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A KRASNOSELSKII TYPE FIXED-POINT THEOREM ON CONVEX CONES

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Dedicated to Professor Ioan A. Rus on the occasion of his 70th birthday

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Abstract. We present in this paper a new variant of a classical fixed-point theorem on cones, due to M. A. Krasnoselskii. For our result it is not necessary to have a generating cone and the topological degree is not used.

Key Words and Phrases: fixed-point, convex cone, scalar asymptotic derivative.
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1. Introduction

It is well known that the fixed-point theorems of nonlinear mappings, with respect to closed convex cones have been studied by several authors [1]-[3], [5], [8]-[11], [13], [16]. The development of fixed-point theory with respect to convex cones is justified by many and interesting applications. For example the fixed-point theorems on cones are used in the study of positive solutions of nonlinear equations, in the study of periodical solutions of dynamical systems in the study of complementarity problems and in the study of some problems considered in economics among others.

We note that in particular the complementarity problems have many applications in economics, in optimization and in engineering, [4-7].

The study of fixed-points of nonlinear mappings with respect to convex cones, was initiated by the Russian mathematical school and in particular by M. A. Krasnoselskii and his students [11]-[13], [16].

For new applications of fixed-point theorems on cones in domain as the complementarity theory and the theory dynamical systems it is useful to have new fixed-point theorems on cones.

In this paper we present a new variant of a classical fixed-point theorem on a convex cone due to M. A. Krasnoselskii [12].

Our fixed-point theorem is with respect to a convex cone in a Hilbert space, but we do not suppose that the cone is generating and the complete continuity of the operator is replaced by the scalar compactness. We do not use the topological degree. We use the notion of scalar asymptotic derivability and the notion of variational inequality.

Perhaps, the ideas used in this paper can be used to obtain new fixed-point theorems on convex cones.

2. Preliminaries

We introduce in this section some notions and we recall some definitions. Let $(H, \langle\cdot\rangle)$ be an arbitrary real Hilbert space. We say that a non-empty subset $\mathbb{K}$ of H is a pointed, convex cone if the following conditions are satisfied:

k_1) $\mathbb{K} + \mathbb{K} \subseteq \mathbb{K}$,

k_2) $\lambda\mathbb{K} \subseteq \mathbb{K}$, for any $\lambda \in \mathbb{R}_+$,

k_3) $\mathbb{K} \cap (-\mathbb{K}) = \{0\}$.

In this paper any pointed convex cone will be closed with respect to the topology defined by the norm of H. We say that $\mathbb{K}$ is *generating* if $H = \mathbb{K} - \mathbb{K}$. By definition the dual cone $\mathbb{K}^*$ of $\mathbb{K}$ is: $\mathbb{K}^*\{y \in H \mid \langle x, y\rangle \geq 0 \text{ for all } x \in \mathbb{K}\}$. It is known that $\mathbb{K}^*$ is a closed convex cone. If $\mathbb{K}$ is total in H, i.e., $H = \overline{\mathbb{K} - \mathbb{K}}$, then $\mathbb{K}^*$ is a pointed cone. The pointed convex cone $\mathbb{K}$ defines an ordering on H by "$x \leq y$" if and only if $y - x \in \mathbb{K}$. If $\mathbb{K}$ is not pointed this relation is only a quasi-ordering. Given a non-empty subset D in H and a mapping $h : H \to H$, the *variational inequality* problem associated to h and D is:

$$VIP(h, D) : \begin{cases} \text{find } x_0 \in D \text{ such that} \\ \langle h(x_0), x - x_0\rangle \geq 0 \text{ for all } x \in D. \end{cases}$$

If in the problem $VIP(h, D)$, the set D is a closed convex cone $\mathbb{K}$, we have the *nonlinear complementarity problem* defined by h and $\mathbb{K}$, i.e.,

$$NCP(h, \mathbb{K}) : \begin{cases} \text{find } x_0 \in \mathbb{K} \text{ such that} \\ h(x_0) \in \mathbb{K}^* \text{ and } \langle x_0, h(x_0) \rangle = 0. \end{cases}$$

From the complementarity theory we know the following facts [6], [7].

