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EXISTENCE RESULTS FOR INTEGRAL EQUATIONS: SPECTRAL METHODS VS. FIXED POINT THEORY

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Dedicated to Professor Ioan A. Rus on the occasion of his 70th birthday

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Abstract. In this note we compare fixed point methods and methods of nonlinear spectral theory in view of their applicability to nonlinear integral equations of Hammerstein or Uryson type.

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The classical fixed point principles of Schauder and Darbo belong to the most powerful (nonconstructive) tools for proving existence of solutions to nonlinear problems. Roughly speaking, to apply these principles one has to transform the specific nonlinear problem under consideration (integral equation, boundary value problem, dynamical system etc.) into an equivalent operator equation involving a compact resp. condensing operator in a suitable Banach space such that the fixed points of this operator coincide with the solutions of the original problem. For the case of noncompact problems, various examples of this type may be found in the book [1] or in the recent survey article [2].

In comparison with fixed point methods, spectral methods provide a relatively new alternative tool for studying existence of solutions to nonlinear problems; for selected applications we refer to Chapter 12 of the monograph [3]. Both the formulation and application of spectral results for nonlinear operators usually require a larger amount of technical details than fixed point results; on the other hand, they make it possible to obtain more precise results. For instance, the two spectra we will consider below give not only existence, but also perturbation results for nonlinear equations, and so apply to a larger variety of problems.

Fixed point and spectral methods are not independent of each other, but are linked by some interesting interconnections. Thus, a nonlinear operator F has a fixed point if $\lambda = 1$ belongs to the spectrum $\sigma(F)$ of F in a sense to be made precise. Conversely, one of the spectra we will consider in what follows is based on the notion of so-called "k-epi maps" (for a precise definition see Section 2 below), and Schauder's [resp., Darbo's] fixed point theorem may be stated by saying that the identity operator is 0-epi [resp. k-epi for $k < 1$] on each ball.

The plan of this paper is as follows. In the first section we briefly recall three fixed point principles which are intimately related to certain nonlinear spectra described in the second section. In the third and fourth section we show how theses methods apply to nonlinear integral equations of Hammerstein or, more generally, Uryson type. It turns out that both methods give essentially the same existence results, but spectral methods provide, in addition, perturbation results which cannot be obtained by fixed point theorems. As already stated, the price to pay for this is the rather heavy technical machinery we have to develop before applying spectral methods.

We point out that this note is, both in contents and style, quite elementary. We do not aim at proving new sophisticated existence theorems; instead, we just want to illustrate the advantages and drawbacks of each of the two methods mentioned in the title when applied to a certain class of simple nonlinear equations.

1. Fixed point theory

In what follows, X denotes a (real or complex) Banach space, and $B_r(X) := \{x \in X : ||x|| \leq r\}$ the closed ball around 0 in X with radius $r > 0$. All operators considered in the sequel are assumed to be continuous and bounded, i.e., to map bounded sets into bounded sets.

Throughout this paper, we follow the notation of the monograph [3]. To begin with, we recall some numerical characteristics for operators from Chapter 2 of [3]. Given some (in general, infinite dimensional) Banach space X and some (in general, nonlinear) operator $F : X \to X$, we consider the metric characteristics

$$[F]_B = \sup_{x\neq 0} \frac{||F(x)||}{||x||}, \qquad [F]_b = \inf_{x\neq 0} \frac{||F(x)||}{||x||} \tag{1.1}$$

and

$$[F]_Q = \limsup_{||x||\to\infty} \frac{||F(x)||}{||x||}, \qquad [F]_q = \liminf_{||x||\to\infty} \frac{||F(x)||}{||x||}. \tag{1.2}$$

Operators F satisfying $[F]_B < \infty$ are usually called *linearly bounded*, those satisfying $[F]_Q < \infty$ *quasibounded*. Clearly, the estimates

$$[F]_b \leq [F]_q \leq [F]_Q \leq [F]_B \tag{1.3}$$

are true, and so every linearly bounded operator is quasibounded (but of course not vice versa). Moreover, if α denotes some measure of noncompactness on X (e.g., the Hausdorff or Kuratowski measure of noncompactness), we consider the topological characteristics

$$[F]_A = \inf\{k > 0 : \alpha(F(M)) \leq k\alpha(M)\} = \sup_{\alpha(M)>0} \frac{\alpha(F(M))}{\alpha(M)} \tag{1.4}$$

and

$$[F]_a = \sup\{k > 0 : \alpha(F(M)) \geq k\alpha(M)\} = \inf_{\alpha(M)>0} \frac{\alpha(F(M))}{\alpha(M)}. \tag{1.5}$$

