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SHADOWING IN PARAMETERIZED IFS

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Dedicated to Professor Ioan A. Rus on the occasion of his 70th birthday

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Abstract. The Shadowing Property is stated for set-valued dynamical systems, generated by parameterized IFS, which are uniformly contracting, or uniformly expanding, or products of such ones. We also prove that a parameterized IFS with "condensation", consisting of an affine function and a constant compact-valued multi-function, has the Shadowing Property if and only if the affine function is a contraction.

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Introduction

Theory of shadowing in discrete or continuous (ordinary) dynamical systems has evolved spectacularly during the last decade (see, i.e. [7, 8] and the bibliography therein). A new discipline coined as "numerical dynamics" has appeared.

As for set-valued dynamical systems some attempts to generalize this theory have been made. E.Sander [10] treated smooth relations and proposed a notion of hyperbolicity for such relations. He stated the Shadowing Property for hyperbolic smooth relations.

The authors proposed a generalization for iterations of multi-functions with bounded and closed values and stated the Shadowing Property for weakly contracting multi-functions [1, 2, 3].

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A concept of shadowing with respect to a given stratification of the phase space has been proposed by Ioan A. Rus [9], and sufficient conditions for a dynamical system to have the Shadowing Property in this sense, with emphasis on weakly Picard operators, have been stated.

In this paper we are concerned with set-valued dynamical systems generated by parameterized IFS. The main results state the Shadowing Property for uniformly expanding or contracting IFS, and for products of such IFS. These results represent a generalization of the Shadowing Property, stated for hyperbolic linear operators (see, i.e. [5, 6]). We give also a criterion of shadowing for affine functions with "condensation".

1. Definitions

Let (X, d) be a complete metric space. Following [4], we call as a *parameterized Iterated Function System* (*IFS*) $\mathcal{F} = \{X; f_\lambda \mid \lambda \in \Lambda\}$ any family of continuous mappings $f_\lambda : X \to X, \lambda \in \Lambda$, where Λ is an arbitrary nonempty set.

We shall say that the parameterized IFS $\mathcal{F} = \{X; f_\lambda \mid \lambda \in \Lambda\}$ is *uniformly contracting* if there exists

$$\beta := \sup_{\lambda \in \Lambda} \sup_{x \neq y} \frac{d(f_\lambda(x), f_\lambda(y))}{d(x, y)},$$

and this number, called also the *contracting ratio*, is less than one.

Respectively, we shall say that $\mathcal{F}$ is *uniformly expanding* if

$$\alpha := \inf_{\lambda \in \Lambda} \inf_{x \neq y} \frac{d(f_\lambda(x), f_\lambda(y))}{d(x, y)} > 1.$$

We call α the *expanding ratio*.

Let $\mathcal{F} = \{X; f_\lambda \mid \lambda \in \Lambda\}$ be a parameterized IFS and let $T = \mathbb{Z}$ or $T = \mathbb{Z}_+ := \{n \in \mathbb{Z} : n \geq 0\}$. A sequence $\{x_n\}_{n \in T}$ in X is called a *chain* of the IFS $\mathcal{F}$ if for any $n \in T$ there exists $\lambda_n \in \Lambda$ such that $x_{n+1} = f_{\lambda_n}(x_n)$. Given $\delta > 0$, a sequence $\{x_n\}_{n \in T}$ in X is called a δ-*chain* of $\mathcal{F}$ if for any $n \in T$ there exists $\lambda_n \in \Lambda$ such that $d(x_{n+1}, f_{\lambda_n}(x_n)) \leq \delta$.

One says that the IFS $\mathcal{F}$ has the *Shadowing Property* (*on* T) if, given $\varepsilon > 0$, there exists $\delta > 0$ such that for any δ-chain $\{x_n\}_{n \in T}$ there exists a chain $\{y_n\}_{n \in T}$, satisfying the inequality $d(x_n, y_n) \leq \varepsilon$ for all $n \in T$. In this case one says that the chain $\{y_n\}_{n \in T}$ ε-*shadows* the δ-chain $\{x_n\}_{n \in T}$.

