Fixed Point Theory, Volume 7, No. 2, 2006, 191-199
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

NOTE ON TOPOLOGICAL DEGREE FOR MONOTONE-TYPE MULTIVALUED MAPS

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Dedicated to Professor Ioan A. Rus on the occasion of his 70th birthday

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Abstract. Topological degree will be indicated for monotone-type available multivalued mappings on open bounded subsets of reflexive separable Banach spaces. This degree is, in particular, a multivalued generalization of the one for single-valued maps developed in [Ber86, BM86].

Key Words and Phrases: topological degree, monotone-type maps, multivalued maps.
2000 Mathematics Subject Classification: 47H04, 47H05, 47H11.

1. Introduction

Monotone multivalued maps in real Hilbert spaces are known to be single-valued on a G_δ-set which is dense in the interior of their domains (see e.g. [Dei92, Proposition 4.23]). This indicates that the notion of monotonicity is rather restrictive for multivalued maps. On the other hand, some sorts of monotonicity often allow us to avoid other restrictions like compactness, typically required in the fixed point theory. The basis of the theory of monotone operators was mainly developed by F. E. Browder, H. Brezis, T. Kato, J. Leray,

The first author was supported by the Council of Czech Government (MSM 6198959214).

J.-L. Lions, G. J. Minty, M. I. Vishik, ... There exist several monographs dealing with monotone operators (see e.g. [Bar76, PS78, SM99, Vai73, Zei90]).

The topological degree theory for single-valued maps of monotone-type in reflexive Banach spaces was developed in the 80's by F. E. Browder [Bro83a]-[Bro83d]. His technique was based on the combination of the Galerkin approximation and the standard Brouwer degree. Using the Leray-Schauder degree and the Browder-Ton embedding theorem, J. Berkovits [Ber86], jointly with V. Mustonen [BM86], defined a new degree allowing them to make more straightforward applications in nonlinear analysis, in particular, for parabolic problems [BM92].

Although some further definitions of degree for both single-valued as well as multivalued maps were given (see e.g. [Kra83, KS99, KS05, Skr86, Skr94, ZC90], from which especially the Skrypnik degree became popular (cf. [KS99, KS05, Skr86, Skr94]), the degree theory for monotone-type maps is far from to be built in a satisfactory way.

In our note, we would only like to indicate a possible extension of the approach in [Ber86, BM86] to multivalued maps. A more systematic and detailed exposition of our results in this field will be published elsewhere.

2. Some preliminaries

For a given reflexive Banach space E, by E^*, we shall denote its dual space. In what follows, the symbol $\langle\,\cdot\,,\,\cdot\,\rangle$ stands for pairing between E and E^*.

We recall the Browder-Ton Theorem (cf. [BM86, Ber86, BT68]).

Theorem 2.1. *For every reflexive separable Banach space E, there exists a separable Hilbert space F and a linear completely continuous injection $h : F \to E$ such that $h(F)$ is dense in E.*

Assume that $h : F \to E$ is the same as in Theorem 2.1. We define $h^* : E^* \to F$ by putting:

$$h^*(f)(v) := f(h(v)), \quad \text{for every } f \in E^* \text{ and } v \in F.$$

We also define a map $\widehat{h} : E^* \to F$ by the formula

$$\widehat{h}(f) := v_f, \quad \text{for every } f \in E^*,$$

where v_f is a unique element of F, for which we have

$$\langle v, v_f \rangle = h^*(f)(v).$$

Let us note that the existence of v_f follows directly from the well-known Fréchet-Riesz Theorem.

Corollary 2.2 (cf. [Ber86, BM86])**.** *The map $\widehat{h} : E^* \to F$ is a linear completely continuous injection.*

Let (X, d) be a metric space and $q : X \to E^*$ be a mapping. The mapping q is called *bounded* if, for every bounded $B \subset X$, the set $q(B)$ is bounded. It is called *demicontinuous* if, for every sequence $\{x_n\} \subset X$, the condition $\{x_n\} \to x$ implies that $\{q(x_n)\} \rightharpoonup q(x)$, i.e. that the sequence $\{q(x_n)\}$ weakly converges to $q(x)$.