I. *The problems $VIP(h, D)$ and $NCP(h, \mathbb{K})$ are equivalent.*

II. *If $h : \mathbb{K} \to H$ has the form $h(x) = x - f(x)$, where $f : \mathbb{K} \to \mathbb{K}$, then f has a fixed-point in $\mathbb{K}$, if and only if, the problem $NCP(h, \mathbb{K})$ has a solution.*

Finally we say that a mapping $f : \mathbb{K} \to H$ is demi-continuous, if for any sequence $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{K}$, is strongly convergent to an element $x_* \in \mathbb{K}$ we have that $\{f(x_n)\}_{n\in\mathbb{N}}$ is weakly convergent to $f(x_*)$. Also, we say that $f : \mathbb{K} \to H$ is *completely continuous* if f is continuous and for any bounded subset $B \subset \mathbb{K}$, we have that $f(B)$ is *relatively compact*.

3. The main result

Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space and $\mathbb{K} \subset H$ a closed, pointed convex cone. We suppose that $\mathbb{K}$ is generating, i.e., $H = \mathbb{K} - \mathbb{K}$. The set of all linear continuous operators from H into H will be denoted by $\mathcal{L}(H)$. The following notion is due to M. A. Krasnoselskii [12]. We say that a mapping $f : \mathbb{K} \to H$ is *asymptotically linear along* $\mathbb{K}$ if there exists $T \in \mathcal{L}(H)$ such that

$$\lim_{\substack{\|x\|\to\infty \\ x\in\mathbb{K}}} \frac{\|f(x) - T(x)\|}{\|x\|} = 0.$$

In this case, because $\mathbb{K}$ is generating in H, we have that T is unique. Moreover, if f is completely continuous we have that T is a linear completely continuous operator and $T(\mathbb{K}) \subseteq \mathbb{K}$. If f is asymptotically linear along $\mathbb{K}$, we denote $f'_\infty = T$ and we say that f'_∞ is the *asymptotic derivative* of f along the cone $\mathbb{K}$. For more information about the notion of asymptotic derivative we recommend [1], [10], [12], [13].

In our main result we will use the notion of *scalar asymptotic derivative*, which is related to the notion of *scalar derivative* due to S. Z. Nemeth [14], [15]. We note that we introduced the notion of scalar asymptotic derivative being inspired by the notion of scalar derivative.

Definition 1. We say that $T \in \mathcal{L}(H)$ is a scalar asymptotic derivative of a mapping $f : \mathbb{K} \to H$, along the cone $\mathbb{K}$ if

$$\lim_{\substack{\|x\|\to\infty \\ x\in\mathbb{K}}} \frac{\langle f(x) - T(x), x\rangle}{\|x\|^2} = 0.$$

If an operator $T \in \mathcal{L}(H)$ satisfies Definition 1 it will be denoted by $f'_s(\infty)$ (i.e., $f'_s(\infty) := T$. We observe that if $T \in \mathcal{L}(H)$ is an *asymptotic derivative* of f along the cone $\mathbb{K}$, then f is also a *scalar asymptotic derivative* of f. This fact is a consequence of the following relation:

$$\limsup_{\substack{\|x\|\to\infty \\ x\in\mathbb{K}}} \frac{\langle f(x) - T(x), x\rangle}{\|x\|^2} \le \lim_{\substack{\|x\|\to\infty \\ x\in\mathbb{K}}} \frac{\|f(x) - T(x)\|}{\|x\|} = 0.$$

Our main result is related to the following classical result due to M. A. Krasnoselskii.