Remark 1.1. The authors [2, 3] considered upper semi-continuous multi-functions with closed and bounded, as well as with compact, values. The parameterized IFS can be regarded as a multi-function, but this multi-function need not be upper semi-continuous, nor with closed or bounded values, because we do not impose any restriction on the parameter set Λ. In this connection we adjust the notions of orbit (chain), pseudo-orbit (pseudo-chain), and we treat the Shadowing Property for parameterized IFS similarly as for linear operators, i.e. in its own rights, without localization of this phenomenon. Of course, in this case one cannot speak about closed, or compact attractors, or about other limit sets (as, for example, fractals as attractors of contracting IFS, or "semi-fractals", as limit sets of IFS with a subsystem of contracting elements [4]).

2. Shadowing in Uniformly Contracting and Uniformly Expanding IFS

Theorem 2.1. *If a parameterized IFS $\mathcal{F} = \{X; f_\lambda \,|\, \lambda \in \Lambda\}$ is uniformly contracting, then it has the Shadowing Property on $\mathbb{Z}_+$.*

Proof. Assume that the IFS $\mathcal{F}$ is uniformly contracting with the contracting ratio β. Given $\varepsilon > 0$ take $\delta = (1-\beta)\varepsilon/2 \leq \varepsilon/2$, and let $\{x_n\}_{n\geq 0}$ be a δ-chain of $\mathcal{F}$. This means that for any $n \geq 0$ there exists $\lambda_n \in \Lambda$ such that $d(x_{n+1}, f_{\lambda_n}(x_n)) \leq \delta$. Consider a chain $\{y_n\}_{n\geq 0}$ such that $d(x_0, y_0) \leq \varepsilon/2$ and $y_{n+1} = f_{\lambda_n}(y_n)$ for all $n \geq 0$. We shall show that $\{y_n\}_{n\geq 0}$ ε-shadows the δ-chain $\{x_n\}_{n\geq 0}$. For, observe that

$$d(x_1, y_1) \leq d(x_1, f_{\lambda_0}(x_0)) + d(f_{\lambda_0}(x_0), f_{\lambda_0}(y_0)) \leq \delta + \beta d(x_0, y_0).$$

Next, one can show by induction that for any $n \geq 1$

$$d(x_n, y_n) \leq \delta(1 + \beta + \cdots + \beta^{n-1}) + \beta^n d(x_0, y_0).$$

The last inequality together with $d(x_0, y_0) \leq \varepsilon/2$ imply the desired one:

$$d(x_n, y_n) \leq \delta \frac{1}{1-\beta} + d(x_0, y_0) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad (n \geq 1).$$

□

Remark 2.1. The authors [2] have proved that any contracting set-valued mapping with closed and bounded values has the Shadowing Property.

The following theorem is one of the main results of the paper.

Theorem 2.2. *If a parameterized IFS $\mathcal{F} = \{X; f_\lambda \mid \lambda \in \Lambda\}$ is uniformly expanding and if each function $f_\lambda (\lambda \in \Lambda)$ is surjective, then the IFS has the Shadowing Property on $\mathbb{Z}_+$.*

Proof. Given $\lambda \in \Lambda$, consider the function $\varphi_\lambda : X \times X \to \mathbb{R}_+$, defined by

$$\varphi_\lambda(x,y) = \begin{cases} \dfrac{d(f_\lambda(x), f_\lambda(y))}{d(x,y)}, & \text{if} \quad x \neq y, \\ \alpha, & \text{if} \quad x = y, \end{cases}$$

where $\alpha > 1$ is the expanding ratio.

One has

$$d(x,y) = \frac{d(f_\lambda(x), f_\lambda(y))}{\varphi_\lambda(x,y)}, \quad \varphi_\lambda(x,y) \geq \alpha \quad (x, y \in X,\ \lambda \in \Lambda). \tag{2.1}$$

Given $\varepsilon > 0$, take $\delta = (\alpha - 1)\varepsilon$ and let $\{x_n\}_{n \geq 0}$ be a δ-chain of $\mathcal{F}$, i.e. for any $n \geq 0$ there exists $\lambda_n \in \Lambda$ such that $d(x_{n+1}, f_{\lambda_n}(x_n)) \leq \delta$. Consider the sequence $\{z_n\}_{n \geq 0}$ in X, defined as follows (recall that an expanding and surjective function is invertible):

$$z_0 = x_0, \qquad z_n = (f_{\lambda_0}^{-1} \circ \cdots \circ f_{\lambda_{n-1}}^{-1})(x_n), \quad \forall\, n \geq 1.$$

