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THREE EXAMPLES IN METRIC FIXED POINT THEORY

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Dedicated to Professor Ioan A. Rus on the occasion of his 70th birthday

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Abstract. We present three examples concerning the relations between various classes of mappings of contractive type defined on complete metric spaces.
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Introduction

Let (X, ρ) be a complete metric space. A mapping $A : X \to X$ is said to be nonexpansive if

$$\rho(Ax, Ay) \leq \rho(x, y) \text{ for all } x, y \in X.$$

It is called a strict contraction if there is a constant $c \in [0, 1)$ such that

$$\rho(Ax, Ay) \leq c\rho(x, y) \text{ for all } x, y \in X.$$

A mapping $A : X \to X$ is called contractive if there exists a decreasing function $\phi^A : [0, \infty) \to [0, 1]$ such that

$$\phi^A(t) < 1 \text{ for all } t > 0$$

and

$$\rho(Ax, Ay) \leq \phi^A(\rho(x, y))\rho(x, y) \text{ for all } x, y \in X.$$

According to Banach's fixed point theorem, every strict contraction has a unique fixed point. In 1962 Rakotch [5] proved that this is also true if the mapping is assumed to be merely contractive. Since it is not difficult to show that if a mapping has a strictly contractive power, then it has a unique fixed point, it is natural to ask if each contractive mapping has a strictly contractive power. In the first section of our paper we answer this question in the negative.

It is also known that if a mapping is contractive, then its iterates converge, uniformly on bounded sets, to its unique fixed point. This is also true even if only a certain power of A is contractive. On the other hand, even if a nonexpansive mapping has a fixed point, its iterates do not necessarily converge to it. Therefore it is also natural to ask if a nonexpansive mapping with uniformly convergent iterates must have a contractive power. In the second section of our paper we answer this second question also in the negative.

When (X, ρ) is compact and the iterates of a nonexpansive self-mapping converge, then these iterates must converge uniformly (Theorem 3.1). Our last example (see Section 3) shows that this is not true in general if X is merely complete.

The notion of a contractive mapping, as well as its modifications and applications, were studied by many authors. See, for example, [3]. We also note that the fixed point problem for such mappings is well-posed [7]. It is also known [6, 8] that most (in the sense of Baire category) nonexpansive mappings are, in fact, contractive. On the other hand, the set of strict contractions is of the first category in the space of all nonexpansive mappings, at least in Hilbert space [1, 2]. For more information on generic aspects of metric fixed point theory see, for instance, [4, 9, 10].

1. A contractive mapping with no strictly contractive powers

Let

$$X = [0, 1] \text{ and } \rho(x, y) = |x - y| \text{ for each } x, y \in X.$$

In this section we will construct a contractive mapping $A : [0, 1] \to [0, 1]$ such that none of its powers is a strict contraction.

We begin by setting

$$A(0) = 0. \tag{1.1}$$

Next, we define, for each natural number n, the mapping A on the interval $[(n+1)^{-1}, n^{-1}]$ by

$$A((n+1)^{-1}+t) = (n+2)^{-1}+t(n^{-1}-(n+1)^{-1})^{-1}((n+1)^{-1}-(n+2)^{-1}) \quad (1.2)$$

$$\text{for all } t \in [0, n^{-1}-(n+1)^{-1}].$$

It is clear that for each natural number n,

$$A(n^{-1}) = (n+1)^{-1}, \quad (1.3)$$

the restriction of A to the interval $[(n+1)^{-1}, n^{-1}]$ is affine, and that the mapping $A : [0,1] \to [0,1]$ is well defined.

First, we show that A is nonexpansive, that is, $|Ax - Ay| \le |x - y|$ for all $x, y \in [0,1]$.

Indeed, if $x \in [0,1]$, then

$$|Ax - A(0)| \le |x|. \quad (1.4)$$

Assume now that n is a natural number and that

$$x, y \in [(n+1)^{-1}, n^{-1}]. \quad (1.5)$$

By (1.2) and (1.5),

$$\begin{aligned}|Ax-Ay| &= |(n+2)^{-1}+(x-(n+1)^{-1})(n^{-1}-(n+1)^{-1})^{-1}((n+1)^{-1}-(n+2)^{-1}) \\ &\quad -[(n+2)^{-1}+(y-(n+1)^{-1})(n^{-1}-(n+1)^{-1})^{-1}((n+1)^{-1}-(n+2)^{-1})]| \\ &= |x-y|(n^{-1}-(n+1)^{-1})^{-1}((n+1)^{-1}-(n+2)^{-1}) \\ &= |x-y|n(n+1)((n+1)(n+2))^{-1} = |x-y|n(n+2)^{-1}.\end{aligned}$$

Thus for each natural number n and each $x, y \in [(n+1)^{-1}, n^{-1}]$,

$$|Ax - Ay| \le |x-y|n(n+2)^{-1}. \quad (1.6)$$

Together with (1.4) this last inequality implies that

$$|Ax - Ay| \le |x - y| \text{ for all } x, y \in [0,1], \quad (1.7)$$

as claimed.

