence $(y_n = x_{i(n)})$ of (x_n) with

$$(y_n) \text{ is strongly } e\text{-asymptotic; and } (\varphi(y_n)) \text{ is descending.}$$

This, along with (3.2), yields an element $y \in M$ fulfilling $y_n \xrightarrow{e} y$ and $\lim_n \varphi(y_n) \geq \varphi(y)$. It is now clear (by the choice of (x_n)) that the point y has all the desired in (3.3) properties. □

Now, call $v \in M$, Brezis-Browder (in short: BB) - *variational* (modulo (e,φ)) provided

$$x\in M, e(v,x)\leq\varphi(v)-\varphi(x) \Longrightarrow \varphi(v)=\varphi(x) \ \ (\text{hence } e(v,x)=0). \tag{3.4}$$

Some basic properties of this concept are collected in

Lemma 3. *Suppose that $v \in M$ is BB-variational (modulo (e,φ)). Then, the following are true*

$$e(v,x)\geq\varphi(v)-\varphi(x), \quad \textit{for all } x\in M \tag{3.5}$$

$$e(v,x)>\varphi(v)-\varphi(x), \quad \textit{for each } x\in M \textit{ with } e(v,x)>0. \tag{3.6}$$

Proof. The latter part is clear, by definition; so, it remains to establish the former one. Assume this would be false; i.e.,

$$e(v,x)<\varphi(v)-\varphi(x), \text{ for some } x\in M. \tag{3.7}$$

This, along with (3.4), yields $\varphi(v)=\varphi(x)$; wherefrom $0 \leq e(v,x) < 0$, contradiction. So, (3.7) cannot hold; and the conclusion follows. □

Finally, let $(\nabla = \nabla_\varphi)$ stand for the transitive relation (over M)

$$(x,y\in M)\ x\nabla y \text{ iff } e(x,y)\leq\varphi(x)-\varphi(y). \tag{3.8}$$

Remember that $u \in M$ is called (∇)-starting if $M(u, \nabla) \neq \emptyset$; i.e.,

$$e(u, x) \leq \varphi(u) - \varphi(x), \quad \text{for at least one } x \in M.$$

This will be also referred to as u is *starting* (modulo (e, φ)). Note that the written property holds under

$$u \text{ is } (\nabla)\text{-}\textit{reflexive}\text{:} \quad u \nabla u;$$

which, in our terms, amounts to $e(u, u) = 0$; and is referred to as: u is *reflexive* (modulo e). Further, call the selfmap $T : M \to M$, (e, φ)-*contractive* provided (1.2) holds (with e in place of d). Clearly, this is equivalent with T being (∇)-progressive (see above).

We are now in position to state a pseudometric variational principle useful in the sequel.

Theorem 1. *Let the general conditions upon (e, φ) be accepted; as well as (3.3) (or equivalently, (3.2)). Then*

i) *for each starting (modulo (e, φ)) $u \in M$ there exists a BB-variational (modulo (e, φ)) $v = v(u) \in M$ with*

$$e(u, v) \leq \varphi(u) - \varphi(v) \text{ (hence } \varphi(u) \geq \varphi(v)). \tag{3.9}$$

ii) *if $T : M \to M$ is (e, φ)-contractive we have (in addition)*

$$\varphi(v) = \varphi(Tv), \quad e(v, Tv) = 0. \tag{3.10}$$

Proof. We claim that (∇, φ) fulfills conditions of Proposition 2 on M. In fact, φ is (∇)-decreasing ; so, it remains to show that (M, ∇) is sequentially inductive. Let (x_n) be an ascending (modulo (∇)) sequence in M:

$$e(x_n, x_m) \leq \varphi(x_n) - \varphi(x_m), \text{ whenever } n < m. \tag{3.11}$$