Obviously, $x_n = (f_{\lambda_{n-1}} \circ \cdots \circ f_{\lambda_0})(z_n)$ $(n \geq 1)$. Given $n \geq 1$ and $0 \leq k \leq n-1$, denote

$$z_n^{(k)} = (f_{\lambda_k} \circ \cdots \circ f_{\lambda_0})(z_n). \tag{2.2}$$

Therefore, for any $n \geq 1$ and $1 \leq k \leq n-1$, one has

$$z_n^{(k)} = f_{\lambda_k}(z_n^{(k-1)}), \qquad x_n = z_n^{(n-1)} = f_{\lambda_{n-1}}(z_n^{(n-2)}). \tag{2.3}$$

We claim that $\{z_n\}_{n \geq 0}$ is a Cauchy sequence. Firstly, fixing $n \geq 1$ and $p \geq 1$, and using (2.1), (2.2) and (2.3), we obtain

$$d(z_n, z_{n+p}) = \frac{d(f_{\lambda_0}(z_n), f_{\lambda_0}(z_{n+p}))}{\varphi_{\lambda_0}(z_n, z_{n+p})} =$$

$$= \frac{d(z_n^{(0)}, z_{n+p}^{(0)})}{\varphi_{\lambda_0}(z_n, z_{n+p})} = \frac{d(z_n^{(1)}, z_{n+p}^{(1)})}{\varphi_{\lambda_0}(z_n, z_{n+p})\varphi_{\lambda_1}(z_n^{(0)}, z_{n+p}^{(0)})} =$$

$$\ldots = \frac{d(z_n^{(n-1)}, z_{n+p}^{(n-1)})}{\varphi_{\lambda_0}(z_n, z_{n+p}) \prod_{i=1}^{n-1} \varphi_{\lambda_i}(z_n^{(i-1)}, z_{n+p}^{(i-1)})} =$$

$$= \frac{d(x_n, z_{n+p}^{(n-1)})}{\varphi_{\lambda_0}(z_n, z_{n+p}) \prod_{i=1}^{n-1} \varphi_{\lambda_i}(z_n^{(i-1)}, z_{n+p}^{(i-1)})}. \tag{2.4}$$

Secondly, by induction on $p \geq 1$ we show that the following inequality holds uniformly with respect to $n \geq 1$:

$$d(x_n, z_{n+p}^{(n-1)}) \leq \delta \sum_{k=1}^{p} \alpha^{-k}. \tag{2.5}$$

Indeed, for $p = 1$ the inequality (2.5) follows from (2.1) and (2.3):

$$d(x_n, z_{n+1}^{(n-1)}) = \frac{d(f_{\lambda_n}(x_n), f_{\lambda_n}(z_{n+1}^{(n-1)}))}{\varphi_{\lambda_n}(x_n, z_{n+1}^{(n-1)})} = \frac{d(f_{\lambda_n}(x_n), x_{n+1})}{\varphi_{\lambda_n}(x_n, z_{n+1}^{(n-1)})} \leq \frac{\delta}{\alpha}.$$

Assume that (2.5) holds for some $p = q \geq 1$ uniformly on $n \geq 1$. Taking into account this assumption, as well as (2.1) and (2.3), we prove (2.5) for $p = q + 1$:

$$d(x_n, z_{n+q+1}^{(n-1)}) = \frac{d(f_{\lambda_n}(x_n), f_{\lambda_n}(z_{n+q+1}^{(n-1)}))}{\varphi_{\lambda_n}(x_n, z_{n+q+1}^{(n-1)})} = \frac{d(f_{\lambda_n}(x_n), z_{n+q+1}^{(n)})}{\varphi_{\lambda_n}(x_n, z_{n+q+1}^{(n-1)})} \leq$$

$$\frac{d(f_{\lambda_n}(x_n), x_{n+1}) + d(x_{n+1}, z_{n+1+q}^{(n)})}{\varphi_{\lambda_n}(x_n, z_{n+1+q}^{(n-1)})} \leq \frac{1}{\alpha}\left[\delta + \delta \sum_{k=1}^{q} \alpha^{-k}\right] \leq \delta \sum_{k=1}^{q+1} \alpha^{-k}.$$

Therefore (2.5) holds for any $p \geq 1$ and any $n \geq 1$.