Next, we show that the power A^m is not a strict contraction for any integer $m \ge 1$. Assume the converse. Then there would exist a natural number m and $c \in (0,1)$ such that for each $x, y \in [0,1]$,

$$|A^m x - A^m y| \le c|x - y|. \quad (1.8)$$

Since

$$(m+i)(m+i+1)i^{-1}(i+1)^{-1} \to 1 \text{ as } i \to \infty,$$

there is an integer $p \geq 4$ such that

$$p(p+1) > (p+m)(p+m+1)c. \tag{1.9}$$

By (1.3), (1.1) and (1.9),

$$A^m(p^{-1}) - A^m((p+1)^{-1}) = (p+m)^{-1} - (p+m+1)^{-1}$$
$$= (p+m)^{-1}(p+m+1)^{-1} > cp^{-1}(p+1)^{-1} = c(p^{-1} - (p+1)^{-1}),$$

which contradicts (1.8).

The contradiction we have reached proves that A^m is not a strict contraction for any integer $m \geq 1$.

Finally, we show that A is contractive. Let $\epsilon \in (0,1)$. We claim that there exists $c \in (0,1)$ such that

$$|Ax - Ay| \leq c|x-y| \text{ for each } x, y \in [0,1] \text{ satisfying } |x-y| \geq \epsilon. \tag{1.10}$$

Indeed, choose a natural number $p \geq 4$ such that

$$p > 18\epsilon^{-2}, \tag{1.11}$$

and assume that

$$x, y \in [0,1],\ |x-y| \geq \epsilon. \tag{1.12}$$

We may assume without loss of generality that

$$y > x. \tag{1.13}$$

There are two cases:

$$x < (4p)^{-1}; \tag{1.14}$$
$$x \geq (4p)^{-1}. \tag{1.15}$$

Assume that (1.14) holds. There exists a natural number n such that

$$(1+n)^{-1} < y \leq n^{-1}. \tag{1.16}$$

By (1.16), (1.13) and (1.12),

$$\epsilon \leq y \leq 1/n,\ (n+2)^{-1} \geq (3n)^{-1} \geq \epsilon/3. \tag{1.17}$$

By (1.16) and (1.2),

$$Ay = (n+2)^{-1} + (y - (n+1)^{-1})(n^{-1} - (n+1)^{-1})^{-1}((n+1)^{-1} - (n+2)^{-1})$$

$= (n+2)^{-1}+(y-(n+1)^{-1})n(n+1)(n+1)^{-1}(n+2)^{-1} \leq y-(n+1)^{-1}+(n+2)^{-1}$
and

$$y - Ay \geq (n+1)^{-1}(n+2)^{-1}.$$

When combined with (1.17), the above inequality implies that

$$Ay - Ax \leq Ay \leq y - (n+1)^{-1}(n+2)^{-1} \leq y - (n+2)^{-2} \leq y - \epsilon^2/9. \quad (1.18)$$

By (1.14), (1.11) and (1.18),

$$(1 - 18^{-1}\epsilon^2)(y-x) \geq (1 - 18^{-1}\epsilon^2)y - x \geq (1 - 18^{-1}\epsilon^2)y - (4p)^{-1}$$

$$\geq y - \epsilon^2/18 - (4p)^{-1} \geq y - \epsilon^2/18 - \epsilon^2/18 \geq Ay - Ax.$$

Thus we have shown that if (1.14) holds, then

$$|Ax - Ay| \leq (1 - \epsilon^2/18)|x - y|. \quad (1.19)$$

Now assume that (1.15) holds. By (1.15) and (1.13),

$$x, y \in [(4p)^{-1}, 1].$$

In view of (1.6), the Lipschitz constant of the restriction of A to the interval $[(4p)^{-1}, 1]$ does not exceed $(4p+2)(4p+4)^{-1}$ and therefore we have

$$|Ax - Ay| \leq (4p+2)(4p+4)^{-1}|x - y|.$$

By this inequality and (1.19), we see that, in both cases,

$$|Ax - Ay| \leq \max\{(1 - \epsilon^2/18),\ (4p+2)(4p+4)^{-1}\}|x - y|.$$

Since this inequality holds for each $x, y \in X$ satisfying (1.12), we conclude that (1.10) is satisfied and therefore A is indeed contractive, as claimed.