Theorem (Krasnoselskii Type Theorem). *Let $(E, \|\cdot\|)$ be a Banach space and $\mathbb{K} \subset E$ a generating, closed pointed convex cone. Let $f : \mathbb{K} \to \mathbb{K}$ be an asymptotically linear and completely continuous mapping. If the spectral radius of the asymptotic derivative $f'_s(\infty)$ of f is strictly less that 1, i.e., $r(f'_\infty) < 1$, then f has a fixed-point.*

Proof. H. Amman [1] gives a proof of this theorem based on the topological index. □

We note that this Krasnoselskii Type Theorem is essentially Theorem 4.7 [12]. The proof given in [12] is complicated and does not use the topological degree. In [12] and [2] are given application of this theorem and of variants of this theorem. Our main result is a variant of this theorem but in Hilbert spaces and our proof is not based on the topological degree. The proof is based on some techniques developed in the theory of Variational Inequalities and in the Complementarity Theory. We will use the following notion.

If $h, g : \mathbb{K} \to H$ are two mappings, the relation $h \underset{(\mathbb{K}^*)}{\le} g$ means, $g(x) - h(x) \in \mathbb{K}^*$ for all $x \in \mathbb{K}$. In this case, in particular we have $\langle h(x), x\rangle \le \langle g(x), x\rangle$ for all $x \in \mathbb{K}$.

Definition 2. We say that a mapping $f : \mathbb{K} \to H$ is scalarly compact if for any sequence $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{K}$, weakly convergent to an element $x_* \in \mathbb{K}$, there exists a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ such that

$$\limsup_{k\to\infty}\langle x_{n_k}-x_*, f(x_{n_k})\rangle \le 0.$$

Examples.

(1) Any completely continuous mapping is scalarly compact.

(2) Given a mapping $f:\mathbb{K}\to H$, if there exists a completely continuous mapping $h:\mathbb{K}\to H$ such that $|\langle y, f(x)\rangle| \le \langle y, h(x)\rangle$ or $\langle y, f(x)\rangle \le |\langle y, h(x)\rangle|$ for any $x, y \in \mathbb{K}$, then f is scalarly compact.

Theorem 1. *Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space, $\mathbb{K}\subset H$ a pointed closed convex cone and $f:\mathbb{K}\to\psi\mathbb{K}$ a mapping.*

If the following assumptions are satisfied:

(i) f is demicontinuous,

(ii) f is scalarly compact,

(iii) there exists a scalar asymptotically derivable mapping $f_0:\mathbb{K}\to H$ such that

$$f \underset{(\mathbb{K}^*)}{\le} f_0 \text{ and } \|f'_{0s}(\infty)\| < 1,$$

then f has a fixed-point in $\mathbb{K}$.

Proof. We define $h = I - f$, where I is the identity mapping. From Complementarity Theory we know that f has a fixed-point in $\mathbb{K}$ if and only if the problem $NCP(h,\mathbb{K})$ has a solution. For every $m\in\mathbb{N}$ we define the set $\mathbb{K}_m = \{x\in\mathbb{K} \mid \|x\| \le m\}$ and we observe that $\mathbb{K}_m$ is closed, convex, weakly closed and $\mathbb{K} = \bigcup_{m=1}^{\infty}\mathbb{K}_m$. Obviously, any set $\mathbb{K}_m$ is bounded. First, we show that for every $m\in\mathbb{N}$, the problem $VIP(I-f,\mathbb{K}_m)$ has a solution $y_m^*\in\mathbb{K}_m$. Indeed, let $m\in\mathbb{N}$ be arbitrary and denote by Λ the family of all finite dimensional subspaces of H ordered by inclusion. Consider the mapping $h(x) = x - f(x)$ for all $x\in\mathbb{K}$ and define $\mathbb{K}_m(E) = \mathbb{K}_m\cap E$ for each $E\in\Lambda$. For each $E\in\Lambda$ we set $A_E = \{y\in\mathbb{K}_m \mid \langle h(y), x-y \ge 0$ for all $x\in\mathbb{K}_m(E)\}$ and we have that A_E is non-empty. Indeed, the solution set of the problem $VI(h,\mathbb{K}_m(E))$ is a subset of A_E, but the solution set of $VI(h,\mathbb{K}_m(E))$ is non-empty. To see this, we consider the mappings $j:E\to H$ and $j^*:H^*\to E^*$, where j is the inclusion and j^* is the adjoint of j. The mapping $j^*\circ h\circ j:\mathbb{K}_m(E)\to E^*$ is continuous and

