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FIXED POINT THEOREMS FOR OCCASIONALLY WEAKLY COMPATIBLE MAPPINGS

G. JUNGCK* AND B. E. RHOADES**

Dedicated to Professor Ioan A. Rus on the occasion of his 70th birthday

*Department of Mathematics
Bradley University
Peoria, Illinois 62625
E-mail: gfj@hilltop.bradley.edu

**Department of Mathematics
Indiana University
Bloomington, IN 47405-7106
E-mail: rhoades@indiana.edu

Abstract. We obtain two fixed point theorems for a class of operators called occasionally weakly compatible maps defined on a symmetric space . These results establish two of the most general fixed point theorems for four maps.
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Prior to 1968 all work involving fixed points used the Banach contraction principle [3]. In 1968 Kannan [10] proved a fixed point theorem for a map satisfying a contractive condition that did not require continuity at each point. This paper was a genesis for a multitude of fixed point papers over the next two decades. (See, e.g., [14] for a listing and comparison of many of these definitions.) Also during this time a number of authors established fixed point theorems for pairs of maps. Then, replacing x and y on the right hand of the inequality condition with continuous functions S and T fixed point theorems were established for four maps. However, it was necessary to add additional hypotheses in addition to the contractive condition in order to obtain fixed points. The first condition was that, given the space X, $A(X) \subset T(X)$ and $B(X) \subset S(X)$. Next it was necessary to add some kind of commutativity condition. First it was assumed that the maps all commuted. Then Sessa [16] defined the concept of weakly commuting. Then the first author generalized this idea, first to compatible mappings [8] and then to weakly compatible mappings [9]. There are examples that show that each of these generalizations of commutativity is a proper extension of the previous definition. We shall list here only the definition of weakly compatible.

Two maps S and T are said to be weakly compatible if they commute at coincidence points.

Definition 1. *Let X be a set, f, g selfmaps of X. A point x in X is called a coincidence point of f and g iff $fx = gx$. We shall call $w = fx = gx$ a point of coincidence of f and g.*

The following concept [2] is a proper generalization of nontrivial weakly compatible maps which do have a coincidence point.

Definition 2. *Two selfmaps f and g of a set X are occasionally weakly compatible (owc) iff there is a point x in X which is a coincidence point of f and g at which f and g commute.*

Lemma 1. *Let X be a set, f, g owc selfmaps of X. If f and g have a unique point of coincidence, $w := fx = gx$, then w is the unique common fixed point of f and g.*

Proof. Since f and g are owc, there exists a point $x \in X$ such that $fx = gx := w$ and $fgx = gfx$. Thus, $ffx = fgx = gfx$, which says that fx is also a point of coincidence of f and g. Since the point of coincidence $w = fx$ is unique by hypothesis, $gfx = ffx = fx$, and $w = fx$ is a common fixed point of f and g. Moreover, if z is any common fixed point of f and g, then $z = fz = gz = w$ by the uniqueness of the point of coincidence. □

Our theorems are proved in symmetric spaces, which are more general than metric spaces.

Definition 3. *Let X be a set. A symmetric on X is a mapping $r : X \times X \to [0, \infty)$ such that*

$$r(x, y) = 0 \quad iff \quad x = y, \quad and \quad r(x, y) = r(y, x) \quad for \quad x, y \in X.$$

Theorem 1. *Let X be a set with a symmetric r. Suppose that f, g, S, T are selfmaps of X and that the pairs $\{f, S\}$ and $\{g, T\}$ are each owc. If*

$$r(fx, gy) < M(x, y) \tag{1}$$

for each $x, y \in X$ for which $fx \neq gy$

$$M(x, y) := \max\{r(Sx, Ty), r(Sx, fx), r(Ty, gy), \\ r(Sx, gy), r(Ty, fx)\}.$$

Then there is a unique point $w \in X$ such that $fw = gw = w$ and a unique point $z \in X$ such that $gz = Tz = z$. Moreover, $z = w$, so that there is a unique common fixed point of f, g, S, and T.