The relations (2.1), (2.4) and (2.5) give us the following estimation for $d(z_n, z_{n+p})$ with any $n \geq 1$ and $p \geq 1$:

$$d(z_n, z_{n+p}) \leq \frac{1}{\varphi_{\lambda_0}(z_n, z_{n+p}) \prod_{i=1}^{n-1} \varphi_{\lambda_i}(z_n^{(i-1)}, z_{n+p}^{(i-1)})} \cdot \delta \sum_{k=1}^{p} \alpha^{-k} \leq$$

$$\frac{1}{\varphi_{\lambda_0}(z_n, z_{n+p}) \prod_{i=1}^{n-1} \varphi_{\lambda_i}(z_n^{(i-1)}, z_{n+p}^{(i-1)})} \cdot \frac{\delta}{\alpha - 1} =$$

$$\frac{\varepsilon}{\varphi_{\lambda_0}(z_n, z_{n+p}) \prod_{i=1}^{n-1} \varphi_{\lambda_i}(z_n^{(i-1)}, z_{n+p}^{(i-1)})} \leq \varepsilon \alpha^{-n}. \tag{2.6}$$

These inequalities demonstrate that $\{z_n\}_{n \geq 0}$ is a Cauchy sequence and, therefore, a convergent one. Let y_0 denote its limit and consider the chain

$\{y_n\}_{n\ge 0}$ which starts at y_0 and is defined as follows:

$$y_{n+1} = f_{\lambda_n}(y_n) = (f_{\lambda_n} \circ \cdots \circ f_{\lambda_0})(y_0) \qquad (n \ge 0).$$

From (2.2) one has for any $k \ge 0$:

$$\lim_{n\to\infty} z_n^{(k)} = (f_{\lambda_k} \circ \cdots \circ f_{\lambda_0})(y_0) = y_{k+1}.$$

Letting $p \to \infty$ in (2.6) yields

$$d(z_n, y_0) \le \frac{\varepsilon}{\varphi_{\lambda_0}(z_n, y_0) \prod_{i=1}^{n-1} \varphi_{\lambda_i}(z_n^{(i-1)}, y_i)} \qquad (n \ge 1),$$

which, in turn, implies

$$d(x_n, y_n) = d((f_{\lambda_{n-1}} \circ \cdots \circ f_{\lambda_0})(z_n), (f_{\lambda_{n-1}} \circ \cdots \circ f_{\lambda_0})(y_0)) =$$

$$\varphi_{\lambda_{n-1}}(z_n^{(n-2)}, y_{n-1}) d((f_{\lambda_{n-2}} \circ \cdots \circ f_{\lambda_0})(z_n), (f_{\lambda_{n-2}} \circ \cdots \circ f_{\lambda_0})(y_0)) = \ldots =$$

$$\varphi_{\lambda_{n-1}}(z_n^{(n-2)}, y_{n-1}) \cdot \cdots \cdot \varphi_{\lambda_1}(z_n^{(0)}, y_1) \varphi_{\lambda_0}(z_n, y_0) d(z_n, y_0) \le \varepsilon \quad (n \ge 1).$$

The lacking case $n = 0$ is treated similarly.

Therefore, the chain $\{y_n\}_{n\ge 0}$ ε-shadows the δ-chain $\{x_n\}_{n\ge 0}$. □

Example 2.1. This example shows that surjectivity of components of the parameterized IFS in Theorem 2.2 is essential. Let $X = \{1\} \cup [2, +\infty)$ be a complete metric space endowed with the standard metric from $\mathbb{R}$. Consider the function $f : X \to X$, $f(x) = 2x$, $\forall\, x \in X$; it is not a surjective one. The function f does not possess the Shadowing Property on $\mathbb{Z}_+$. Indeed, for $\varepsilon = 1/4$ and for any $\delta > 0$ one can construct a δ-chain $\{x_n\}_{n\ge 0}$ in X such that $x_0 = 1$ and for some natural $k \ge 1$ the fractional part of x_k is equal to $1/2$. Assume that there exists a chain $\{y_n\}_{n\ge 0}$ in X, which ε-shadows $\{x_n\}_{n\ge 0}$. Then, $y_0 = 1$ and y_n is an integer for any $n \ge 0$. Therefore, $d(x_k, y_n) \ge 1/2 > \varepsilon$ for any $n \ge 0$, a contradiction.