Operators F satisfying $[F]_A < \infty$ are usually called *α-Lipschitz*, in case $[F]_A < 1$ *α-condensing*. Obviously, an operator F satisfies $[F]_A = 0$ precisely if it is *compact*, i.e., maps every bounded set into a precompact set. Using these characteristics, the three classical fixed point principles by Schauder [13], Darbo [5], and Vignoli [16] which we will use in the sequel may be formulated as follows.

Theorem 1. *Let $B \subset X$ be a nonempty, convex, bounded, closed set, and let $F : B \to B$ be an operator satisfying $[F]_A = 0$. Then F has a fixed point in B.*

Theorem 2. *Let $B \subset X$ be a nonempty, convex, bounded, closed set, and let $F : B \to B$ be an operator satisfying $[F]_A < 1$. Then F has a fixed point in B.*

Theorem 3. *Let $F : X \to X$ be an operator satisfying $[F]_A < 1$ and $[F]_Q < 1$. Then F has a fixed point in X.*

A crucial condition in Theorems 1 and 2 is the existence of the invariant set B for F; this is trivially satisfied, of course, if $[F]_B \leq 1$. On the other hand, the "asymptotic growth condition" $[F]_Q < 1$ in Theorem 3 replaces not only this invariance assumption in Theorems 1 and 2, but gives a stronger result. In fact, under the hypotheses $[F]_Q < 1$ and $[F]_A < 1$ one may even claim that the operator $I - F$, with I denoting the identity operator, is surjective. To see this, fix $y \in X$ and put $F_y(x) := F(x) + y$; then we may choose q satisfying $[F_y]_Q = [F]_Q < q < 1$ and $b > 0$ such that $||F_y(x)|| \leq q||x|| + b$ for all $x \in X$. Consequently, for $R \geq b/(1-q)$ the operator F_y maps the closed ball $B_R(X)$ into itself and satisfies $[F_y]_A = [F]_A < 1$, and so Theorem 2 implies that the equation $x = F_y(x)$ has a solution $x \in X$ (actually, $x \in B_R(X)$).

2. Nonlinear spectral theory

In this section we recall some definitions and results from the monograph [3]. Unfortunately, this requires some technical notation which will be justified by the applications in the second part of this paper; we request the reader's indulgence until then.

Given $k \geq 0$, we call an operator $F : X \to X$ *k-stably solvable* if, for any operator $G : X \to X$ with $[G]_Q \leq k$ and $[G]_A \leq k$, the coincidence equation

$$F(x) = G(x) \tag{2.1}$$

admits a solution $x \in X$. In particular, in case $k = 0$ (i.e., the right-hand side G in (2.1) has to be compact with quasinorm 0) we get the class of *stably solvable* operators introduced in [9]. It is easy to see that every stably solvable operator F is surjective (take $G(x) \equiv y$ in (2.1)); for linear operators F, stable solvability is even equivalent to surjectivity [9].

Since any k-stably solvable operator is also k'-stably solvable for $k' < k$, the characteristic

$$\mu(F) := \inf\{k \geq 0 : F \text{ is } \textit{not } k\text{-stably solvable}\} \tag{2.2}$$

which measures in a certain sense the "degree of solvability" of (2.1), is of some interest. If $\mu(F) > 0$ (i.e., F is k-stably solvable for some $k > 0$), the operator F is sometimes called *strictly stably solvable*.

The *Furi-Martelli-Vignoli spectrum* [10] of an operator $F : X \to X$ is defined by

$$\sigma_{FMV}(F) = \sigma_\mu(F) \cup \sigma_q(F) \cup \sigma_a(F), \tag{2.3}$$

where $\lambda \in \sigma_\mu(F)$ if $\mu(\lambda I - F) = 0$, $\lambda \in \sigma_q(F)$ if $[\lambda I - F]_q = 0$, and $\lambda \in \sigma_a(F)$ if $[\lambda I - F]_a = 0$. In spite of its technical definition, this spectrum has many natural properties: in case of a *linear* operator F it coincides with the familiar spectrum, and it is always closed. Moreover, if F is quasibounded and α-Lipschitz, then $\sigma_{FMV}(F)$ is contained in the closed disc in the complex plane with radius $r = \max\{[F]_A, [F]_Q\}$, hence compact.