The sequence $(\varphi(x_n))$ is descending in R_+; hence a Cauchy one. In addition, by (3.11), (x_n) is an e-Cauchy sequence in M. Putting these together, it follows (via (3.3)) that there must be some $y \in M$ with

$$x_n \xrightarrow{e} y \text{ and } \lim_n \varphi(x_n) \geq \varphi(y). \tag{3.12}$$

Fix some rank n. By (3.11) and the triangular property of e, one has

$$e(x_n, y) \leq e(x_n, x_m) + e(x_m, y) \leq \varphi(x_n) - \varphi(x_m) + e(x_m, y), \forall m > n.$$

This, along with (3.12), yields by a limit process (relative to m)

$$e(x_n, y) \leq \varphi(x_n) - \lim_m \varphi(x_m) \leq \varphi(x_n) - \varphi(y) \quad \text{(i.e.: } x_n \nabla y).$$

As n was arbitrarily chosen, one deduces that $(x_n)\nabla y$; and this proves our claim. By Proposition 2 it then follows that conclusions **b)** and **bb)** given there must hold with respect to (∇, φ)-maximal points $v \in M$. It suffices now remarking that

$$v \text{ is } (\nabla, \varphi)\text{-maximal (on } M) \Longleftrightarrow v \text{ is BB-variational (modulo } (e, \varphi))$$

to get all the conclusions above. □

Now, the regularity condition (3.2) holds under

$$\begin{aligned}&(e, \varphi) \text{ is weakly complete:}\\ &\text{for each strongly } e\text{-asymptotic sequence } (x_n) \text{ in } M\\ &\text{there exists } x \in M \text{ with } x_n \xrightarrow{e} x \text{ and } \liminf_n \varphi(x_n) \geq \varphi(x).\end{aligned} \tag{3.13}$$

This, in the particular case when

$$e \text{ is (in addition) } \textit{reflexive } (e(x, x) = 0, \forall x \in M),$$

tells us that the variational portion of Theorem 1 includes the ordering principle in Tataru [37]. Moreover, the fixed point portion of the same includes a related statement in Turinici [40] when e is sufficient ($e(x, y) = 0 \Longrightarrow x = y$). The question of the converse inclusions being also true remains open; we conjecture that the answer is positive.

Let us now return to our initial setting. An interesting completion of Theorem 1 is to be done under the lines of Section 2. Precisely, let us say that v is Ekeland (in short: E) - *variational* (modulo (e, φ)) provided

$$x \in M, \quad e(v, x) \leq \varphi(v) - \varphi(x) \quad \Longrightarrow v = x. \tag{3.14}$$

This concept is stronger than the one introduced via (3.4). To get a corresponding form of Theorem 1 involving such points we have to impose (in addition to (3.2)/(3.3)) a regularity condition like

$$e \text{ is } \textit{transitively sufficient } \; (e(z, x) = e(z, y) = 0 \Longrightarrow x = y). \tag{3.15}$$

Theorem 2. *Let the specified conditions be in force. Then*

j) *for each starting (modulo (e, φ)) $u \in M$ there exists an E-variational (modulo (e, φ)) $w = w(u) \in M$ with the property (3.9) (relative to w)*

jj) *if $T : M \to M$ is (e, φ)-contractive, we have (in addition)*

$$w = Tw \text{ (hence, } w \text{ is fixed under } T\text{)} \quad \text{and } e(w, w) = 0 \tag{3.16}$$

jjj) *each (e, φ)-contractive selfmap is strongly fp-admissible.*

Proof. As already shown, conditions of Proposition 2 hold over M for the couple (∇, φ). On the other hand, the special regularity condition (2.10) also holds, via (3.15). Summing up, Proposition 3 is applicable to (M, ∇) and φ; wherefrom, the conclusions **c)**, **cc)** and **ccc)** stated there must hold with respect to (∇)-maximal points $w \in M$. It suffices now remarking that

$$w \text{ is } (\nabla)\text{-maximal} \Longleftrightarrow w \text{ is E-variational (modulo } (e, \varphi))$$

to derive the written conclusions. □

As before, the regularity condition (3.3) holds under

$$\begin{array}{l}(e, \varphi) \text{ is complete: for each } e\text{-Cauchy sequence } (x_n) \text{ in } M \\ \text{there exists } x \in M \text{ with } x_n \stackrel{e}{\longrightarrow} x \text{ and } \liminf_n \varphi(x_n) \geq \varphi(x).\end{array} \tag{3.17}$$