Given two complete metric spaces (X, d_X) and (Y, d_Y), consider the product set $X \times Y$ endowed with the metric $\max\{d_X(x_1, x_2), d_Y(y_1, y_2)\}$.

Let $\mathcal{F} = \{X; f_\lambda \,|\, \lambda \in \Lambda\}$ and $\mathcal{G} = \{Y; g_\mu \,|\, \mu \in M\}$ be two parameterized IFS. The IFS $\mathcal{H} = \{X \times Y; f_\lambda \times g_\mu \,|\, \lambda \in \Lambda, \mu \in M\}$, defined by $(f_\lambda \times g_\mu)(x, y) := (f_\lambda(x), g_\mu(y))$ is called the *product* of the IFS $\mathcal{F}$ and $\mathcal{G}$.

The proof of the following theorem is straightforward.

Theorem 2.3. *The product of two parameterized IFS has the Shadowing Property if and only if each projection has.*

As a consequence of Theorems 2.1, 2.2 and 2.3 one has the following result.

Theorem 2.4. *The product of an uniformly contracting with an uniformly expanding parameterized IFS has the Shadowing Property.*

3. Shadowing Property on $\mathbb{Z}$

Theorem 3.1. *If a parameterized IFS $\mathcal{F} = \{X; f_\lambda \,|\, \lambda \in \Lambda\}$ has the Shadowing Property on $\mathbb{Z}$, then it has the Shadowing Property on $\mathbb{Z}_+$, provided it satisfies the equality:*

$$\bigcup_{\lambda \in \Lambda} \bigcup_{x \in X} f(x) = X. \tag{3.1}$$

Proof. Assume that $\mathcal{F}$ satisfies (3.1) and has the Shadowing Property on $\mathbb{Z}$. Thus, given $\varepsilon > 0$ there exists $\delta > 0$ such that for any δ-chain $\{\tilde{x}_n\}_{n\in\mathbb{Z}}$ there exists a chain $\{\tilde{y}_n\}_{n\in\mathbb{Z}}$ such that $d(\tilde{x}_n, \tilde{y}_n) \leq \varepsilon$ for all $n \in \mathbb{Z}$.

Fix any $\varepsilon > 0$ and take $\delta > 0$ that has been mentioned above. Any δ-chain $\{x_n\}_{n\geq 0}$ can be embedded in a δ-chain $\{\tilde{x}_n\}_{n\in\mathbb{Z}}$ in the following way: take $\tilde{x}_n = x_n$ for any $n \geq 0$, and choose consecutively $\tilde{x}_n \in X$ and $\lambda_n \in \Lambda$ such that $\tilde{x}_{n+1} = f_{\lambda_n}(\tilde{x}_n)$, if $n < 0$; this choice is possible due to (3.1).

For the δ-chain $\{\tilde{x}_n\}_{n\in\mathbb{Z}}$ there exists a chain $\{\tilde{y}_n\}_{n\in\mathbb{Z}}$ such that $d(\tilde{x}_n, \tilde{y}_n) \leq \varepsilon$ for all $n \in \mathbb{Z}$. It follows that the chain $\{\tilde{y}_n\}_{n\geq 0}$ ε-shadows the δ-chain $\{x_n\}_{n\geq 0}$. $\square$

Remark 3.1. A result, similar to Theorem 3.1, holds for any relation $f : X \to \mathcal{P}(X)$ (see, e.g. [2]), which satisfies the equality $\bigcup_{x\in X} f(x) = X$.

Theorem 3.2. *If a parameterized IFS $\mathcal{F} = \{X; f_\lambda \,|\, \lambda \in \Lambda\}$ is uniformly contracting or uniformly expanding, then $\mathcal{F}$ has the Shadowing Property on $\mathbb{Z}$, provided that all functions $f_\lambda, \lambda \in \Lambda$, are surjective.*

Proof. Assume that $\mathcal{F}$ is uniformly expanding (the case of a uniformly contracting IFS is treated similarly), thus all functions $f_\lambda, \lambda \in \Lambda$, are invertible.

It is easy to see that the parameterized IFS $\mathcal{F}^{-1} := \{X; f_\lambda^{-1}, \lambda \in \Lambda\}$ is uniformly contracting.

Fix $\varepsilon > 0$ and let $\{x_n\}_{n\in\mathbb{Z}}$ be an arbitrary δ-chain of $\mathcal{F}$ with $\delta > 0$ to be chosen later.