We point out that the Furi-Martelli-Vignoli spectrum (2.3) is *asymptotic* in its nature, since we used the asymptotic characteristic (1.2) in its definition. Taking instead into account the *global* behaviour of F leads to another spectrum which was defined by Feng in 1997 and is based on the notion of so-called epi operators defined in the following way.

Given $k \geq 0$, we call an operator $F : B_r(X) \to X$ *k-epi* on $B_r(X)$ if $F(x) \neq 0$ for $||x|| = r$ and, for any operator $G : B_r(X) \to X$ with $[G|_{B_r(X)}]_A \leq k$ and $G(x) \equiv 0$ for $||x|| = r$, the coincidence equation (2.1) admits a solution $x \in B_r(X)$. In particular, in case $k = 0$ (i.e., the right-hand side G in (2.1) has to be compact) we get the class of *0-epi* operators introduced in [11]. Similarly as above, since any k-epi operator is also k'-epi for $k' < k$, the characteristic [14]

$$\nu_r(F) := \inf\{k \geq 0 : F \text{ is } \textit{not } k\text{-epi on } B_r(X)\} \tag{2.4}$$

which measures again the "degree of solvability" of (2.1) (but this time locally in the ball $B_r(X)$), is of some interest. If the number

$$\nu(F) := \inf_{r>0} \nu_r(F) \tag{2.5}$$

is positive (i.e., F is k-epi for some $k > 0$ on *every* ball), the operator F is usually called *strictly epi*. The characteristics (2.2) and (2.5) are related by the inequality $\mu(F) \leq \nu(F)$, which may be easily verified by extending the operator G in (2.1) to be zero outside $B_r(X)$. In particular, every strictly stably solvable operator is strictly epi (but not vice versa).

It is instructive to see what the characteristics (2.2) and (2.5) mean in case of a *linear* operator $L : X \to X$. For example, one may show that $\nu(L) \geq]L[$ in any infinite dimensional Banach space, where

$$]L[:= \begin{cases} ||L^{-1}||^{-1} & \text{if } L \text{ is an isomorphism,} \\ 0 & \text{otherwise} \end{cases}$$

denotes the so-called "inner norm" of the operator L. In particular, $\mu(I) = \nu(I) =]I[= 1$ in every infinite dimensional Banach space; this equality is essentially an equivalent formulation of Darbo's and Vignoli's fixed point theorems (on balls) stated in the preceding section.

Now we recall another important spectrum, the *Feng spectrum* [6] of an operator $F : X \to X$, which is defined by

$$\sigma_F(F) = \sigma_\nu(F) \cup \sigma_b(F) \cup \sigma_a(F), \tag{2.6}$$

where $\lambda \in \sigma_\nu(F)$ if $\nu(\lambda I - F) = 0$, and $\lambda \in \sigma_b(F)$ if $[\lambda I - F]_b = 0$. Also this spectrum shares many natural properties with the Furi-Martelli-Vignoli spectrum: in case of a linear operator F it coincides again with the familiar spectrum, and it is always closed. Moreover, if F is linearly bounded and α-Lipschitz, then $\sigma_F(F)$ is contained in the closed disc in the complex plane with radius $r = \max\{[F]_A, [F]_B\}$, hence compact. Finally, it is not hard to see that the inclusion

$$\sigma_{FMV}(F) \subseteq \sigma_F(F) \tag{2.7}$$

holds true which may be strict if F is nonlinear, even in the scalar case $X = \mathbb{R}$. For example, for the real function $F(x) := \sqrt{|x|}$ we have $\sigma_{FMV}(F) = \{0\}$ and $\sigma_F(F) = \mathbb{R}$.

We point out that the characteristics introduced above are not independent of each other. For example, a highly nontrivial result due to Väth [15] asserts that an epi operator $F : X \to X$ with $[F]_a > 0$ actually satisfies $\nu(F) \geq [F]_a$, i.e., is even k-epi for sufficiently small $k > 0$. This result is quite surprising:

it means that, if $[F]_a > 0$ and equation (2.1) is solvable for compact operators G, then it is also solvable for "slightly noncompact" operators G.

