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A CLASS OF NON-CONTRACTIVE OPERATORS WITH A UNIQUE FIXED POINT

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Dedicated to Professor Ioan A. Rus on the occasion of his 70th birthday

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Abstract. In this paper, we prove the following result: Let X be a real Hilbert space and let $J : X \to \mathbb{R}$ be a C^1 functional, such that 0 is a global maximum of J and J' is Lipschitzian with Lipschitz constant less than 2. Then, 0 is the unique fixed point of J'.
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Here and in the sequel, $(X, \langle\cdot,\cdot\rangle)$ is a real Hilbert space and $J : X \to \mathbb{R}$ is a C^1 functional whose derivative is Lipschitzian in X, with Lipschitz constant L.

In this paper, we are interested in the question of knowing when the operator J' has a unique fixed point.

The classical answer to such a question is, of course, to assume that $L < 1$, in which case the fact that J' is the derivative of a functional has no role at all.

When $L \geq 1$, the question becomes quite delicate, the most immediate counterexample being provided by $J(x) = \frac{1}{2}\|x\|^2$.

The contribution that we offer in this paper is as follows:

Theorem 1. *If $L < 1$ and if $u \in X$ is a global maximum of J, then u is the unique solution of the equation*

$$x = J'(x) + P(J'(x)) + u \ ,$$

where $P : X \to X$ is an arbitrary operator satisfying

$$\|P(x)\| \leq \|x\|$$

for all $x \in X$.

Clearly, from Theorem 1, we get

Corollary 1. *If $L < 2$ and if 0 is a global maximum of J, then 0 is the unique fixed point of J'.*

We will draw Theorem 1 from the following more general result:

Theorem 2. *Assume that $L \leq 1$ and that $u \in X$ is a global maximum of J.*

Then, for every $y \in X \setminus \{u\}$ there exists a global minimum $\hat{x}$ of the functional

$$x \to \tfrac{1}{2}\|x - y\|^2 - J(x)$$

such that

$$\|\hat{x} - y\| < \|u - y\| \ .$$

So, in particular, one has

$$\hat{x} = J'(\hat{x}) + y \ .$$

Proof. Fix $y \in X \setminus \{u\}$. For each $x \in X$, set

$$f(x) = \tfrac{1}{2}\|x - y\|^2 - J(x)$$

and

$$g(x) = \tfrac{1}{2}\|x - y\|^2 - \tfrac{1}{2}\|u - y\|^2 \ .$$

Of course, the functions f, g are C^1 and their derivatives are

$$f'(x) = x - J'(x) - y$$

and

$$g'(x) = x - y \ .$$

For each $x, z \in X$, we have

$$\langle f'(x) - f'(z), x - z\rangle = \|x - z\|^2 - \langle J'(x) - J'(z), x - z\rangle \geq (1 - L)\|x - z\|^2 \ .$$

Therefore, since $L \leq 1$, the operator f' is monotone and hence f is convex. Thus, f is also weakly lower semicontinuous, being continuous. Then, since $g^{-1}(]-\infty,0])$ is weakly compact, there exists $\hat{x} \in g^{-1}(]-\infty,0])$ such that

$$f(\hat{x}) = \inf_{g^{-1}(]-\infty,0])} f \ .$$

Since $g(y) < 0$, the Slater condition is verified, and so, by a classical result ([2], Corollary 2.9.4) there exists $\lambda \geq 0$ such that

$$f'(\hat{x}) + \lambda g'(\hat{x}) = 0 \ .$$

So

$$(1+\lambda)(\hat{x} - y) - J'(\hat{x}) = 0 \ .$$

Note that $\hat{x} \neq u$. Otherwise, if $\hat{x} = u$, we would have $J'(\hat{x}) = J'(u) = 0$, and so $\hat{x} = y$ which yields the contradiction $y = u$. Since $g(u) = 0$, the above argument shows that

$$f(\hat{x}) < f(u) \ ,$$

that is

$$\tfrac{1}{2}\|\hat{x} - y\|^2 - J(\hat{x}) < \tfrac{1}{2}\|u - y\|^2 - J(u) \ .$$

Recalling that u is a global maximum of J, we then infer that

$$\tfrac{1}{2}\|\hat{x} - y\|^2 < \tfrac{1}{2}\|u - y\|^2 \ .$$

This shows that $\hat{x}$ is a local minimum of f. Hence, by convexity, $\hat{x}$ is a global minimum of f, and the proof is complete. □

Continue to assume that $L \leq 1$ and that u is a global maximum of J.