By Theorem 2.2 $\mathcal{F}$ has the Shadowing Property on $\mathbb{Z}_+$, i.e. given $\varepsilon > 0$ there exists $\delta_1 > 0$ such that for any δ_1-chain $\{x_n\}_{n\geq 0}$ there exists a chain $\{z_n\}_{n\geq 0}$, such that $d(x_n, z_n) \leq \varepsilon/2$ for all $n \geq 0$.

Consider the sequence $\{u_n\}_{n\geq 0}$, where $u_n = x_{-n}$ for all $n \geq 0$. It is easy to verify that $\{u_n\}_{n\geq 0}$ is a $\tilde{\delta}$-chain of $\mathcal{F}^{-1}$, where $\tilde{\delta} = \delta\alpha^{-1}$ and α is the expanding ratio for $\mathcal{F}$. By Theorem 2.1 there exists $\delta_2 > 0$ such that any δ_2-chain $\{u_n\}_{n\geq 0}$ is ε-shadowed by any chain $\{v_n\}_{n\geq 0}$, such that $d(u_0, v_0) \leq \varepsilon/2$.

Take $\delta = \min\{\delta_1, \alpha\delta_2\}$ and choose two chains $\{z_n\}_{n\geq 0}$ and $\{v_n\}_{n\geq 0}$ of $\mathcal{F}$ and $\mathcal{F}^{-1}$ respectively with $v_0 = z_0$ and such that $d(x_n, z_n) \leq \varepsilon/2$ (for all $n \geq 0$) and $d(x_{-n}, v_n) \leq \varepsilon$ (for all $n \geq 0$). The sequence $\{y_n\}_{n\in\mathbb{Z}}$, defined by

$$y_n = \begin{cases} z_n, & \text{if} \quad n \geq 0, \\ v_{-n}, & \text{if} \quad n < 0, \end{cases}$$

is a chain of $\mathcal{F}$ we are looking for. Thus, $\mathcal{F}$ has the Shadowing Property on $\mathbb{Z}$. □

Example 3.1. This example is similar to that from 2.1 and shows, that the condition, imposed on the functions to be surjective, is essential. Take $X = \{-1\} \cup [0, +\infty)$ endowed with the standard metric from $\mathbb{R}$ and the function $f : X \to X$ with $f(-1) = 1$ and $f(x) = 2x$ for $x \geq 0$. Observe, that f is not surjective. Following arguments similar to those from 2.1, one can show that f has the Shadowing Property on $\mathbb{Z}$, but it lacks this property on $\mathbb{Z}_+$.

4. Shadowing in affine parameterized IFS

Given nonempty subsets $A, B \subset \mathbb{C}$ consider the affine parameterized IFS $\mathcal{F} = \{\mathbb{C}; f_{a,b} \,|\, a \in A, b \in B\}$, where $f_{a,b}(z) = az + b$.

Theorem 4.1. *Given a closed nonempty subset $A \subset \mathbb{C}$, situated strictly inside or strictly outside the unit circle, and a subset $B \subset \mathbb{C}$, the IFS $\mathcal{F} = \{\mathbb{C}; f_{a,b} \,|\, a \in A, b \in B\}$, $f_{a,b}(z) = az + b$, has the Shadowing Property on $\mathbb{Z}_+$.*

Proof. Let $A \subset \{z \in \mathbb{C} : |z| < 1\}$ (the case $A \subset \{z \in \mathbb{C} : |z| > 1\}$ is treated similarly). Due to closeness of A, there exists a real $c > 0$ such that

$\max\{|a| : a \in A\} < c < 1$. The IFS $\mathcal{F}$ is uniformly contracting and by Theorem 2.1 has the Shadowing Property. □

The following examples and the Theorem 4.2 show that the condition on A to be situated inside or outside the unit circle is essential, but can be relaxed in some cases.

Example 4.1. Let $A \subset \mathbb{C}$ be a compact such that $A \subset \{z \in \mathbb{C} : |z| \leq 1\}$ and $A \cap \{z \in \mathbb{C} : |z| = 1\} \neq \emptyset$. Consider the parameterized IFS $\mathcal{F} = \{\mathbb{C}; f_a \,|\, a \in A\}$ with $f_a(z) = az$. Let $a_1 \in A$, $|a_1| = 1$. Observe, that every chain is bounded. At the same time, the sequence $\{z_n\}_{n \geq 0}$, $z_{n+1} = a_1 z_n + \delta \frac{a_1 z_n}{|z_n|}$ $(n \geq 0, \delta > 0)$, with $z_0 \neq 0$, represents an unbounded δ-chain, impossible to shadow by any chain.