Väth's theorem has two important consequences. First, it implies that an epi operator F which is not k-epi for any $k > 0$ must satisfy $[F]_a = 0$. It was an open problem for some time to find such an operator; an example was found quite recently by Furi [8]. Second, Väth's theorem shows that $\nu(F) = 0$ implies $[F]_a = 0$, and so the subspectrum $\sigma_a(F)$ in the definition (2.6) of the Feng spectrum is superfluous. Of course, Väth's result was still unknown when Feng defined her spectrum in 1997.

There are also some important perturbation results for the characteristics (2.2) and (2.5) which could be called *Rouché type theorems.* To state these theorems which will be needed later, we recall first two *homotopy invariance theorems* for strictly stably solvable and strictly epi operators which are due to Furi, Martelli and Vignoli [9] and Tarafdar and Thompson [14], respectively. Such theorems may be viewed as analogues to the classical *Leray-Schauder continuation principle* for compact operators.

Lemma 1. *Suppose that* $F_0 : X \to X$ *is* k_0*-stably solvable. Moreover, assume that* $H : X \times [0,1] \to X$ *satisfies* $H(x,0) \equiv 0$ *and*

$$\alpha(H(M \times [0,1])) \leq k\alpha(M) \tag{2.8}$$

for all bounded sets $M \subset X$ *and some* $k \leq k_0$*. Finally, suppose that*

$$\sup_{0 \leq t \leq 1} \limsup_{||x|| \to \infty} \frac{||H(x,t)||}{||x||} \leq k.$$

Then the operator $F_1 := F_0 + H(\cdot,1)$ *is* k_1*-stably solvable for* $k_1 \leq k_0 - k$*.*

Lemma 2. *Suppose that* $F_0 : B_r(X) \to X$ *is* k_0*-epi on* $B_r(X)$*. Moreover, assume that* $H : X \times [0,1] \to X$ *satisfies* $H(x,0) \equiv 0$ *and* (2.8) *for all sets* $M \subseteq B_r(X)$ *and some* $k \leq k_0$*. Finally, suppose that the set*

$$\Sigma := \{x \in B_r(X) : F_0(x) + H(x,t) = 0 \text{ for some } t \in [0,1]\} \tag{2.9}$$

is disjoint from the boundary of $B_r(X)$*, i.e., does not contain any element* x *with* $||x|| = r$*. Then the operator* $F_1 := F_0 + H(\cdot,1)$ *is* k_1*-epi on* $B_r(X)$ *for* $k_1 \leq k_0 - k$*.*

A scrutiny of the proof of these lemmas allows us to deduce the following two perturbation results of Rouché type.

Lemma 3. *Suppose that $\mu(F) > 0$, i.e., F is strictly stably solvable. Let $G : X \to X$ be an operator satisfying $[G]_A < \mu(F)$ and $[G]_Q < \mu(F)$. Then*

$$\mu(F+G) \geq \mu(F) - \max\{[G]_A, [G]_Q\}, \tag{2.10}$$

i.e., $F+G$ is also strictly stably solvable.

Lemma 4. *Suppose that $\nu(F) > 0$, i.e., F is strictly epi on each ball. Let $G : X \to X$ be an operator satisfying $[G|_{B_r(X)}]_A < \nu_r(F)$ and*

$$\sup_{||x||=r} ||G(x)|| < \inf_{||x||=r} ||F(x)|| \tag{2.11}$$

for each $r > 0$. Then

$$\nu(F+G) \geq \nu(F) - [G]_A, \tag{2.12}$$

i.e., $F+G$ is also strictly epi on each ball.