2. A power convergent mapping with no contractive powers

Once again, let $X = [0,1]$ and let $\rho(x,y) = |x - y|$ for all $x, y \in X$. We will construct a mapping $A : [0,1] \to [0,1]$ such that

$$|Ax - Ay| \leq |x - y| \text{ for all } x, y \in [0,1],$$

$$A^n x \to 0 \text{ as } n \to \infty, \text{ uniformly on } [0,1],$$

and for each integer $m \geq 0$, the power A^m is not contractive.

To this end, let

$$A(0) = 0 \quad (2.1)$$

and for $t \in [2^{-1}, 1]$, set

$$A(t) = t - 1/4. \tag{2.2}$$

Clearly,

$$A(1) = 3/4 \text{ and } A(1/2) = 1/4. \tag{2.3}$$

For $t \in [4^{-1}, 2^{-1})$, set

$$A(t) = 4^{-1} - 16^{-1} + (t - 4^{-1})4^{-1}. \tag{2.4}$$

Clearly, A is continuous on $[4^{-1}, 1]$ and

$$A(4^{-1}) = 4^{-1} - 16^{-1}. \tag{2.5}$$

Now let $n \geq 2$ be a natural number. We will define the mapping A on the interval $[2^{-2^n}, 2^{-2^{n-1}}]$. For each $t \in [2^{-2^n+1}, 2^{-2^{n-1}}]$, set

$$A(t) = t - 2^{-2^n}. \tag{2.6}$$

Clearly,

$$A(2^{-2^n+1}) = 2^{-2^n} \text{ and } A(2^{-2^{n-1}}) = 2^{-2^{n-1}} - 2^{-2^n}. \tag{2.7}$$

For $t \in [2^{-2^n}, 2^{-2^n+1})$, set

$$A(t) = 2^{-2^n} - 2^{-2^{n+1}} + (t - 2^{-2^n})2^{2^n}(2^{-2^{n+1}}) = 2^{-2^n} - 2^{-2^{n+1}} + 2^{-2^n}(t - 2^{-2^n}). \tag{2.8}$$

It is clear that

$$A(2^{-2^n}) = 2^{-2^n} - 2^{-2^{n+1}}$$

and

$$\lim_{t \to (2^{-2^n+1})^+} A(t) = 2^{-2^n} - 2^{-2^{n+1}} + 2^{-2^n}(2^{-2^n+1} - 2^{-2^n}) = 2^{-2^n}. \tag{2.9}$$

It follows from (2.6)-(2.9) that the mapping A is continuous on each one of the intervals $[2^{-2^n}, 2^{-2^{n-1}}]$, $n = 2, 3, \ldots$. It is not difficult to check that A is well defined on $[0, 1]$ and that it is increasing.

By (2.2) and (2.4), for each $x \in [1/4, 1]$ we have $Ax < x$. We will now show that this inequality holds for all $x \in (0, 1]$.

Let $n \geq 2$ be an integer and let $x \in [2^{-2^n}, 2^{-2^{n-1}}]$. It is clear that $Ax < x$ if $x \in [2^{-2^n+1}, 2^{-2^{n-1}}]$. If $x \in [2^{-2^n}, 2^{-2^n+1})$, then by (2.6) and (2.7),

$$Ax < A(2^{-2^n+1}) \leq 2^{-2^n} \leq x.$$

Thus $Ax < x$ for all $x \in [2^{-2^n}, 2^{-2^{n-1}}]$ and for any integer $n \geq 2$. Therefore we have indeed shown that

$$Ax < x \text{ for all } x \in (0,1], \tag{2.10}$$

as claimed.

Next, we will show that

$$|Ax - Ay| \leq |x - y| \text{ for each } x, y \in [0,1]. \tag{2.11}$$

If $x = 0$ and $y > 0$, then

$$|Ay - Ax| = Ay \leq y = |y - x|. \tag{2.12}$$

Assume that $x, y \in (0,1]$. Note that the restrictions of the mapping A to the interval $[1/4, 1]$ and to all the intervals $[2^{-2n}, 2^{-2^{n-1}}]$, where $n \geq 2$ is an integer, are Lipschitz with Lipschitz constant one. This obviously implies that the mapping A is 1-Lipschitz on all of $(0,1]$. Therefore (2.11) is true.