Proof. Since the pairs $\{f, S\}$ and $\{g, T\}$ are each owc, there exist points $x, y \in X$ such that $fx = Sx$ and $gy = Ty$. We claim that $fx = gy$. For, otherwise, by (1),

$$r(fx, gy) < r(M(x, y)) = r(fx, gy),$$

a contradiction. Therefore, $fx = gy$; i.e., $fx = Sx = gy = Ty$. Moreover, if there is another point z such that $fz = Sz$, then, using (1) it follows that $fz = Sz = gy = Ty$, or $fx = fz$ and $w = fx = Sx$ is the unique point of coincidence of f and S. By Lemma 1, w is the only common fixed point of f and S. By symmetry there is a unique point $z \in X$ such that $z = gz = Tz$.

Suppose that $w \neq z$. Using (1),

$$r(w, z) = r(fw, gz) < r(M(x, y)) = r(w, z),$$

a contradiction. Therefore $w = z$ and w is a common fixed point. By the preceding argument it is clear that w is unique. □

Corollary 1. *Let X be a set with a symmetric r. Suppose that $f, g,$ S, T are selfmaps of X such that $\{f, S\}$ and $\{g, T\}$ are owc. If*

$$r(fx, gy) \leq hm(x, y) \quad \textit{for all} \quad x, y \in X, \tag{2}$$

where

$$m(x, y) := \max\{r(Sx, Ty), r(Sx, fx), r(Ty, gy), \\ [r(Sx, gy) + r(Ty, fx)]/2\},$$

and $0 \leq h < 1$, then f, g, S, T have a unique common fixed point.

Proof. Since (2) is a special case of (1), the result follows immediately from Theorem 1. □

The proofs of fixed point theorems for four compatible or weakly compatible maps all have the same pattern. Step one is to show that there exists a common coincidence point for one pair of maps. The second step is to show that step one gives rise to a common coincidence point for the second pair of maps. In step three it is shown that these pairwise coincidence points are equal. In step four it is shown that this common coincidence point is a common fixed point. Uniqueness is established in step five.

Theorems 1 and its variants, provides a new proof for steps two through four. We shall illustrate this fact with the following results.

Let (X, d) be a complete metric space, f a continuous selfmap of X. Then a selfmap g is called an f-contraction if $\overline{g(X)} \subset f(X)$ and

$$d(gx, gy) \leq hd(fx, fy) \quad \text{for all} \quad x, y \in X \quad \text{where} \quad 0 \leq h < 1. \tag{3}$$

Corollary 2. *If g is an f-contraction of a complete metric space and f and g are weakly compatible, then f and g have a unique common fixed point.*

Proof. Uniqueness of the common fixed point follows from (2). From [7] f and g have a coincidence point. Condition (3) is a special case of condition (2) with $S = T = I, \psi(s) = hs$. Therefore the result follows from Corollary 1. □

Corollary 2 is Theorem 2.4 of [4].

Corollary 3. *Let (X, d) be a metric space, f, g, T selfmaps of X such that $f(X) \subset T(X)$ and*

$$d(fx, gy) \leq \phi(m_1(x, y)) \quad forall \quad x, y \in X, \tag{4}$$

where

$$m_1(x, y) = \max\{d(Tx, Ty), d(fx, Tx), d(gy, Ty), \\ [d(fx, Ty) + d(gy, Sx)]/2\},$$

where $\phi : \mathbb{R}^+ \to \mathbb{R}^+, \phi$ upper semicontinuous with $\phi(t) < t$ for each $t > 0$. If one of $f(X), g(X)$ or $T(X)$ is complete, then f, g and T have a coincidence point. If $\{f, T\}$ and $\{g, T\}$ are weakly compatible, then $f, g,$ and T have a unique common fixed point.

Proof. Uniqueness of the common fixed point follows from (4). From [19], Theorem 2.1, $\{f, S\}$ and $\{g, T\}$ have a common coincidence point. The result follows from Corollary 1. □

Theorem 2. *Let (X, d) be a symmetric space with symmetric r and f, S selfmaps of X such that $f(X) \subset S(X), f$ and S are owc, and*

$$r(fx, fy) \leq ar(Sx, Sy) + b\max\{r(fx, Sx), r(fy, Sy)\} + c\max\{r(Sx, Sy), r(Sx, fx), r(Sy, fy)\} \tag{5}$$

for all $s, y \in X$, where $a, b, c > 0, a + b + c = 1$ and $a + c < \sqrt{a}$. Then f and S have a unique common fixed point.

