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ON AN EXISTENCE RESULT OF LERAY FOR NONLINEAR INTEGRAL EQUATIONS

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Dedicated to Professor Ioan A. Rus on the occasion of his 70th birthday

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Abstract. We show that an overlooked existence theorem on nonlinear integral equations proved by Leray in his 1933 PhD thesis is related to results of Brezis-Browder and suggests some generalization.

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1. Introduction

In his PhD thesis of 1933 [7], Leray introduced a generalization of the continuation method based upon Lyapunov-Schmidt's local analysis and compactness arguments, that he called *Arzelá-Schmidt's method* (see [10]). Among several applications, he proved that the nonlinear integral equation

$$u(x) + \int_0^1 K(s,t)F(u(t))\,dt = 0$$

(nowadays called a Hammerstein integral equation) has at least one solution when $K(s,t) + K(t,s)$ is a positive symmetric kernel (all eigenvalues are positive), K is continuous on $[0,1] \times [0,1]$, F is analytical on $\mathbb{R}$ and $|F(u)|$ is bounded by some constant A on the set of u such that $uF(u) < 0$. Consequently, there exists $M > 0$ such that

$$|uF(u)| \leq uF(u) + M|u| \quad (u \in \mathbb{R}). \tag{1}$$

Conversely, if condition (1) holds, then, when $uF(u) \leq 0$, $2|uF(u)| \leq M|u|$, i.e. $|F(u)| \leq M/2$, so that Leray's condition upon F and (1) are equivalent.

This result of Leray seems to have been completely ignored by the literature on nonlinear integral equations. It is not mentioned in fundamental books on nonlinear integral equations like [4, 6] or in surveys like [2, 5]. We show in this paper that Leray's condition has been partially rediscovered, and can still inspire further work. First, we give a simpler proof of a slight extension of Leray's result, before comparing it to some results of Brezis-Browder [1], and finally prove an abstract version of Leray's theorem.

2. An existence theorem

For $A \subset \mathbb{R}^N$ a bounded measurable subset, let $K \in L^\infty(A \times A)$ be such that the mapping

$$\mathcal{K} : L^1(A) \to L^\infty(A),\ v \mapsto \int_A K(\cdot, y)v(y)\, dy \tag{2}$$

is defined and compact, and let $f : A \times \mathbb{R} \to \mathbb{R}$ be a L^1-Carathéodory function.

Theorem 1. *If $K \in L^\infty(A \times A)$ is such that*

$$\int_{A\times A} \varphi(x)K(x, y)\varphi(y)\, dx\, dy \geq 0 \tag{3}$$

for all $\varphi \in L^1(A)$, and if f satisfies the growth condition

$$|uf(x, u)| \leq uf(x, u) + \mu(x)|u| \quad (a.e.\ x \in A,\ u \in \mathbb{R}) \tag{4}$$

for some nonnegative $\mu \in L^1(A)$, then, for each $h \in L^\infty(A)$, equation

$$u(x) + \int_A K(x, y)f(y, u(y))\, dy = h(x) \tag{5}$$

has at least one solution $u \in L^\infty(A)$.

Proof. Under the assumptions above, the mapping $T : L^\infty(A) \to L^\infty(A)$ defined for almost every $x \in A$ by

$$T(u)(x) = -\int_A K(x, y)f(y, u(y))\, dy + h(x)$$

is completely continuous. Hence the problem is equivalent to finding a fixed point of T in $L^\infty(A)$ and Leray-Schauder's fixed point theorem in its simplest form [8] implies that it will be the case if the set of possible solutions $u \in L^\infty(A)$ of the family of equations

$$u(x) + \lambda \int_A K(x,y) f(y, u(y))\, dy = \lambda h(x) \quad (0 \leq \lambda \leq 1) \tag{6}$$

is *a priori* bounded independently of $\lambda \in [0,1]$. So, for such a λ, let u be a possible solution of (6). Then we have

$$\|u\|_\infty \leq \|K\|_\infty \int_A |f(y,u(y))|\, dy + \|h\|_\infty. \tag{7}$$

On the other hand, the identity obtained from (6) after multiplication of both members by $f(x,u(x))$ and integration over A

$$\begin{aligned}\int_A u(x) f(x,u(x))\, dx \quad &+ \quad \lambda \int_{A\times A} f(x,u(x)) K(x,y) f(y,u(y))\, dx\, dy \\ &= \quad \lambda \int_A h(x) f(x,u(x))\, dx,\end{aligned}$$

and assumption (3) imply that

$$\int_A u(x) f(x,u(x))\, dx \leq \|h\|_\infty \int_A |f(x,u(x))|\, dx. \tag{8}$$