To see how Lemmas 3 and 4 follow from Lemmas 1 and 2, let us briefly sketch the proof of Lemma 4. Put $F_0 := F$ in Lemma 2, fix $r > 0$, and choose k with $[G|_{B_r(X)}]_A < k < \nu_r(F)$. We define a (linear) homotopy H by $H(x,t) := tG(x)$, so the set (2.9) is here

$$\Sigma = \{x \in B_r(X) : F(x) = -tG(x) \text{ for some } t \in [0,1]\}.$$

Suppose that there exists $\hat{x} \in \Sigma$ of norm $||\hat{x}|| = r$. Then

$$\inf_{||x||=r} ||F(x)|| \leq ||F(\hat{x})|| = t||G(\hat{x})|| \leq ||G(\hat{x})|| \leq \sup_{||x||=r} ||G(x)||,$$

contradicting (2.11). So from Lemma 2 we conclude that the operator $F_1 = F_0 + H(\cdot,1) = F + G$ is k-epi on $B_r(X)$ for $0 \leq k \leq \nu_r(F) - [G|_{B_r(X)}]_A$ as claimed.

In what follows, we will apply these lemmas to the very special case when $F = \lambda I$ for some scalar λ, and G is some nonlinear integral operator.

3. Hammerstein integral equations

Now we start the second part of this paper in which we apply the abstract results given above to some specific nonlinear problems. We begin with an application to a nonlinear Hammerstein integral equation of the form

$$\lambda x(s) - \int_0^1 k(s,t)f(t,x(t))\,dt = y(s) \qquad (0 \le s \le 1). \tag{3.1}$$

The nonlinear Hammerstein operator

$$H(x)(s) = \int_0^1 k(s,t)f(t,x(t))\,dt \tag{3.2}$$

defined by the left-hand side of (3.1) may be viewed as composition $H = KF$ of the nonlinear Nemytskij operator

$$F(x)(t) = f(t,x(t)) \tag{3.3}$$

generated by the nonlinear function f and the linear integral operator

$$Ky(s) = \int_0^1 k(s,t)y(t)\,dt \tag{3.4}$$

generated by the kernel function k. We suppose that $k : [0,1] \times [0,1] \to \mathbb{R}$ is continuous and $f : [0,1]\times\mathbb{R} \to \mathbb{R}$ satisfies a Carathéodory condition. Moreover, we assume that f satisfies a growth condition of the form

$$|f(t,u)| \le a(t) + b(t)|u| \qquad (0 \le t \le 1,\ u \in \mathbb{R}), \tag{3.5}$$

with two functions $a, b \in L_1[0,1]$. In what follows, we write $||x||_1$ for the L_1-norm and $||x||_\infty$ for the C-norm of a function x. Moreover, we define a scalar function κ by

$$\kappa(t) := \max_{0\le s\le 1} |k(s,t)| \qquad (0 \le t \le 1). \tag{3.6}$$

Observe that, by well-known formulas for the norm of a linear integral operator [4], the L_1-norm $||\kappa||_1$ of (3.6) gives then an upper estimate for the operator norm of the linear operator (3.4) in the space $C[0,1]$.

In the following proposition we apply the Feng spectrum to the operator H defined by (3.2). Afterwards we will show that we get almost the same result if we apply Schauder's fixed point principle, but a slightly stronger result if we use the Furi-Martelli-Vignoli spectrum.

Proposition 1. *Suppose that* $|\lambda| > ||\kappa b||_1$*, where* $\kappa(t)$ *is given by* (3.6) and b *is from* (3.5). *Then the equation* (3.1) *has a solution* $x \in C[0,1]$ *for* $y(s) \equiv 0$. *Moreover, if* $a(t) \equiv 0$ *in* (3.5), *then equation* (3.1) *has a solution* $x \in C[0,1]$ *for every* $y \in C[0,1]$.

We discuss essentially three different proofs of this proposition, where the first proof is in the spirit of Feng's paper [7]. Since the operator (3.2) is compact in the space $X = C[0,1]$, we see that $[\lambda I - H]_a = |\lambda| > 0$. Now we distinguish two cases for λ.

Suppose first that $\lambda \notin \sigma_b(H)$, i.e., $[\lambda I - H]_b > 0$. Consider the set

$$\Sigma = \{x \in X : \lambda x = tH(x) \text{ for some } t \in [0,1]\}. \tag{3.7}$$

By (3.5), for $x \in \Sigma$ we have

$$|\lambda|\,||x||_\infty = t||H(x)||_\infty \leq ||H(x)||_\infty \leq ||\kappa a||_1 + ||\kappa b||_1||x||_\infty, \tag{3.8}$$

hence

$$||x||_\infty \leq \frac{||\kappa a||_1}{|\lambda| - ||\kappa b||_1},$$

which shows that the set (3.7) is bounded. From Lemma 2 we conclude that the operator $\lambda I - H$ is k-epi on X for $k < |\lambda|$, i.e., $\nu(\lambda I - H) > 0$. Together with our assumption $[\lambda I - H]_b > 0$ this implies that $\lambda \notin \sigma_F(H)$, and so the equation $H(x) = \lambda x$ has a solution.