Proof. By hypothesis there exists a point $x \in X$ such that $fx = Sx$. Suppose that there exists another point $y \in X$ for which $fy = Ty$. Then, from (5),

$$\begin{aligned} r(fx, fy) &\leq ar(fx, fy) + b\max\{0, 0\} \\ &\qquad + c\max\{r(fx, fy), 0, 0\} \\ &= (a + c)r(fx, fy). \end{aligned}$$

Since $a + c < 1$, the above inequalty implies that r(fx, fy) = 0, which, in turn implies that $fx = fy$. Therefore fx is unique. From Lemma 1, f and S have a unique fixed point. □

Theorem 3. *Let X be a symmetric space with symmetric r, f, g, S, and T selfmaps of X with $f(X) \subset T(X), g(X) \subset S(X)$, and satisfying*

$$\begin{aligned} (r(fx, gy))^p \leq a(r(fx, Ty))^p + \\ &+ (1 - a)\max\{(r(fx, Sx))^p, (r(gy, Ty))^p\}, \\ &(r(fx, Sx))^{p/2}(r(fx, Ty))^{p/2} \\ &(r(Ty, fx))^{p/2}(r(Sx, gy))^{p/2}\} \end{aligned} \tag{6}$$

for all $x, y \in X$, where $0 < a, \alpha, \beta \leq 1$, and $p \geq 1$. If $\{f, S\}$ and $\{g, T\}$ are owc, then f, g, S, and T have a unique common fixed point.

Proof. By hypothesis there exist points x and y such that $fx = Sx$ and $gy = Ty$. Suppose that $fx \neq gy$. Then, from (6),

$$\begin{aligned}(r(fx, gy))^p &\leq a(r(fx, gy))^p \\ &\quad + (1 - a) \max\{0, 0, 0, (r(fx, gy))^p\} \\ &= (r(fx, gy))^p) < (r(fx, gy))^p,\end{aligned}$$

a contradiction. Therefore $r(fx, gy) = 0$, which implies that $fx = gy$. Suppose that there exists another point z such that $fz = Sz$. Then, using (6) one obtains $fz = Sz = gy = Ty = fx = Sx$ and hence $w = fx = fz$ is the unique point of coincidence of f and S. By symmetry there exists a unique point $v \in X$ such that $v = gz = Tv$. It then follows that $w = v$, w is a common fixed point of f, g, S, and T, and w is unique. $\square$

A special case of Theorem 3 is Theorem of 1 of [20].

Define $\mathcal{G} = \{g : \mathbb{R}^+ \to \mathbb{R}^+\}$ such that

(g_1) g is nondecreasing in the 4th and 5th variables,

(g_2) If $u, v, \in \mathbb{R}^+$ are such that $u \leq g(v, v, u, u + v, 0)$, or $u \leq g(v, u, v, u + v, 0)$ or $v \leq g(u, u, v, u + v, 0)$, or $u \leq g(v, u, v, u, u + v)$, then $u \leq hv$, where $0 < h < 1$ is a constant,

(g_3) If $u \in \mathbb{R}^+$ is such that $u \leq g(u, 0, 0, u, u)$ or $u \leq g(0, u, 0, u, u)$ or $u \leq g(0, 0, u, u, u)$, then $u = 0$.

Theorem 4. *Let X be a set, r a symmetric on X. Let f, g, S, T be selfmaps of X satisfying $f(X) \subset T(X), g(X) \subset S(X)$, and*

$$\begin{aligned}r(fx, gy) \leq g(&r(Sx, Ty), r(fx, Sx), r(gy, Ty) \\ &r(fx, Ty), r(gy, Sx))\end{aligned} \tag{7}$$

for all $x, y \in X$, where $g \in \mathcal{G}$. If $\{f, S\}$ and $\{g, T\}$ are owc, then f, g, S, T have a unique common fixed point.

Proof. By hypothesis there exist points $x, y \in X$ such that $fx = SX$ and $gy = Ty$. Suppose that $fx \neq gy$. Then, from (7),

$$r(fx, gy) \leq g(r(fx, gy), 0, 0, r(fx, gy), r(gy, fx)),$$

which, from (g_3) implies that $r(fx, gy) = 0$. Hence $fx = gy$. As in the previous theorems it can then be shown that fx is unique and that $u = fx$ is a common fixed point of the four mappings. Condition (7) implies uniqueness. □

Two maps S and T are said to be pointwise R-commuting if, for each $x \in X$ there exists an $R > 0$ such that $d(STx, TSx) \leq Rd(Sx, Tx)$. The definition of R-pointwise commuting is equivalent to S and T commuting at coincidence points; i.e., S and T are weakly compatible. The maps S and T are said to be reciprocally continuous if $\lim_n STx_n = St$ and $\lim_n TSx_n = Tt$ whenever $\{x_n\}$ is a sequence in X such that $\lim_n Sx_n = \lim_n Tx_n = t$ for some $t \in X$.