The corresponding variant of Theorem 2 may be viewed as an "absolute" version of the variational (and its subsequent fixed point) principle in Kada, Suzuki and Takahashi [20] (cf. Section 5). In particular, (3.17) holds under

$$e \text{ is complete and } \varphi \text{ is } e\text{-lsc (cf. (1.1)).} \tag{3.18}$$

Note that, in such a case, the fixed point portion of Theorem 2 includes directly the Caristi-Kirk fixed point theorem [9] (Theorem CK(ex)). The reciprocal of this also holds (cf. Jachymski [19]); but the resulting metric structure is strongly connected with the initial selfmap T. Further aspects of structural nature may be found in Isac [17] and Nemeth [26]; see also Hamel [15, Ch 4] and Khanh [22].

4. Normal functions

(A) Let $b : R_+ \to R_+$ be some *normal* function (cf. (1.6)+(1.7)). In particular, it is Riemann integrable on each compact interval of R_+ and

$$\int_p^q b(\xi)d\xi = (q - p)\int_0^1 b(p + \tau(q - p))d\tau, \quad 0 \leq p < q < \infty. \tag{4.1}$$

Some basic facts involving the couple (b, B) (where $B : R_+ \to R_+$ is that of (1.7)) are specified in

Lemma 4. *The following are true*

$$\begin{aligned}&B \text{ is a continuous order isomorphism of } R_+;\\ &\text{hence, so is } B^{-1}(=\text{its functional inverse})\end{aligned} \tag{4.2}$$

$$b(s) \leq (B(s) - B(t))/(s-t) \leq b(t), \ \forall t, s \in R_+, t < s \tag{4.3}$$

$$\begin{aligned}&B \text{ is almost concave:}\\ &t \vdash [B(t+s) - B(t)] \text{ is decreasing on } R_+, \forall s \in R_+\end{aligned} \tag{4.4}$$

$$\begin{aligned}&B \text{ is concave: } B(t + \lambda(s-t)) \geq (1-\lambda)B(t) + \lambda B(s),\\ &\text{for all } t, s \in R_+ \text{ with } t < s \text{ and all } \lambda \in [0,1]\end{aligned} \tag{4.5}$$

$$B \text{ is sub-additive (hence } B^{-1} \text{ is super-additive).} \tag{4.6}$$

The proof is immediate, by (4.1) above; so, we do not give details. Note that the properties (4.4) and (4.5) are equivalent to each other, under (4.2). This follows at once from the (non-differential) mean value theorem in Bantas and Turinici [4].

(B) Now, let M be some nonempty set; and $e : M \times M \to R_+$, a triangular pseudometric over it (cf. Section 3). Further, let $\Gamma : M \to R_+$ be chosen according to

$$\Gamma \text{ is almost } e\text{-nonexpansive } (\Gamma(x) - \Gamma(y) + e(x,y) \geq 0, \forall x, y \in M) \tag{4.7}$$

$$\Gamma \text{ is } e\text{-continuous } (\Gamma(x_n) \to \Gamma(x), \text{ whenever } x_n \xrightarrow{e} x). \tag{4.8}$$

Given the function $\varphi : M \to R_+$, let us attach it the function $\psi = \psi(B, \Gamma; \varphi)$ from M to R_+ according to

$$\psi(x) = B^{-1}[B(\Gamma(x)) + (\varphi(x) - \varphi_*)] - \Gamma(x), \quad x \in M; \tag{4.9}$$

or equivalently (in the implicit way)

$$\varphi(x) = \varphi_* + [B(\Gamma(x) + \psi(x)) - B(\Gamma(x))], \quad x \in M. \tag{4.10}$$

(Here, as usually, $\varphi_* = \inf[\varphi(M)]$). An essential question to be solved is that of transferring the regularity properties (3.2)/(3.3) from φ to ψ (or vice versa). Unfortunately, this is not (in general) possible; because the descending sequence $(\varphi(x_n))$ is not in general transformed into a descending sequence $(\psi(x_n))$. This forces us working with the "non-descending" counterparts of these properties; i.e., with (3.13)/(3.17). The following answer to this question is available.