Suppose now that $\lambda \in \sigma_b(H)$, i.e., $[\lambda I - H]_b = 0$. Then we may find, by definition (1.1), a sequence $(x_n)_n$ in X such that

$$||\lambda x_n - H(x_n)||_\infty \leq \frac{1}{n}||x_n||_\infty$$

and thus

$$|\lambda|\,||x_n||_\infty - ||\kappa a||_1 - ||\kappa b||_1||x_n||_\infty \leq \frac{1}{n}||x_n||_\infty,$$

hence

$$\left(|\lambda| - ||\kappa b||_1 - \frac{1}{n}\right)||x_n||_\infty \leq ||\kappa a||_1.$$

This shows that the sequence $(x_n)_n$ is bounded, because $|\lambda| > ||\kappa b||_1$. Consequently, $||\lambda x_n - H(x_n)||_\infty \to 0$ as $n \to \infty$. Denoting $M := \{x_1, x_2, x_3, \ldots\}$, from the estimate

$$[\lambda I - H]_a \alpha(M) \leq \alpha((\lambda I - H)(M)) = 0$$

we see that $(x_n)_n$ has a convergent subsequence, and the limit of this subsequence is certainly a solution of the equation $H(x) = \lambda x$.

To prove the last assertion, assume that $a(t) \equiv 0$. Then we get the estimate

$$\max\{[H]_A, [H]_B\} = [H]_B = \sup_{x\neq 0} \frac{||H(x)||_\infty}{||x||_\infty} \leq ||\kappa b||_1.$$

By what we observed before on the boundedness of the Feng spectrum, this implies that $\lambda \notin \sigma_F(H)$ for $|\lambda| > ||\kappa b||_1$, and so we are done.

Now we show how the second assertion of Proposition 1 may be obtained immediately from the estimate (2.12) in Lemma 4. In fact, for $F := \lambda I$ and $G := -H$ the hypotheses of Lemma 4 simply read

$$[H]_A < |\lambda|, \qquad \sup_{||x||=r} ||H(x)|| < |\lambda| r. \tag{3.9}$$

The first estimate in (3.9) follows from the compactness of H, while the second estimate follows from

$$\sup_{||x||_\infty = r} ||H(x)||_\infty \leq ||b\kappa||_1 r < |\lambda| r, \tag{3.10}$$

where we have used the hypothesis $||b\kappa||_1 < |\lambda|$, the last inequality in (3.8) and the fact that $a(t) \equiv 0$.

In case $a(t) \not\equiv 0$ it is not possible to apply Lemma 4. However, in this case one may use the Furi-Martelli-Vignoli spectrum and apply Lemma 3 instead of Lemma 4. In fact, let $F := \lambda I$ and $G := -H$ as before. Then instead of (3.10) we obtain

$$[H]_Q = \limsup_{||x||_\infty \to \infty} \frac{||H(x)||_\infty}{||x||_\infty} \leq \limsup_{||x||_\infty \to \infty} \left(\frac{||a\kappa||_1}{||x||_\infty} + ||b\kappa||_1 \right) = ||b\kappa||_1 < |\lambda|, \tag{3.11}$$

and so the hypothesis $[G]_Q < \mu(F)$ in Lemma 3 is satisfied. We conclude that

$$\mu(\lambda I - H) = \mu(F + G) \geq \mu(F) - \max\{[G]_A, [G]_Q\} = |\lambda| - [H]_Q,$$

i.e., the operator $\lambda I - H$ is k-stably solvable (and hence surjective) for $0 \leq k < |\lambda| - [H]_Q$.