Let $x \in (0,1]$. By (2.10), the sequence $\{A^n x\}_{n=1}^{\infty}$ is decreasing and there exists the limit

$$x_* = \lim_{n\to\infty} A^n x.$$

Clearly, $Ax_* = x_*$. If $x_* > 0$, then by (2.10), $Ax_* < x_*$, a contradiction. Thus $x_* = 0$ and $\lim_{n\to\infty} A^n(1) = 0$. Since the mapping A is increasing, this implies that

$$A^n x \to 0 \text{ as } n \to \infty, \text{ uniformly on } [0,1].$$

Finally, we will show that for each integer $m \geq 1$, the power A^m is not contractive.

Indeed, let $m \geq 1$ be an integer. It is sufficient to show that there exist $x, y \in [0,1]$ such that

$$x \neq y \text{ and } |A^m x - A^m y| = |x - y|.$$

To this end, choose a natural number $n \geq m + 4$ such that

$$2^{2^{n-1}} - 3 \geq m + 2. \tag{2.13}$$

Using induction and (2.6), we can show that for each integer $i \in \{1, \dots, 2^{2^{n-1}} - 2\}$,

$$A^i(2^{-2^{n-1}}) = 2^{-2^{n-1}} - i2^{-2^n} \geq 2^{-2^n+1}$$

and

$$A^i(2^{-2^{n-1}}) \in [2^{-2^n+1}, 2^{-2^n-1}].$$

Put

$$x = 2^{-2^{n-1}} \text{ and } y = A(2^{-2^{n-1}}).$$

Then for $i = 1, \dots, 2^{2^{n-1}} - 3$, we have

$$|A^i x - A^i y| = |x - y|,$$

and in view of (2.13),

$$|A^m x - A^m y| = |x - y|.$$

Thus the power A^m is not contractive, as asserted.

3. A nonexpansive mapping with nonuniformly convergent powers

Theorem 3.1. *Let (X, ρ) be a compact metric space, let a mapping $A : X \to X$ satisfy*

$$\rho(Ax, Ay) \le \rho(x, y) \text{ for each } x, y \in X, \tag{3.1}$$

and let $x_A \in X$ satisfy

$$A^n x \to x_A \text{ as } n \to \infty, \text{ for each } x \in X.$$

Then $A^n x \to x_A$ as $n \to \infty$, uniformly on X.

Proof. Let $\epsilon > 0$. For each $x \in X$, there is a natural number $n(x)$ such that

$$\rho(A^n x, x_A) \le \epsilon/2 \text{ for all integers } n \ge n(x). \tag{3.2}$$

Let

$$x, y \in X \text{ with } \rho(x, y) < \epsilon/2. \tag{3.3}$$

By (3.2) and (3.3), for each integer $n \ge n(x)$,

$$\rho(A^n y, x_A) \le \rho(A^n y, A^n x) + \rho(A^n x, x_A) < \epsilon/2 + \epsilon/2.$$

Thus the following property holds:

(P) For each $x \in X$, each integer $n \ge n(x)$, and each $y \in X$ satisfying $\rho(x, y) < \epsilon/2$, we have

$$\rho(A^n y, x_A) < \epsilon.$$

Since X is compact, there exist finitely many points $x_1, \dots, x_q \in X$ such that

$$\cup_{i=1}^q \{y \in X : \rho(y, x_i) < \epsilon/2\} = X.$$

Assume that $y \in X$ and that the integer $n \geq \max\{n(x_i) : i = 1, \dots, q\}$. Then there is $j \in \{1, \dots, q\}$ such that $\rho(y, x_j) < \epsilon/2$. By property (P),

$$\rho(A^n y, x_A) < \epsilon.$$

This completes the proof of Theorem 3.1. $\square$

Example. Let X be the set of all sequences $(x_1, x_2, \dots, x_n, \dots)$ such that $\sum_{i=1}^{\infty} |x_i| \leq 1$, and set

$$\rho(x, y) = \rho((x_i), (y_i)) = \sum_{i=1}^{\infty} |x_i - y_i|.$$

In other words, (X, ρ) is the closed unit ball of ℓ_1. Clearly, (X, ρ) is a complete metric space. Define

$$A(x_1, x_2, \dots, x_n, \dots) = (x_2, x_3, \dots, x_n, \dots),\ x = (x_1, x_2, \dots) \in X.$$

Then the mapping A is nonexpansive, and for each $x \in X$, $A^n x \to 0$ as $n \to \infty$.

However, if n is a natural number and e_n is the n-th unit vector of X, then $\rho(A^n e_{n+1}, 0) = 1$.

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