Example 4.2. Let A be a segment on $\mathbb{R}$, which intersects the interior and the exterior of the unit circle in $\mathbb{C}$. Consider the parameterized IFS $\mathcal{F} = \{\mathbb{C}; f_a \,|\, a \in A\}$ with $f_a(z) = az$. We shall show that $\mathcal{F}$ does not possess the Shadowing Property on $\mathbb{Z}_+$. Fix $\varepsilon = 1$. Given $\delta > 0$ one constructs a δ-chain $\{z_n\}_{n \geq 0}$ such that $z_0 = 2$, $z_{n+1} = a_1 z_n$ with some $a_1 \in A$, $|a_1| < 1$, until one obtains $0 < |z_m| < \delta/\sqrt{2}$ for some m. After this one takes $z_{m+1} = a_1 z_m + \delta_m$ with $\delta_m = a_1 z_m(\mathrm{i}\ - 1)$, $|\delta_m| < \delta$, and for any $n > m$ one takes $z_{n+1} = a_2 z_n$ with some $a_2 \in A$, $|a_2| > 1$. For this δ-chain one obtains that $|z_n| \to \infty$ as $n \to \infty$ and for any $n > m$ one has that $\arg z_n$ equals $-\pi/2$ or $\pi/2$. Assume that a chain $\{w_n\}_{n \geq 0}$ ε-shadows $\{z_n\}_{n \geq 0}$. Since $|w_0 - z_0| \leq 1$ it follows that $-\pi/6 \leq \arg w_0 \leq \pi/6$ and $\arg w_n$ equals $\arg w_0$ or $\arg w_0 + \pi$. This contradicts the assumption that $\{w_n\}_{n \geq 0}$ ε-shadows $\{z_n\}_{n \geq 0}$. Therefore, there is no chain to ε-shadow $\{z_n\}_{n \geq 0}$.

Theorem 4.2. *For any closed disc centered at* 0 *with radius* $r > 1$ *in* $\mathbb{C}$ *and any subset* $B \subset \mathbb{C}$ *the parameterized IFS* $\mathcal{F} = \{\mathbb{C}; f_{a,b} \,|\, a \in A, b \in B\}$, *with* $f_{a,b}(z) = az + b$, *has the Shadowing Property on* $\mathbb{Z}_+$.

Proof. Given $\varepsilon > 0$, put $\delta = (r - 1)\varepsilon > 0$ and take any δ-chain $\{z_n\}_{n \geq 0}$:

$$z_{n+1} = a_n z_n + b_n + \delta_n \quad (n \geq 0), \tag{4.1}$$

(here $a_n \in A, b_n \in B$ and $|\delta_n| \leq \delta$ for all $n \geq 0$).

Consider the sequence $\{w_n\}_{n\geq 0}$, defined as follows:

$$w_n = \begin{cases} \varepsilon, \text{ if } & z_n = 0, \\ z_n + \varepsilon \dfrac{z_n}{|z_n|}, \text{ if } & z_n \neq 0. \end{cases}$$

It is easily seen that

$$|w_n| = |z_n| + \varepsilon \geq \varepsilon \quad (n \geq 0), \tag{4.2}$$

$$|w_n - z_n| = \varepsilon \quad (n \geq 0), \tag{4.3}$$

and

$$|z_{n+1} - b_n| \leq |a_n||z_n| + |\delta_n| \leq r|z_n| + \delta \quad (n \geq 0),$$

which, in turn, imply

$$|w_{n+1} - b_n| \leq |w_{n+1} - z_{n+1}| + |z_{n+1} - b_n| \leq$$
$$\varepsilon + r|z_n| + \delta = \varepsilon + r(|w_n| - \varepsilon) + (r-1)\varepsilon = r|w_n| \quad (n \geq 0).$$