Using assumption (4), (7) and (8), we get

$$\begin{aligned}\int_A |u(x)||f(x,u(x))|\, dx \quad &\leq \quad \int_A u(x) f(x,u(x))\, dx + M\|u\|_\infty \\ &\leq \quad (M\|K\|_\infty + \|h\|_\infty) \int_A |f(y,u(y))|\, dy + M\|h\|_\infty,\end{aligned} \tag{9}$$

where

$$M = \int_A \mu(x)\, dx.$$

Therefore, if

$$A_1 := \{x \in A : |u(x)| \geq M\|K\|_\infty + \|h\|_\infty + 1\}, \quad A_2 := A \setminus A_1,$$

we deduce from (9) that

$$\begin{aligned}
& (M\|K\|_\infty + \|h\|_\infty + 1) \int_{A_1} |f(x,u(x))|\,dx \\
\leq\ & \int_A |u(x)||f(x,u(x))|\,dx \\
\leq\ & (M\|K\|_\infty + \|h\|_\infty) \int_A |f(y,u(y))|\,dy + M\|h\|_\infty \qquad (10) \\
\leq\ & (M\|K\|_\infty + \|h\|_\infty) \int_{A_1} |f(y,u(y))|\,dy + (M\|K\|_\infty + \|h\|_\infty)F_2 + M\|h\|_\infty
\end{aligned}$$

where

$$F_2 = \int_{A_2} \Phi(x)\,dx$$

and $\Phi \in L^1(A)$ is such that (L^1-Carathéodory condition)

$$|f(x,s)| \leq \Phi(s) \quad \text{whenever} \quad |s| \leq 2(M\|K\|_\infty + \|h\|_\infty).$$

Therefore,

$$\int_{A_2} |f(x,u(x))|\,dx \leq \int_{A_2} \Phi(x)\,dx = F_2, \qquad (11)$$

and, using (10),

$$\int_{A_1} |f(x,u(x))|\,dx \leq (M\|K\|_\infty + \|h\|_\infty)F_2 + M\|h\|_\infty, \qquad (12)$$

so that, using (7), (11) and (12), we have

$$\|u\|_\infty \leq \|K\|_\infty[(M\|K\|_\infty + \|h\|_\infty + 1)F_2 + M\|h\|_\infty] + \|h\|_\infty.$$

■

Remark 1. If there exists $R > 0$ such that

$$uf(x,u) \geq 0 \quad \text{whenever} \quad |u| \geq R, \qquad (13)$$

then, for $|u| \geq R$ and almost every $x \in \mathbb{R}$, one has

$$|uf(x,u)| = uf(x,u)$$

and, for $|u| \leq R$ and almost every $x \in \mathbb{R}$, one has

$$|uf(x,u)| - uf(x,u) \leq 2|uf(x,u)| \leq 2|u|\Phi_R(x),$$

where $\Phi_R \in L^1(A)$ is such that

$$|f(x,s)| \leq \Phi_R(s) \quad \text{whenever} \quad |s| \leq R$$

(L^1-Carathéodory condition). Consequently, condition (4) holds with $\mu = 2\Phi_R$.

Remark 2. If $f(x,\cdot)$ is monotone for a.e. $x \in A$, namely

$$(f(x,u) - f(x,v))(u-v) \geq 0 \quad (a.e.\ x \in A,\ u,v \in \mathbb{R}), \tag{14}$$

then

$$f(x,u)u - f(x,0)u = |f(x,u)u - f(x,0)u| \quad (a.e.\ x \in A,\ u,v \in \mathbb{R}),$$

so that

$$|uf(x,u)| \leq uf(x,u) + 2|f(x,0)||u|$$

and condition (4) holds. Notice that, as shown in [1], the compactness assumption upon K can be weakened when f satisfies condition (14) by replacing Leray-Schauder fixed point theorem by monotonicity methods.