Lemma 5. *Let the sequence* (x_n) *in* M *and the point* $x \in M$ *be such that* $x_n \xrightarrow{e} x$*. Then*

$$\varphi(x) \leq \liminf_n \varphi(x_n) \Longleftrightarrow \psi(x) \leq \liminf_n \psi(x_n). \tag{4.11}$$

As a consequence, the completeness properties above are retainable in passing from φ *to* ψ *(and vice versa); i.e.,*

$$(e, \varphi) \text{ is weakly complete} \Longleftrightarrow (e, \psi) \text{ is weakly complete} \tag{4.12}$$

$$(e, \varphi) \text{ is complete} \Longleftrightarrow (e, \psi) \text{ is complete.} \tag{4.13}$$

Proof. Assume that the left part of (4.11) holds; but the right part of the same would be false:

$$0 \leq \liminf_n \psi(x_n) < \beta < \psi(x), \quad \text{for some } \beta.$$

By the definition of lim inf, there must be a sequence (y_n) of (x_n) with $y_n \xrightarrow{e} x$ and $0 \leq \psi(y_n) < \beta$, for all n. This, in turn, yields a subsequence (z_n) of (y_n) (hence of (x_n)) with

$$z_n \xrightarrow{e} x \text{ and } \lambda := \lim_n \psi(z_n) \text{ exists (hence } 0 \leq \lambda \leq \beta < \psi(x)).$$

So, by the implicit formula (4.10) and the continuity of Γ (cf. (4.8))

$$\lim_n \varphi(z_n) = \varphi_* + [B(\Gamma(x) + \lambda) - B(\Gamma(x))] <$$
$$\varphi_* + [B(\Gamma(x) + \psi(x)) - B(\Gamma(x))] = \varphi(x);$$

in contradiction with the initial choice of our data. Hence, the left to right implication of (4.11) is retainable. The right to left implication is deductible in a similar way; wherefrom, (4.11) follows. The remaining part is clear, by the definition of the involved concepts. □

An interesting property of the couple (φ, ψ) to be discussed here is of variational nature. Precisely, we have:

Lemma 6. *Under these conventions,*

$$b(\Gamma(x))e(x, y) \lhd \varphi(x) - \varphi(y) \Longrightarrow e(x, y) \lhd \psi(x) - \psi(y). \tag{4.14}$$

(Here, $(\lhd)$ *is either of the relations* $\{\leq, <\}$*).*

Proof. Let the points $x, y \in M$ be as in the premise of this implication. By (4.3) and the implicit formula (4.10), this yields

$$B(\Gamma(x)+e(x,y))-B(\Gamma(x))\triangleleft$$
$$[B(\Gamma(x)+\psi(x))-B(\Gamma(x))]-[B(\Gamma(y)+\psi(y))-B(\Gamma(y))];$$

or equivalently (by a simple re-arrangement)

$$B(\Gamma(x)+e(x,y))+[B(\Gamma(y)+\psi(y))-B(\Gamma(y))]\triangleleft B(\Gamma(x)+\psi(x)).$$

On the other hand, the almost e-nonexpansivity condition (4.7) gives $\Gamma(x)+e(x,y)\geq\Gamma(y)$; so, by (4.4) above

$$B(\Gamma(x)+e(x,y)+\psi(y))-B(\Gamma(x)+e(x,y))\leq B(\Gamma(y)+\psi(y))-B(\Gamma(y)).$$