Take $\tilde{a}_n = \dfrac{w_{n+1} - b_n}{w_n}$. Since $|\tilde{a}_n| = \dfrac{|w_{n+1} - b_n|}{|w_n|} \leq r$, each $\tilde{a}_n$ belongs to A. Obviously, $\{w_n\}_{n\geq 0}$, $w_{n+1} = \tilde{a}_n w_n + b_n$, is a chain and it ε-shadows $\{z_n\}_{n\geq 0}$, by (4.3). □

Consider the multi-function $z \mapsto \{az + b\} \bigcup K$ for some nonempty compact $K \subset \mathbb{C}$. This multi-function can be regarded as a parameterized IFS $\{\mathbb{C}; f_1, \tilde{f}_\lambda \,|\, \lambda \in K\}$, where $f_1(z) = az + b$ and $\tilde{f}_\lambda(z) \equiv \lambda$. The notions of pseudo-chain, chain and shadowing for such parameterized IFS are the same as for relations (see, e.g. [2]).

Theorem 4.3. *The IFS with "condensation" $\mathcal{F} = \{\mathbb{C}; f_1, f_2\}$, where $f_1(z) = az + b$ and $f_2(z) \equiv K$ for some nonempty compact $K \subset \mathbb{C}$, has the Shadowing Property iff $|a| < 1$.*

Proof. If $|a| < 1$, then $\mathcal{F}$ is uniformly contracting, so it has the Shadowing Property, by Theorem 2.1.

Conversely, let $\mathcal{F}$ have the Shadowing Property. One has to prove that $|a| < 1$.

Firstly, notice that the function $f(z) = z + b$ has not the Shadowing Property. Indeed, the δ-chain $\{z_n\}_{n\geq 0}$, $z_{n+1} = z_n + b + \delta = z_0 + n(b + \delta)$, cannot be shadowed by any chain $\{w_n\}_{n\geq 0}$, since the latter has the form $w_n = w_0 + nb$ and $|z_n - w_n| \to \infty$ as $n \to \infty$.

Adding the compact "condensation" $f_2(z) \equiv K$ does not change the situation, since one can take the same δ-chain as above, but with starting point z_0 far enough from K. Thus, one can exclude the case $a = 1$.

Secondly, making the translation $z \mapsto z - \frac{b}{a-1}$, one can reduce the affine IFS $\mathcal{F}$ to a linear IFS with "condensation" $\mathcal{G} = \{\mathbb{C}; g_1, g_2\}$, with $g_1(z) = az$ and $g_2(z) = Q = K + \frac{b}{a-1}$.

It is easy to check that this translation do not affect the Shadowing Property.

Let's return to the proof. Assuming the contrary, i.e. assuming $|a| \geq 1$, with exception of $a = 1$, we will construct for any $\delta > 0$ a δ-chain, which admits no chain to ε-shadow it with ε small enough. For, take any $q \in Q$ with the maximal modulus, any $u_0 \in \mathbb{C}$ with $|u_0| > |q| + 2\varepsilon$, and define the δ-chain $\{u_n\}_{n\geq 0}$ with starting point u_0 as follows:

$$u_1 = q \in g_2(u_0), \quad u_{n+1} = g_1(u_n) + \delta_n = au_n + \delta\frac{au_n}{|au_n|} \quad (n \geq 1)$$

(if $q = 0$ take simply $u_2 = au_1 + \delta$).

Thus, $|u_{n+1}| = |a||u_n| + \delta$, or, more explicitly,

$$|u_{n+1}| = |a|^n|q| + \delta\sum_{k=0}^{n-1}|a|^k \quad (n \geq 1). \tag{4.4}$$

On the other hand, every chain $\{v_n\}_{n\geq 0}$, ε-shadowing $\{u_n\}_{n\geq 0}$, must satisfy

$$|v_0| \geq |u_0| - \varepsilon > |q| + \varepsilon, \quad |v_1| \leq |u_1| + |u_1 - v_1| \leq |q| + \varepsilon,$$

which, in turn, imply that $v_1 \in g_2(v_0) = Q$.

As a consequence we obtain that $|v_n| \leq |a|^{n-1}|q|$, and, by (4.4), we have that

$$|u_n - v_n| \geq |u_n| - |v_n| \geq \delta\sum_{k=0}^{n-2}|a|^k \to \infty, \quad \text{as } n \to \infty.$$

The latest means that there is no chain to ε-shadow the δ-chain $\{u_n\}_{n\geq 0}$. This contradiction finishes the proof.

$\square$

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