Corollary 4. *Let $\{f, S\}$ and $\{g, T\}$ be pointwise R-weakly commuting pairs of selfmaps of a complete metric space (X, d) satisfying $f(X) \subset T(X)$, $g(X) \subset S(X)$, and*

$$d(fx, gy) \leq g(d(Sx, Ty), d(fx, Sx), d(gy, Ty), \\ d(fx, Ty), d(gy, Sx)) \tag{8}$$

for all $x, y \in X$, where $g \in \mathcal{G}$. Suppose that $\{f, S\}$ or $\{g, T\}$ is a pair of reciprocally continuous mappings. Then f, g, S, and T have a unique common fixed point in X.

Proof. The first part of the proof of Theorem 3.1 of [11] verifies that $\{f, S\}$ and $\{g, T\}$ each have an owc point. The result then follows from Theorem 4. □

A control function Φ is defined by $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ which is continuous, monotonically increasing, $\Phi(2t) \leq 2\Phi(t)$ and $\Phi(0) = 0$ iff $t = 0$. Two maps f and S are said to be Φ-compatible if $\lim_n \Phi(d(fSx_n, Sfx_n)) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_n fx_n = \lim_n Sx_n = t$ for some $t \in X$. Let $\psi : \mathbb{R}+ \to \mathbb{R}+$ such that $\psi(t) < t$ for each $t > 0$.

Theorem 5. *Let $\{f, S\}$ and $\{g, t\}$ be owc pairs of selfmaps of a space X, with symmetric r, on which a control function ψ is defined, satisfying $f(X) \subset T(X), g(X) \subset S(X)$, and*

$$\Phi(r(fx, gy)) \leq \psi(M_\Phi(x, y)), \tag{9}$$

where

$$M_\Phi(x,y) := \max\{\Phi(r(Sx,Ty)), \Phi(r(Sx,fx)), \Phi((r(gy,Ty)), [\Phi(r(fx,Ty)), +\Phi(r(Sx,gy))]/2\}.$$

Then $f, g, S,$ and T have a unique common fixed point.

Proof. By hypothesis there exist points $x, y \in X$ for which $fx = Sx$ and $gy = Ty$. Suppose that $fx \neq gy$. Then, from (9),

$$\begin{aligned} 0 < \Phi((r(fx,gy)) &\le \psi(M_\Phi(x,y)) \\ &= \psi(\Phi(r(fx,gy)) \\ &< \Phi((r(fx,gy)), \end{aligned}$$

a contradiction. Therefore

$$\Phi((r(fx,gy)) = 0,$$

which implies that $(r(fx,gy) = 0$, which implies that $fx = gy$. It then follows that f, g, S, and T have a common fixed point. Condition (9) gives uniqueness. □

Theorem 2.1 of [15] is a special case of Theorem 5 with $\psi(t) = ht$.

In proving fixed point theorems for four maps, step one is by far the most difficult part of the proof. In this paper we have imposed the condition owc, which automatically gives the result of step one. Other authors have circumvented this difficulty by hypothesizing a property, known as property (E, A), which implies owc.

Two maps f, S are said to satisfy property (E, A) if there exists a sequence $\{x_n\}$ such that $\lim_n Sx_n = \lim_n fx_n = t$ for some $t \in X$. Some papers in which this property have appeared are [1], [12], [13], and [18].

Corollary 5. *Let S and T be two weakly compatible selfmappings of a metric space (X, d) such that T and S satisfy property (E, A), $T(X) \subset S(X)$, and and*

$$d(Tx,Ty) < \max\{d(Sx,Sy), [d(Tx,Sx) + d(Ty,Sy)]/2, [d(Ty,Sx) + d(Tx,Sy)]/2\}. \tag{10}$$

If SX or TX is a complete subspace of X, then T and S have a unique common fixed point.

Proof. Condition (10) is a special case of condition (1). Property (E, A) implies that S and T have owc. The conclusion now follows from Theorem 1. □

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