$$\langle j^*\circ h\circ j(y), x-y\rangle = \langle h(j(y)), j(x-y)\rangle = \langle h(y), x-y\rangle, \text{ for all } x, y\in\mathbb{K}_m(E).$$

Applying the classical Hartman-Stampacchia Theorem to the set $\mathbb{K}_m(E)$ and to the mapping $j^* \circ h \circ j$ we obtain that the problem $VI(h, \mathbb{K}_m(E))$ has at least a solution. For every $E \subset \Lambda$, we denote $\overline{A}_E^\sigma$ the weak closure of A_E. We have $\bigcap_{E\in\Lambda} \overline{A}_E^\sigma$ is non-empty. Indeed, let $\overline{A}_{E_1}^\sigma, \overline{A}_{E_2}^\sigma, \dots, \overline{A}_{E_n}^\sigma$ be a finite subfamily of the family $\{\overline{A}_E^\sigma\}_{E\in\Lambda}$. Let M be the finite dimensional subspace in H generated by $E_1, E_2, \dots, E_n$. Because $E_k \subseteq M$ for all $k \in \{1, 2, \dots, n\}$ we have that $\mathbb{K}_m(E_k) \subseteq E_m(M)$ for all $k \in \{1, 2, \dots, n\}$, which implies that $\bigcap_{k=1} \overline{A}_{E_k}^\sigma$ is non-empty. The weak compactness of $\mathbb{K}_m$ implies that $\bigcap_{E\in\Lambda} \overline{A}_E^\sigma \neq \emptyset$. Let $y_m^* \in \bigcap_{E\in\Lambda} \overline{A}_E^\sigma$ be arbitrary and let $x \in \mathbb{K}_m$ be any element of this set. There exists some $E \in \Lambda$ such that $x, y_m^* \in E$. Since $y_m^* \in \overline{A}_E^\sigma$ there exists a sequence $\{y_n\}_{n\in\mathbb{N}} \subset A_E$ such that $\{y_n\}_{n\in\mathbb{N}}$ is weakly convergent to y_m^*. (We applied Şmulian's Theorem). We have

$$\langle h(y_n), y_m^* - y_n\rangle \geq 0,$$

and

$$\langle h(y_n), x - y_n\rangle \geq 0,$$

or

$$\langle y_n, y_n - y_m^*\rangle \leq \langle f(y_n), y_n - y_m^*\rangle, \tag{1}$$

and

$$\langle y_n, x - y_n\rangle \geq \langle f(y_n), x - y_n\rangle. \tag{2}$$

From (1) and assumption (ii) we have that $\{y_n\}$ has a subsequence denoted again by $\{y_n\}$ such that

$$\limsup_{n\to\infty} \langle y_n, y_n - y_m^*\rangle \leq 0, \tag{3}$$

which implies

$$\begin{aligned} 0 \leq \limsup_{n\to\infty} \|y_n - y_m^*\|^2 &= \limsup_{n\to\infty} \langle y_n - y_m^*, y_n - y_m^*\rangle \\ &\leq \limsup_{n\to\infty} \langle y_n, y_n - y_m^*\rangle + \limsup_{n\to\infty} [-\langle y_m^*, y_n - y_m^*\rangle] \leq 0. \end{aligned}$$

We deduce that $\{y_n\}$ is strongly convergent to y_m^*. Because f is demicontinuous we have that $\{f(y_n)\}_{n\in\mathbb{N}}$ is weakly convergent to $f(y_m^*)$.