Consider the operator $T : X \to X$ defined by

$$T(x) = x - J'(x)$$

for all $x \in X$, and consider also the set

$$A_u = \{x \in X : \|J'(x)\| < \|x - J'(x) - u\|\} \ .$$

With these notations, from Theorem 2, we clearly get

$$X \setminus \{u\} \subseteq T(A_u) \ . \tag{1}$$

When T is injective, we have

Corollary 2. *If the operator T is injective (in particular, if $L < 1$), then*

$$A_u = X \setminus \{u\} \ . \tag{2}$$

Proof. Clearly, $u \notin A_u$. So, let $x \in X \setminus \{u\}$. Since $T(u) = u$ and T is injective, we have $T(x) \neq u$. Consequently, by (1), $T(x) \in T(A_u)$, from which, by injectivity again, we get $x \in A_u$, as claimed. □

It is worth noticing the following consequence of Corollary 2.

Corollary 3. *If the operator T is injective and if $u \neq 0$, then the norm of any possible solution of the equation $J'(x) = u$ is strictly greater than $\|u\|$.*

Proof. Let $x \in X$ be such that $J'(x) = u$. Since $u \neq 0$, it clearly follows that $x \neq u$. Consequently, thanks to (2), we have

$$\|u\| = \|J'(x)\| < \|x - J'(x) - u\| = \|x\| \ ,$$

as claimed. □

We now give the

Proof of Theorem 1. Let $x^* \in X$ be such that

$$x^* = J'(x^*) + P(J'(x^*)) + u \ .$$

We then get

$$\|J'(x^*)\| \geq \|P(J'(x^*))\| = \|x^* - J'(x^*) - u\| \ .$$

Since $L < 1$, the operator T is injective and so, by Corollary 2, we have $x^* = u$, as claimed. □

Two remarks on the previous results are now in order.

The first remark concerns the validity of (2) when u is not a global maximum of J. It is clear that if (2) holds for some u and if $L < 1$, then $J'(u) = 0$. Indeed, in this case, the operator T is surjective and so $T(x) = u$ for some $x \in X$. By (2), we necessarily have $x = u$, and so $J'(u) = 0$. However, it can happen that, though (2) holds for some u and $L < 1$, the point u is not a global maximum of J. For instance, take $X = \mathbb{R}$ and

$$J(x) = -\frac{1}{2 + x^2}$$

for all $x \in \mathbb{R}$. In this case, it is easy to see that $L \leq \frac{1}{2}$, and (2) is clearly satisfied with $u = 0$ which is a global minimum of J. However, it may happen that $L < 1$, that u is a global minimum of J and that (2) is violated. In this connection, the simplest example is provided by $J(x) = \frac{\lambda}{2}\|x\|^2$, with $\frac{1}{2} \leq \lambda < 1$. On the basis of the previous observations, it would then be interesting to characterize, when $L < 1$, the set U of all $u \in X$ for which (2) holds.

The second remark concerns Corollary 1. For any real Hilbert space H, denote by $\mathcal{A}_H$ the set of all C^1 functionals $I : H \to \mathbb{R}$ such that 0 is a global maximum of I and I' is Lipschitzian with Lipschitz constant less than 1. Set

$$\gamma_H = \inf_{I \in \mathcal{A}_H} \inf\{\lambda > 0 : x = \lambda I'(x) \text{ for some } x \neq 0\} .$$

Corollary 1 tells us that

$$2 \leq \gamma_H$$

for any H. It would be interesting, of course, to know the exact value of γ_H. We do not know the answer in general. But, in the case $H = \mathbb{R}$, we have it. This is provided by the following proposition which is due to M. Romeo ([1]).