A simple combination with the previous relation yields

$$B(\Gamma(x)+e(x,y)+\psi(y))\triangleleft B(\Gamma(x)+\psi(x)).$$

It suffices now taking (4.2) into account to get the desired conclusion. □

A qualitative type version of this may be given as follows. Call the point $u\in M$, *starting* (modulo $(b,\Gamma;e,\varphi)$) when

$$b(\Gamma(u))e(u,x)\leq\varphi(u)-\varphi(x),\text{ for some } x\in M. \tag{4.15}$$

Then, the conclusion of Lemma 6 gives a relation (useful in the sequel)

$$\text{starting (modulo } (b,\Gamma;e,\varphi)) \Longrightarrow \text{starting (modulo } (e,\psi)). \tag{4.16}$$

In particular, when $e: M\times M\to R_+$ is symmetric over M, the regularity conditions (4.7)+(4.8) may be written as

$$|\Gamma(x)-\Gamma(y)|\leq e(x,y),\ \forall x,y\in M \text{ (non-expansiveness)}. \tag{4.17}$$

And then, the choice

$$\Gamma(x)=e(a,x),\quad x\in M,\qquad\text{for some } a\in M, \tag{4.18}$$

is in agreement with it. Note that, in such a case Lemma 6 includes the statement in Park and Bae [27]. Further aspects may be found in Suzuki [34]; see also Turinici [40].

5. Relative CK theorems

We are now in position to get an appropriate "relative" answer to the questions in Section 1.

(A) Let (M,d) be a complete metric space; and $e : M \times M \to R_+$, some triangular pseudometric over M. We shall say that this object is a KST-*metric* (modulo d) provided

$$\text{each } e\text{-Cauchy sequence is } d\text{-Cauchy (hence } d\text{-convergent)} \tag{5.1}$$

$$[(y_n) \text{ is } e\text{-Cauchy}, \; y_n \xrightarrow{d} y] \Longrightarrow \liminf_n e(x,y_n) \geq e(x,y), \forall x \in M. \tag{5.2}$$

If, in addition, e is transitively sufficient; i.e.,

$$e(z,x) = e(z,y) = 0 \Longrightarrow x = y \quad \text{(cf. (3.15))}$$

then e will be referred to as a *strong* KST-metric (modulo d). Further, take some function $\varphi : M \to R_+$. The following auxiliary fact will be needed.

Lemma 7. *Assume that e is some KST-metric (modulo d) over M and φ is d-lsc (cf. (1.1)). Then, (e,φ) is descending complete (in the sense of (3.3)); as well as complete (cf. (3.17)).*

Proof. Let (x_n) be some e-Cauchy sequence in M with $(\varphi(x_n))$ descending. From (5.1), (x_n) is d-Cauchy; so, by completeness,

$$x_n \xrightarrow{d} y \;\text{ as } n \to \infty, \quad \text{for some } y \in M.$$

We claim that this is our desired point for (3.3). In fact, let $\gamma > 0$ be arbitrary fixed. By the choice of (x_n), there exists $k = k(\gamma)$ so that

$$e(x_p,x_m) \leq \gamma, \quad \text{for each } p \geq k \text{ and each } m > p.$$

Passing to $\liminf$ upon m one gets (via (5.2) and the relation above)

$$e(x_p,y) \leq \gamma, \quad \text{for each} \quad p \geq k(= k(\gamma));$$

and since $\gamma > 0$ was arbitrarily chosen, $x_n \xrightarrow{e} y$. On the other hand,

$$\lim_n \varphi(x_n) \geq \varphi(y) \;\; \text{(if one takes (1.1) into account).}$$

Hence the conclusion. □

Further, let $T : M \to M$ be a self-map. By simply combining these facts with Theorem 2, one derives the following fixed point statement.