From inequality (2) we have $\langle y_m^* - f(y_m^*), x - y_m^*\rangle \leq 0$ for any $x \in \mathbb{K}_m$, that is i.e., y_m^* is a solution of the problem $VI(I - f, \mathbb{K}_m)$. (We note that to obtain the last inequality we use also the following fact: "if $\{u_n\}$ is weakly convergent to an element u^* and $\{v_n\}$ is strongly convergent to an element v^*, then $\lim\limits_{n\to\infty}\langle u_n, v_n\rangle = \langle u_*, v_*\rangle$".)

Now, we pass to the second part of the proof. In the first part we proved that for every $m \in \mathbb{N}$, the problem $VI(I-f, \mathbb{K}_m)$ has a solution y_m, i.e.,

$$\langle y_m - f(y_m), x - y_m\rangle \geq 0, \text{ for all } x \in \mathbb{K}_m. \tag{4}$$

Taking $x = 0$ in (4) we obtain

$$\langle y_m, y_m\rangle \leq \langle f(y_m), y_m\rangle. \tag{5}$$

The sequence $\{y_m\}_{m\in\mathbb{N}}$ is bounded.

Indeed, if this is false, we may assume that $\|y_m\| \to \infty$ as $m \to \infty$, which implies (using (5) and assumption (iii))

$$\begin{aligned}
1 = \frac{\langle y_m, y_m\rangle}{\|y_m\|^2} &\leq \limsup_{\|y_m\|\to\infty} \frac{\langle f(y_m), y_m\rangle}{\|y_m\|^2} \leq \limsup_{\|y_m\|\to\infty} \frac{\langle f_0(y_m), y_m\rangle}{\|y_m\|^2} \\
&\leq \limsup_{\|y_m\|\to\infty} \frac{\langle f_0(y_m) - f'_{0s}(\infty)(y_m), y_m\rangle}{\|y_m\|^2} + \limsup_{\|y_m\|\to\infty} \frac{\langle f'_{0s}(\infty)(y_m), y_m\rangle}{\|y_m\|^2} \\
&\leq \limsup_{\|y_m\|\to\infty} \frac{\|f'_{0s}(\infty)\|\|y_m\|^2}{\|y_m\|^2} = \|f'_{0s}(\infty)\| < 1.
\end{aligned}$$

We have a contradiction and therefore $\{y_m\}_{m\in\mathbb{N}}$ is a bounded sequence. By the reflexivity of H and the weak closedness of $\mathbb{K}$ we have that there exists a subsequence $\{y_{m_k}\}_{k\in\mathbb{N}}$ of the sequence $\{y_m\}_{m\in\mathbb{N}}$, weakly convergent to an element $y_0 \in \mathbb{K}$. For all $x \in \mathbb{K}$, there exists a natural number m_0 (i.e. $m_0 \in \mathbb{N}$) such that y_0 and x are in $\mathbb{K}_{m_0}$.

Thus, for all $m \geq m_0$ we have $y_0, x \in \mathbb{K}_m$.

We have

$$\langle y_m - f(y_m), y_0 - y_m\rangle \geq 0 \tag{6}$$

and

$$\langle y_m - f(y_m), x - y_m\rangle \geq 0. \tag{7}$$

Using inequalities (6) and the scalar compactness of f (i.e.), assumption (ii)) we have that there exists a subsequence $\{y_{m_k}\}_{k\in\mathbb{N}}$ of the sequence $\{y_m\}_{m\in\mathbb{N}}$ such that

$$\limsup_{k\to\infty}\langle y_{m_k}, y_{m_k} - y_0\rangle \leq \limsup_{k\to\infty}\langle f(y_{m_k}), y_{m_k} - y_0\rangle \leq 0,$$

which implies that $\{y_{m_k}\}_{k\in\mathbb{N}}$ is strongly convergent to y_0, as we can see considering the following inequalities

$$0 \le \limsup_{k\to\infty} \|y_{m_k} - y_0\|^2 = \limsup_{k\to\infty} \langle y_{m_k} - y_0, y_{m_k} - y_0\rangle$$
$$\le \limsup_{k\to\infty} \langle y_{m_k}, y_{m_k} - y_0\rangle + \limsup_{k\to\infty} [-\langle y_0, y_{m_k} - y_0\rangle] \le 0.$$