Proposition 1. *One has*

$$\gamma_{\mathbb{R}} = 3 .$$

Proof. Let $I \in \mathcal{A}_{\mathbb{R}}$ and let $L < 1$ be the Lipschitz constant of I'. Replacing, if necessary, I by $I - I(0)$, we can assume that $I(0) = 0$. Fix $\lambda \in]0, 3]$. Let us prove that 0 is the unique solution of the equation

$$x = \lambda I'(x) .$$

Arguing by contradiction, assume that

$$x_0 = \lambda I'(x_0)$$

for some $x_0 \neq 0$. It is not restrictive to assume that $x_0 > 0$ (otherwise, we would work with the function $x \to I(-x)$). Consider now the function $g : \mathbb{R} \to \mathbb{R}$ defined by

$$g(x) = \begin{cases} -\frac{1}{2}x^2 & \text{if } \; x < \frac{1}{3}x_0, \\ \frac{1}{2}x^2 - \frac{2}{3}x_0 x + \frac{1}{9}x_0^2 & \text{if } \; \frac{1}{3}x_0 \leq x \leq x_0, \\ -\frac{1}{2}x^2 + \frac{4}{3}x_0 x - \frac{8}{9}x_0^2 & \text{if } \; x_0 < x. \end{cases}$$

Clearly, $g \in C^1(\mathbb{R})$. Let $x > 0$ with $x \neq x_0$. Let us prove that

$$g'(x) < I'(x) .$$

We distinguish two cases. If $0 < x \le \frac{1}{3}x_0$, we have

$$g'(x) = -x < -Lx \le I'(x) \ .$$

If $x > \frac{1}{3}x_0$, we have

$$g'(x) = \tfrac{1}{3}x_0 - |x - x_0| < \tfrac{1}{3}x_0 - L|x - x_0|$$
$$= \tfrac{1}{3}\lambda I'(x_0) - L|x - x_0| \le I'(x_0) - L|x - x_0| \le I'(x) \ .$$

So we get

$$I\left(\tfrac{4}{3}x_0\right) = \int_0^{\frac{4}{3}x_0} I'(x)dx > \int_0^{\frac{4}{3}x_0} g'(x)dx = g\left(\tfrac{4}{3}x_0\right) = 0$$

which contradicts the fact that the function I is non-positive, since 0 is a global maximum of I. From what we have just proven, it clearly follows that

$$3 \le \gamma_{\mathbb{R}} \ .$$

Now, fix any $\mu > 1$. Continue to consider the function g defined above (for a fixed $x_0 > 0$). Clearly, the function $h = \mu^{-1}g$ belongs to $\mathcal{A}_{\mathbb{R}}$ and

$$x_0 = 3\mu h'(x_0) \ .$$

Of course, from this we infer that

$$\gamma_{\mathbb{R}} \le 3\mu$$

and the conclusion clearly follows. □

We conclude proving the following

Theorem 3. *For any real Hilbert space H, with $H \neq \{0\}$, one has*

$$2 \le \gamma_H \le 3 \ .$$

Proof. As we have already observed, the inequality $2 \le \gamma_H$ is a direct consequence of Corollary 1. To prove the other inequality, let us fix any $\varphi \in \mathcal{A}_{\mathbb{R}}$ and any $\lambda > 0$ such that

$$t = \lambda\varphi'(t)$$

for some $t \neq 0$. Fix also $u \in H$, with $\|u\| = 1$, and consider the functional I defined by

$$I(x) = \varphi(\langle u, x\rangle)$$

for all $x \in X$. It is readily seen that $I \in \mathcal{A}_H$. In particular, note that

$$I'(x) = \varphi'(\langle u, x\rangle)u \ .$$

Finally, set

$$\hat{x} = tu = \lambda\varphi'(t)u \ .$$

Of course, $\hat{x} \neq 0$ and

$$\langle u, \hat{x}\rangle = t,$$

and so

$$\hat{x} = \lambda I'(\hat{x}) \ .$$

From this, it clearly follows that

$$\gamma_H \leq \gamma_{\mathbb{R}}$$

and so the desired inequality follows now from Proposition 1. □

References

[1] M. Romeo, Personal communication.

[2] C. Zălinescu, *Convex Analysis in General Vector Spaces,* World Scientific, Singapore (2002).

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