3. Multivalued mappings

In this paper, by homology, we shall understand the Čech homology functor H with compact carriers and coefficients in the field of rational numbers $\mathbb{Q}$ (for details, see [AG03] or [G06]). A metric space X is called *acyclic* if

$$H_n(X) = \begin{cases} \mathbb{Q}, & \text{for } n = 0, \\ 0, & \text{for } n > 0. \end{cases}$$

A mapping $p : X \to Y$ between two metric spaces is called a *Vietoris map* if:

(i) p is onto,
(ii) p is proper, i.e; for any compact $K \subset Y$, the set $p^{-1}(K)$ is compact,
(iii) $p^{-1}(y)$ is acyclic, for every $y \in Y$.

The symbol $p : X \Longrightarrow Y$ will be reserved for Vietoris mappings.

Let us recall the following properties of Vietoris mappings:

(i) If $X \overset{p}{\Longrightarrow} Y \overset{p_1}{\Longrightarrow} Z$ are two Vietoris mappings, then the composition $p_1 \circ p : X \Longrightarrow Z$ is a Vietoris map, too.
(ii) If $p : X \Longrightarrow Y$ is a Vietoris map, then the map $\widetilde{p} : p^{-1}(B) \Longrightarrow B$, $\widetilde{p}(x) = p(x)$, for every $x \in p^{-1}(B)$, is a Vietoris map, for every $B \subset Y$.

In what follows, by a multivalued map from X to Y, we shall understand the diagram

$$X \overset{p}{\Longleftarrow} \Gamma \overset{q}{\longrightarrow} Y$$

in which p is a Vietoris map, q is an arbitrary map and X, Γ, Y are metric spaces. We shall write, for the sake of simplicity, $(p, q) : X \to Y$ (for more details, see [AG03] or [G06]). A multivalued map $(p, q) : X \to Y$ is called *admissible* (see again [AG03] or [G06]) if q is continuous and compact, i.e. if $q(\Gamma)$ is relatively compact in Y.

Note that the map $(p, q) : X \to Y$ induces the set-valued mapping $\Phi(p, q) = \Phi : X \multimap Y$ defined by $\Phi(x) := q(p^{-1}(x))$, for every $x \in X$. Moreover, if (p, q) is admissible, then Φ is compact and upper semicontinuous.

Now, until the end of this paper, by U, we shall denote an open bounded subset of a reflexive separable Banach space E and, by $\overline{U}$, the closure of U in E. Moreover, we shall consider a multivalued map

$$\overline{U} \overset{p}{\Longleftarrow} \Gamma \overset{q}{\longrightarrow} E^*, \quad ((p, q) : \overline{U} \to E^*).$$

Definition 3.1. A multivalued map $(p, q) : \overline{U} \to E^*$ is called *monotone* (resp. *strongly monotone*) if, for every $x_1, x_2 \in \overline{U}$ and for every $y_1 \in q(p^{-1}(x_1))$, $y_2 \in q(p^{-1}(x_2))$, we have

$$\langle x_1 - x_2, y_1 - y_2 \rangle \geq 0 \tag{1}$$

(resp.

$$\langle x_1 - x_2, y_1 - y_2 \rangle \geq c \left\| x_1 - x_2 \right\|^2, \tag{2}$$

where $c > 0$ is a suitable constant).

For more details concerning monotone-type mappings, see e.g. [Bar76, G06, PS78, SM99, Vai73, Zei90].

Let us associate with any multivalued map

$$\overline{U} \overset{p}{\Longleftarrow} \Gamma \overset{q}{\longrightarrow} E^*,$$

the following multivalued map

$$\overline{U} \overset{p}{\Longleftarrow} \Gamma \overset{h \circ \widehat{h} \circ q}{\longrightarrow} E, \tag{3}$$

where h and $\widehat{h}$ were defined in Section 2.