Theorem 3. *Let e be some strong KST-metric (modulo d), φ be d-lsc and T be (e,φ)-contractive (i.e., (1.2) holds with e in place of d). Then*

i) *for each starting (modulo (e,φ)) $u \in M$ there exists an E-variational (modulo (e,φ)) $w = w(u) \in M$ with the properties*

$$e(u,w) \le \varphi(u) - \varphi(w),\ w = Tw,\ e(w,w) = 0. \tag{5.3}$$

ii) *T is strongly fp-admissible.*

A functional version of this result is now available by the developments in Section 4. Precisely, take some normal function $b : R_+ \to R_+$ (cf. (1.6)+(1.7)); and $\Gamma : M \to R_+$ be chosen according to (4.7)+(4.8). Let $\psi = \psi(B;\Gamma;\varphi)$ stand for the associated (to φ) function (from M to R_+) introduced as in (4.9)/(4.10).

Theorem 4. *Let the couple (e,φ) be taken as in Theorem 3; and $T : M \to M$ be $(b,\Gamma;e,\varphi)$-contractive*

$$b(\Gamma(x))d(x,Tx) \le \varphi(x) - \varphi(Tx),\ \textit{for all}\ x \in M. \tag{5.4}$$

Then

j) *for each starting (modulo $(b,\Gamma;e,\varphi)$) $u \in M$ there exists an E-variational (modulo (e,ψ)) $w = w(u) \in M$ with the properties*

$$e(u,w) \le \psi(u) - \psi(w),\ w = Tw,\ e(w,w) = 0. \tag{5.5}$$

jj) *T is strongly fp-admissible.*

Proof. By Lemma 7, (e,φ) is complete over M; hence (cf. Lemma 5) so is (e,ψ) (over the same). Moreover, as $u \in M$ is starting (modulo $(b,\Gamma;e,\varphi)$) over M, we have

$$b(\Gamma(u))e(u,x) \le \varphi(u) - \varphi(x),\ \text{for some } x \in M;$$

and this, by Lemma 6, yields

$$e(u,x) \le \psi(u) - \psi(x)\quad \text{(for the same } x\text{)};$$

wherefrom u is starting (modulo (e,ψ)) over M. On the other hand, (5.4) gives us (via Lemma 6) that T is (e,ψ)-contractive. Summing up, Theorem 2 is applicable to (M,e), ψ and T; wherefrom, the proof is complete. □

(B) In the following, we shall give some particular cases of our developments, with a methodological value.

B1) Let $e : M \times M \to R_+$ be some triangular pseudometric over M. According to Kada, Suzuki and Takahashi [20], we say that it is a *w-distance* (modulo d) when

$$\begin{aligned} &e \text{ is strongly } d\text{-sufficient: for each } \varepsilon > 0, \text{ there exists} \\ &\delta > 0 \text{ such that: } e(z,x), e(z,y) \le \delta \Longrightarrow d(x,y) \le \varepsilon \end{aligned} \tag{5.6}$$

$$y \vdash e(x,y) \text{ is } d\text{-lsc on } M \text{ (see above)}, \forall x \in M. \tag{5.7}$$

Clearly, any such object is a strong KST-metric (modulo d). Hence the related fixed point statement in the quoted paper is a particular case of Theorem 3 above; in addition, this shows that any recursion to the nonconvex minimization theorem in Takahashi [35] is avoidable. The functional version of it (i.e., the variant of Theorem 4 involving w-distances) seems to be new; it solves an open problem raised by Petrusel [29].

B2) Let $G : R_+ \to R_+$ be a function with the properties

$$G \text{ is continuous increasing and } G^{-1}(0) = \{0\} \tag{5.8}$$

$$G \text{ is subadditive } (G(t+s) \le G(t) + G(s),\ \forall t, s \in R_+); \tag{5.9}$$

it will be referred to as a FL-function. Clearly, $G(\infty) > 0$, in view of (5.8) (the last part). We also note the useful property

$$\text{for each } \varepsilon > 0 \text{ there exists } \delta > 0 \text{ such that } G(\tau) < \delta \Longrightarrow \tau < \varepsilon. \tag{5.10}$$

As a consequence of this, the (standard) metric over M

$$e(x,y) = G(d(x,y)), \quad x, y \in M$$

has all the properties of a strong KST-metric (modulo d). The corresponding version of Theorem 3 (involving FL-functions) is just the statement in Feng and Liu [13]; which, in turn, rephrases the one in Jachymski [18]. (See the quoted paper for details). Further aspects of vectorial nature were discussed in Rozoveanu [32].