Remark 3. If condition (4) holds, then, for $|u| \geq k$ and a.e. $x \in A$,

$$k|f(x,u)| \leq uf(x,u) + \mu(x)|u|,$$

and hence, if Φ_k is the L^1-function associated to k by the L^1-Carathéodory condition,

$$k|f(x,u)| - uf(x,u) \leq 2k\Phi_k(x) \quad (a.e.\ x \in A,\ |u| \leq k),$$

so that

$$k|f(x,u)| \leq uf(x,u) + 2k\Phi_k(x) + \mu(x)|u| \quad (a.e.\ x \in A,\ u \in \mathbb{R}).$$

Consequently, for each $u \in L^\infty(A)$,

$$k\int_A |f(x,u(x))|\,dx \leq \int_A u(x)f(x,u(x))\,dx + c(k) + \|\mu\|_1\|u\|_\infty \tag{15}$$

where

$$c(k) = 2k\int_A \Phi_k(x)\,dx.$$

Condition (15) with $\mu \equiv 0$, and its abstract version

$$k\|F(u)\|_Y \leq (F(u),u) + c(k) \quad (k \geq 0,\ u \in X),$$

where X, Y are Banach space for which a continuous pairing (y, x) exists, have been introduced and used by Brezis and Browder [1] (see also [4], chapter 4) to state and prove some existence results for Hammerstein integral and abstract equations. We show in the next section that such a result still holds under an abstract version of condition (15).

3. Abstract Hammerstein equations

We state and prove here the following generalization of one of the results of Brezis and Browder [1].

Theorem 2. *Let X and Y be real normed spaces with a bilinear real pairing (y, x), such that $|(y, x)| \leq \|y\|_Y \|x\|_X$ for all $x \in X$ and $y \in Y$. Assume that $F : X \to Y$ is continuous and takes bounded sets into bounded sets. Further assume that $K : Y \to X$ is linear, compact, and such that*

$$(v, Kv) \geq 0 \tag{16}$$

for all $v \in Y$. Assume finally that there exists $M \geq 0$ and that, for each $k \geq 0$, there exists $c(k) \geq 0$ such that

$$k\|Fu\|_Y \leq (Fu, u) + c(k) + M\|u\|_X \tag{17}$$

for all $u \in X$. Then equation

$$u + KFu = h \tag{18}$$

has at least a solution for each $h \in X$.

Proof. As the mapping $KF : X \to X$ is completely continuous, we again apply to (18) Leray-Schauder's fixed point theorem in its simplest form [8], and have to prove that the set of possible solutions of the family of equations

$$u + \lambda KFu = \lambda h \quad (0 \leq \lambda \leq 1) \tag{19}$$

is *a priori* bounded independently of λ. If u is a possible solution of (19) for some $\lambda \in [0, 1]$, then

$$\|u\|_X \leq \|K\| \|Fu\|_Y + \|h\|_X \tag{20}$$

Furthermore,

$$(Fu, u) + \lambda(Fu, KFu) = \lambda(Fu, h),$$

and hence, using assumption (16)

$$(Fu, u) \leq \|Fu\|_Y \|h\|_X. \tag{21}$$

Combining (21) with assumption (17) gives

$$k\|Fu\|_Y \leq \|Fu\|_Y \|h\|_X + c(k) + M\|u\|_X$$

and hence, using (20) and taking $k = \|h\|_X + M\|K\| + 1$, we get

$$\|Fu\|_Y \leq M\|h\|_X + c(\|h\|_X + M\|K\|)$$

and therefore

$$\|u\|_X \leq \|K\|[M\|h\|_X + c(\|h\|_X + M\|K\|)] + \|h\|_X.$$

■

Remark 4. In a paper [3] devoted to the study of abstract semilinear equations

$$Lx + Nx = 0 \tag{22}$$

where $L : D(L) \subset X \to Z$ is a noninvertible Fredholm operator of index zero, $N : X \to Z$ a nonlinear mapping satisfying some compactness assumption, and X, Z are real normed spaces, Cañada and Ortega have introduced and used growth conditions of the form

$$\|Nu\|_Z \leq \langle Nu, \zeta u\rangle + \alpha\|u\|_X + \beta \quad (u \in X) \tag{23}$$

where α, β are nonnegative constants, $\zeta : X \to Z^*$ and $\langle z, z^*\rangle$ is the usual pairing between Z and Z^*. When $\langle Lu, \zeta u\rangle \geq 0$, they use coincidence degree [9] to find, under further conditions upon N, existence theorems for (22), and apply it to Picard and periodic boundary value problems for second order differential systems at resonance. Condition (23) can also been seen as some abstract version of Leray's condition for equations of type (22) instead of (18).

4. Conclusion

In their epoch making paper [8], Leray and Schauder started Section IV 'Applications' with the sentence :

> Let us first mention that the existence theorems established through Arzelá-Schmidt's method are all special cases of the fundamental theorem stated above.

This sentence have been overlooked by most readers of Leray-Schauder's paper, and very few people if any came back to the examples given in Leray's interesting thesis [7]. We have shown in [10] that most of those examples anticipated some fundamental results of nonlinear functional analysis proved in the second half of the XXth century. This short note is another example.

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