Considering (7) for all $m_k \ge m_0$ we have

$$\langle y_{m_k} - f(y_{m_k}), x - y_{m_k}\rangle \ge 0.$$

Computing the limit in the previous inequality we obtain

$$\langle y_0 - f(y_0), x - y_0\rangle \ge 0 \text{ for any } x \in \mathbb{K}.$$

Therefore $f(y_0) = y_0$ and the proof is complete. □

Corollary 1. *Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space, $\mathbb{K} \subset H$ a pointed closed convex cone and $f : \mathbb{K} \to \mathbb{K}$ a mapping. If the following assumptions are satisfied:*

(i) f is demicontinuous,

(ii) f is scalarly compact,

(iii) f has a scalar asymptotically derivative $f'_s(\infty)$ and $\|f'_s(\infty)\| < 1$,

then f has a fixed-point in $\mathbb{K}$.

Corollary 2. *Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space and $\mathbb{K} \subset H$ a generating closed pointed convex cone. Let $f : \mathbb{K} \to \mathbb{K}$ be a completely continuous mapping. If f is asymptotically linear and $\|f'(\infty)\| < 1$, then f has a fixed-point in $\mathbb{K}$.*

Corollary 3. *Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space and $\mathbb{K} \subset H$ a generating closed pointed convex cone. Let $f : \mathbb{K} \to \mathbb{K}$ be a completely continuous mapping. If there exists an asymptotically linear mapping $f_0 : \mathbb{K} \to \mathbb{K}$ such that $f \underset{(\mathbb{K}^*)}{\le} f_0$ and $\|f'_0(\infty)\| < 1$ then f has a fixed-point in $\mathbb{K}$.*

4. Application to Complementarity Theory

Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space, $\mathbb{K} \subset H$ a closed pointed convex cone and $f : \mathbb{K} \to \mathbb{K}$ a mapping. We consider the general nonlinear complementarity problem

$$NCP(f, \mathbb{K}) : \begin{cases} \text{find } x_0 \in \mathbb{K} \text{ such that} \\ f(x_0) \in \mathbb{K}^* \text{ and } \langle x_0, f(x_0)\rangle = 0. \end{cases}$$

We say that f is a completely continuous field if f has a representation of the form $f(x) = x - T(x)$, for any $x \in H$, where $T : H \to H$ is a completely continuous mapping. Also, we say that f is an *asymptotically derivable field* with respect to $\mathbb{K}$, if f has a representation of the form $f(x) = x - T(x)$, for any $x \in H$, where $T : H \to H$ has an asymptotic derivative T'_∞ along $\mathbb{K}$. We have the following result related to the problem $NCP(f, \mathbb{K})$.

Theorem 4. *Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space and $\mathbb{K} \subset H$ a generating, pointed, closed convex cone. Let $f : H \to H$ a mapping. The mapping f is supposed to be a completely continuous and asymptotically derivable field of the form $f(x) = x - T(x)$ for any $x \in H$. If $\|T'_\infty(\infty)\| < 1$ and $T'_\infty(\mathbb{K}) \subseteq \mathbb{K}$, then the problem $NCP(f, \mathbb{K})$ has a solution.*