B3) Let again $e : M \times M \to R_+$ be a triangular pseudometric over M. According to Suzuki [34], we say that it is a *τ-distance* (modulo d) over M when there exists a function $\eta = \eta(e)$ from $M \times R_+$ to R_+ with the properties

$$\begin{aligned} &t \vdash \eta(x,t) \text{ is increasing on } R_+ \text{ and} \\ &\lim_{t \to 0} \eta(x,t) = 0 = \eta(x,0), \text{ for all } x \in M \end{aligned} \tag{5.11}$$

$$\begin{aligned}&\lim_n \sup_{m\geq n} \eta(z_n, e(z_n, y_m)) = 0 \text{ and } y_n \xrightarrow{d} y\\ &\text{imply } [\liminf_n e(x, y_n) \geq e(x, y), \text{ for each } x \in M]\end{aligned} \tag{5.12}$$

$$\begin{aligned}&\lim_n \sup_{m\geq n} e(x_n, y_m) = 0 \text{ and } \lim_n \eta(x_n, t_n) = 0\\ &\text{imply } \lim_n \eta(y_n, t_n) = 0\end{aligned} \tag{5.13}$$

$$\begin{aligned}&\lim_n \eta(z_n, e(z_n, x_n)) = 0 \text{ and } \lim_n \eta(z_n, e(z_n, y_n)) = 0\\ &\text{imply } \lim_n d(x_n, y_n) = 0.\end{aligned} \tag{5.14}$$

Clearly, any w-distance (modulo d) is a τ-distance (modulo d); it will suffice noting that, in such a case, (5.11)-(5.14) hold with $\eta(x,t) = t, x \in M, t \in R_+$. On the other hand, as we already remarked, any w-distance (modulo d) is a strong KST-metric (modulo d). So, it is natural asking of what can be said about the relationships between these enlargements of our initial concept. The answer to this is given in

Proposition 4. *Let the specified conventions be in use. Then, each τ-distance (modulo d) is necessarily a strong KST-metric (modulo d). So (combining with a previous claim) we have the generic inclusions (over transitive pseudometrics)*

$$w\text{-distance} \Longrightarrow \tau\text{-distance} \Longrightarrow \text{strong KST-metric (modulo } d).$$

Proof. Let $e : M \times M \to R_+$ be a τ-distance (modulo d); and $\eta = \eta(e)$ stand for some associated map fulfilling (5.11)-(5.14).

i) Call the sequence (x_n) in M, (η, e)-*Cauchy* provided

$$\lim_n \sup_{m\geq n} \eta(z_n, e(z_n, x_m)) = 0, \quad \text{for some sequence } (z_n) \subseteq M.$$

By [34, Lemma 3] we have the (generic) inclusion

$$[\text{for each sequence}]\ \ e\text{-Cauchy} \Longrightarrow (\eta, e)\text{-Cauchy}. \tag{5.15}$$

Adding the fact that (5.12) may be written as

$$\begin{aligned}&(y_n) \text{ is } (\eta, e)\text{-Cauchy and } y_n \xrightarrow{d} y \text{ imply}\\ &\liminf_n e(x, y_n) \geq e(x, y), \forall x \in M,\end{aligned}$$

proves (5.2). On the other hand, by [34, Lemma 1] the (generic) inclusion is (in addition) true

$$\text{[for each sequence]} \quad (\eta, e)\text{-Cauchy} \Longrightarrow d\text{-Cauchy}; \tag{5.16}$$

and this, coupled with a previous relation, yields (5.1); wherefrom e is a KST-metric (modulo d).