Finally, we show how the existence result from Proposition 1 may be obtained in a quite straightforward way by means of Schauder's (or Vignoli's) fixed point principle. In fact, supposing again that f satisfies the growth estimate (3.5), the assumption $|\lambda| > ||\kappa b||_1$ implies that

$$\sup_{||x||_\infty \le R} \frac{||H(x)||_\infty}{|\lambda| R} \le \frac{1}{|\lambda| R} (R||\kappa b||_1 + ||\kappa a||_1) < 1 + \frac{||\kappa a||_1}{|\lambda| R}.$$

So the first expression can be made ≤ 1 for sufficiently large $R > 0$, and hence the (compact) operator H/λ maps the ball $B_R(X)$ into itself. From Theorem 1 it follows that equation (3.1) has a solution in this ball.

A comparison of the methods used so far for proving Proposition 1 one could be inclined to prefer throughout fixed point methods, because of their simplicity, to spectral methods. However, the fixed point principles cited in Section 1 give only *existence*, while the spectral theorems discussed in Section 2 give also *perturbation results.* Indeed, our "sprectral proofs" based on Lemmas 1-4 above yield the following refinement of Proposition 1.

Proposition 2. *Suppose that* $|\lambda| > ||\kappa b||_1$, *where* $\kappa(t)$ *is given by* (3.6) and b *is from* (3.5). *Then the equation*

$$\lambda x(s) - \int_0^1 k(s,t) f(t, x(t))\, dt = G(x)(s) \qquad (0 \le s \le 1). \tag{3.12}$$

has a solution $x \in C[0,1]$ *for any operator* $G : C[0,1] \to C[0,1]$ *which satisfies* $[G]_A < ||\kappa b||_1$ *and* $[G]_Q < ||\kappa b||_1$.

Of course, the crucial requirement in each proof of Proposition 1 above was the inequality $|\lambda| > ||\kappa b||_1$. It is not hard to show that this inequality is sharp. In fact, for the *linear* equation with degenerate kernel

$$\lambda x(s) - \int_0^1 t x(t)\, dt = y(s) \qquad (0 \le s \le 1),$$

i.e., for $k(s,t) = t$ and $f(t,u) = u$ in (3.1), and $a(t) = 0$ and $b(t) = 1$ in (3.5), one easily sees that $\lambda = 1/2$ is the only eigenvalue of the linear operator H; on the other hand, $||\kappa b||_1 = 1/2$ in this example.

4. Uryson integral equations

As second example let us consider the nonlinear Uryson integral equation of second kind

$$\lambda x(s) - \int_0^1 k(s,t,x(t))\,dt = 0 \qquad (0 \le s \le 1). \tag{4.1}$$

We are going to study the nonlinear Uryson operator

$$U(x)(s) = \int_0^1 k(s,t,x(t))\,dt \tag{4.2}$$

generated by the left-hand side of (4.1) in the space $L_2[0,1]$, whose norm we denote by $||\cdot||_2$. To this end, we make the following assumptions on the (continuous) nonlinear kernel function $k : [0,1] \times [0,1] \times \mathbb{R} \to \mathbb{R}$:

$$\sup_{|u|\le r} |k(s,t,u)| \le \beta_r(s,t), \qquad M_r := \sup_{0\le s\le 1} \int_0^1 \beta_r(s,t)\,dt < \infty, \tag{4.3}$$

$$\sup_{|u|\le r} |k(s,t,u) - k(\sigma,t,u)| \le \gamma_r(s,\sigma,t), \qquad \lim_{s\to\sigma} \int_0^1 \gamma_r(s,\sigma,t)\,dt = 0, \tag{4.4}$$

and

$$|k(s,t,u)| \le \psi(s,t)(1+|u|), \qquad M := \int_0^1 \int_0^1 \psi(s,t)^2\,dt\,ds < \infty. \tag{4.5}$$

Proposition 3. *Suppose that $|\lambda|^2 > 2M$, where M is given by* (4.5)*. Then the equation* (4.1) *has a solution $x \in L_2[0,1]$.*

Again, we prove Proposition 3 first by applying Lemma 2 in $X = L_2[0,1]$. It is well-known [12] that, under our assumptions (4.3)-(4.5), the operator (4.2) is compact in X. Moreover, for any $x \in X$ one has the estimates

$$|U(x)(s)|^2 \le \left(\int_0^1 k(s,t,x(t))\,dt\right)^2 \le \left(\int_0^1 \psi(s,t)(1+|x(t)|)\,dt\right)^2$$

$$\le \left(\int_0^1 \psi(s,t)^2\,dt\right)\left(\int_0^1 (1+|x(t)|)^2\,dt\right) \le 2\left(\int_0^1 \psi(s,t)^2\,dt\right)(1+||x||_2^2).$$