Proof. From the Complementarity Theory it is known that the problem $NCP(f, \mathbb{K})$ has a solution, if and only if the mapping $\Phi(x) = P_{\mathbb{K}}[x - f(x)] = P_{\mathbb{K}}[T(x)]$ has a fixed-point. Obviously, $\Phi(\mathbb{K}) \subseteq \mathbb{K}$ and Φ is a completely continuous mapping. Therefore, Φ is demicontinuous and scalarly compact. Because $T'_\infty(\mathbb{K}) \subseteq \mathbb{K}$, we have that for any $x \in \mathbb{K}$, $T'_\infty(x) = P_{\mathbb{K}}[T'_\infty(x)]$, and consequently,

$$\lim_{\substack{\|x\| \to \infty \\ x \in \mathbb{K}}} \frac{\|P_{\mathbb{K}}[T(x)] - T'_\infty(x)\|}{\|x\|} \leq \lim_{\substack{\|x\| \to \infty \\ x \in \mathbb{K}}} \frac{\|P_{\mathbb{K}}[T(x)] - P_{\mathbb{K}}[T'_\infty(x)]\|}{\|x\|}$$

$$\leq \lim_{\substack{\|x\| \to \infty \\ x \in \mathbb{K}}} \frac{\|T(x) - T'_\infty(x)\|}{\|x\|} = 0.$$

We have that T'_∞ is also the asymptotic derivative of the mapping Φ, which implies that $\Phi'_s(\infty) = T'_\infty$. Because the assumptions of Theorem 1 are satisfied, our theorem is proved. $\square$

Remarks.

(1) The assumption $T'_\infty(\mathbb{K}) \subseteq \mathbb{K}$ is satisfied if $T(\mathbb{K}) \subseteq \mathbb{K}$. (See [1], [12])

(2) Theorem 4 is applicable to complementarity problems defined by completely continuous fields of the form $f(x) = x - T(x)$, where T is an integral operator. It is known that many nonlinear integral operators (as for example, Hammerstein operators or Urysohn operators) are under some conditions asymptotically derivable [11], [12] and [13].

(3) Theorem 4 is also applicable to complementarity problems $NCP(f, \mathbb{K})$, where f has a representation of the form $f(x) = \alpha x - T(x)$, where α is

a a positive real number and $T : H \to H$ is a completely continuous and asymptotically derivable operator. In this case, in the proof of Theorem 4 we consider the mapping

$$\Psi(x) = P_{\mathbb{K}}\left[x - \frac{1}{\alpha}f(x)\right] = P_{\mathbb{K}}\left[\frac{1}{\alpha}T(x)\right],$$

and we must suppose that $\|T'_\infty\| < \alpha$.

Another interesting application of Theorem 1 to complementarity problems is when the cone $\mathbb{K}$ is an isotone projection cone (see [7] and [4]) and $\mathbb{K}$ is self-adjoint, i.e., $\mathbb{K} = \mathbb{K}^*$. In this case if $f(x) = x - T(x)$ and there exists a mapping T_0 such that $T_0 : H \to H$ and $T(x) \underset{(\mathbb{K})}{\leq} T_0(x)$ for any $x \in H$, we have that $P_{\mathbb{K}}[T(x)] \leq P_{\mathbb{K}}[T_0(x)]$. If T_0 has an asymptotic derivative $(T_0)'_\infty$ such that $(T_0)'_\infty(\mathbb{K}) \subseteq \mathbb{K}$, and the mapping f is a completely continuous field, then by Theorem 1 we have that the problem $NCP(f, \mathbb{K})$ has a solution if in addition $\|(T_0)'_\infty\| < 1$.

Comments. We presented in this paper a fixed-point theorem which is a variant of a classical fixed-point theorem on convex cones, due to M. A. Krasnoselskii. We note two aspects related to our theorem.

First, the proof is not based on the topological degree but on some techniques developed in complementarity theory.

Second, because the comparison between f and f_0, assumption (iii), enlarges the applicability of Krasnoselskii type fixed-point theorems on cones. In this sense, the applicability of this theorem to the study of complementarity problems defined by asymptotically derivable fields opens a new research direction in Complementarity Theory.

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