ii) Let $x, y, z \in M$ be such that $e(z, x) = e(z, y) = 0$. By (5.11), we have $\eta(z, e(z, x)) = \eta(z, e(z, y)) = 0$; and this, added to (5.14), gives $d(x, y) = 0$ (hence $x = y$); so, (3.15) holds too. □

As a consequence of this, the fixed point statement (involving τ-distances) obtained by the quoted author is deductible from Theorem 3 above. Its functional version (i.e., the variant of Theorem 4 involving τ-distances) seems to be new. Note that the proposed proofs are still depending on Suzuki's reasoning concerning (5.15)+(5.16). It would be interesting to have alternate proofs of these, so as to avoid Proposition 4 above; further aspects will be discussed elsewhere.

B4) Let $e : M \times M \to R_+$ be a triangular pseudometric over M. According to Lin and Du [24] we say that it is a *τ-function* (modulo d) provided (3.15) holds and

$$\begin{aligned} &x \in M,\ y_n \to y \text{ and } e(x, y_n) \le M,\ \forall n \\ &(\text{for some } M = M(x) > 0) \text{ imply } e(x, y) \le M \end{aligned} \tag{5.17}$$

$$\begin{aligned} &\lim_n \sup_{m>n} e(x_n, x_m) = 0 \text{ and } \lim_n e(x_n, y_n) = 0 \\ &\text{imply } \lim_n d(x_n, y_n) = 0. \end{aligned} \tag{5.18}$$

By [24, Remark 1] each w-distance (modulo d) is a τ-function (modulo d). So (as before) it is natural asking of the relationships between this last concept and that of strong KST-metric (modulo d). The answer is contained in

Proposition 5. *Let the specified conventions hold. Then, each τ-function (modulo d) is a strong KST-metric (modulo d). So (combining with the above) we have the generic inclusions (over transitive pseudometrics)*

$$w\text{-}distance \Longrightarrow \tau\text{-}function \Longrightarrow strong\ KST\text{-}metric\ (modulo\ d).$$

Proof. Let $e : M \times M \to R_+$ be some τ-function (modulo d). By definition, it fulfills (3.15); so, it remains to prove that (5.1)+(5.2) hold. The latter of these is immediate via (5.17). To verify the former, call the sequence (x_n), *almost e-Cauchy* when $\lim_n \sup_{m>n} e(x_n, x_m) = 0$. The implication below is clear, by definition

$$\text{[for each sequence]}\ \ e\text{-Cauchy} \Longrightarrow \text{almost } e\text{-Cauchy.}$$

On the other hand, by [24, Lemma 2.1]

$$\text{[for each sequence]}\ \ \text{almost } e\text{-Cauchy} \Longrightarrow d\text{-Cauchy.} \tag{5.19}$$

Combining with the above gives (5.1); and the claim follows. □

As a consequence of this, the fixed point statement (involving τ-functions) obtained by the quoted authors is deductible from Theorem 3 above. Its functional version (i.e., the variant of Theorem 4 involving τ-functions) seems to be new. Note that the proposed proofs are still depending on Lin-Du's reasoning concerning (5.19). As before, it would be interesting to have alternate proofs of such an implication. Concerning this aspect, note that the proof of (5.19) runs as follows. Assume that (x_n) is almost e-Cauchy. Putting $(y_n = x_{n+1}; n \in N)$ we have $\lim_n e(x_n, y_n) = 0$. This, along with (5.18) yields

$$d(x_n, x_{n+1}) \to 0;\ \text{ hence } (x_n) \text{ is } d\text{-Cauchy.}$$

However, the last inference seems to be not in general true; and this conclusion is transferrable upon Lemma 2.1 (of the authors).

Finally, note that further extensions of such statements to multivalued maps $T : M \to \mathcal{P}(M)$ are available. These extend some related contributions due to Petrusel and Sintamarian [30]; see also Birsan [5].

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