Consequently,

$$||U(x)||_2^2 \le 2\left(\int_0^1 \int_0^1 \psi(s,t)^2\,dt\,ds\right)(1+||x||_2^2) \le 2M(1+||x||_2^2).$$

We now distinguish as before the two cases $[\lambda I - U]_b > 0$ and $[\lambda I - U]_b = 0$. In the first case the set

$$\Sigma = \{x \in X : \lambda x = tU(x) \text{ for some } t \in [0,1]\}$$

is again bounded, because every $x \in \Sigma$ satisfies

$$|\lambda|^2 \, ||x||_2^2 \leq ||U(x)||_2^2 \leq 2M(1 + ||x||_2^2),$$

hence

$$||x||_2^2 \leq \frac{2M}{|\lambda|^2 - 2M}.$$

By Lemma 2, the operator $\lambda I - U$ is k-epi for $k < |\lambda|$, and so $\lambda \notin \sigma_F(U)$ as before.

Assume now that $[\lambda I - U]_b = 0$. Then we may find a sequence $(x_n)_n$ in X such that

$$||\lambda x_n - U(x_n)||_2 \leq \frac{1}{n}||x_n||_2.$$

We claim that the sequence $(x_n)_n$ is bounded. In fact, the estimate

$$\frac{1}{n}||x_n||_2 \geq |\lambda| \, ||x_n||_2 - ||U(x_n)||_2 \geq |\lambda| \, ||x_n||_2 - \sqrt{2M}\sqrt{1 + ||x_n||_2^2}$$

implies that

$$|\lambda| - \sqrt{2M}\left(\frac{1}{||x_n||_2^2} + 1\right)^{1/2} \leq \frac{1}{n}.$$

Letting $n \to \infty$, the unboundedness of $(x_n)_n$ would give $|\lambda| \leq \sqrt{2M}$, contradicting our choice of λ. So we have proved that the sequence $(x_n)_n$ is bounded. The remaining part of the proof goes as in Proposition 1.

Now we give again an alternative proof building on Lemma 4. For $F := \lambda I$ and $G := -U$ the hypotheses of Lemma 4 read

$$[U]_A < |\lambda|, \qquad \sup_{||x||_2 = r} ||U(x)||_2 < |\lambda| r. \tag{4.6}$$

The first estimate in (4.6) follows trivially from the compactness of U. If we choose r so large that $2M < r^2(|\lambda|^2 - 2M)$ we obtain

$$\sup_{||x||_2 = r} ||U(x)||_2^2 \leq 2M(1 + r^2) < |\lambda|^2 r^2, \tag{4.7}$$

where we have used the hypothesis $|\lambda|^2 > 2M$. From (4.7) we conclude that the second estimate in (4.6) holds as well, and so the operator $\lambda I - U$ is k-epi on $B_r(X)$ for $0 \leq k < |\lambda| - \sqrt{2M}$.

Finally, Proposition 3 may also be proved, in rather the same way as Proposition 1, by a straightforward application of Schauder's (or Vignoli's) fixed point principle. In fact, the estimate

$$\sup_{||x||_2 \leq R} \frac{||U(x)||_2^2}{|\lambda|^2 R^2} \leq \frac{2M(1+R^2)}{|\lambda|^2 R^2} < \frac{1+R^2}{R^2}$$

shows that the first expression can be made again ≤ 1 for sufficiently large $R > 0$, and so the (compact) operator H/λ maps the ball $B_R(X)$ into itself.

At this point we can make the same remark as at the end of Section 3; indeed, the spectral method gives again the following refinement of Proposition 3:

Proposition 4. *Suppose that* $|\lambda|^2 > 2M$*, where* M *is given by* (4.5)*. Then the equation*

$$\lambda x(s) - \int_0^1 k(s,t,x(t))\,dt = G(x)(s) \qquad (0 \leq s \leq 1) \tag{4.8}$$

has a solution $x \in L_2[0,1]$ *for any operator* $G : L_2[0,1] \to L_2[0,1]$ *which satisfies* $[G]_A < |\lambda| - \sqrt{2M}$ *and* $[G]_Q < |\lambda| - \sqrt{2M}$*.*

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