Definition 3.2. A multivalued map $(p, q) : \overline{U} \to E^*$ is called *d-admissible* if p is Vietoris and q is demicontinuous and bounded.

As an easy consequence of Theorem 2.1 and Corollary 2.2, we get:

Proposition 3.3. *If* $(p, q) : \overline{U} \to E^*$ *is a d-admissible map, then the map* $(p, h \circ \widehat{h} \circ q) : \overline{U} \to E$ *defined in (3) is admissible.*

4. Degree for monotone-type Multivalued maps

We shall introduce the class of multivalued mappings for which we shall be able to define the topological degree. We shall be particularly interested in monotone resp. strongly monotone maps satisfying conditions (1) resp. (2). But since our considerations have also meaning with conditions not necessarily satisfying (1) or (2), we shall rather speak about monotone-type than monotone maps.

Definition 4.1. A multivalued map $(p, q) \colon \overline{U} \to E^*$ is said to be *available* if the following conditions are satisfied:

(i) (p, q) is d-admissible,

(ii) for every sequence $\{x_n\} \subset \overline{U}$ such that $\{x_n\} \rightharpoonup x$ if there exists a sequence $\{y_n\} \subset E^*$ with $y_n \in q(p^{-1}(x_n))$ and such that $\langle x_n, y_n \rangle \leq 0$, for every $n = 1, 2, \ldots$, then $\{x_n\} \to x$.

We put:

$$\begin{aligned} \mathcal{A}(\overline{U}, E^*) \quad &:= \quad \{(p, q) \colon \overline{U} \to E^* \mid (p, q) \text{ is available}\}, \\ \mathcal{A}_{\partial U}(\overline{U}, E^*) \quad &:= \quad \{(p, q) \in \mathcal{A}(\overline{U}, E^*) \mid 0 \notin q(p^{-1}(\partial U))\}. \end{aligned}$$

Proposition 4.2. *Assume that* $(p, q) \in \mathcal{A}(\overline{U}, E^*)$. *Then:*

(i) *the map*

$$\varphi_\varepsilon \colon \overline{U} \multimap E, \quad \varphi_\varepsilon(x) := \left\{ x + y \;\middle|\; y \in \frac{1}{\varepsilon} \cdot h(\widehat{h}(q(p^{-1}(x)))) \right\}$$

is, for every $\varepsilon > 0$, *a compact admissible vector-field of the type* $\mathrm{id}_E - (\widetilde{p}, \widetilde{q})$, *i.e. a compact admissible perturbation of identity, where* h, $\widehat{h}$ *are defined in Section 2 and* $\widetilde{p}$, $\widetilde{q}$ *are induced by* p, q, h, $\widehat{h}$,

(ii) *if for* $\{\varepsilon_n\} \to 0$ *there exists* $x_n \in \overline{U}$ *such that* $0 \in \varphi_{\varepsilon_n}(x_n)$, *then there exists* $x \in \overline{U}$, *for which* $0 \in q(p^{-1}(x))$.

Proof. Observe that assertion (i) is a simple consequence of Proposition 3.3 (cf. [AG03, G06]). Therefore, we shall restrict ourselves to prove (ii). Let us assume, for the sake of simplicity, that $\varepsilon_n = 1/n$ and $\varphi_{\varepsilon_n} = \varphi_{\frac{1}{n}}$. Let $0 \in \varphi_{\varepsilon_n}(x_n)$, for every $n = 1, 2, \ldots$, and $y_n \in q(p^{-1}(x_n))$ be such that $0 =$

$x_n + nh(\widehat{h}(y_n))$. Since U is bounded and (p,q) is bounded, we can assume without any loss of generality that $\{x_n\} \rightharpoonup x$ and $\{y_n\} \rightharpoonup y$. From

$$0 = x_n + n \cdot (h(\widehat{h}(y_n))),$$

we receive

$$h(\widehat{h}(y_n)) = -\frac{1}{n} x_n.$$

Thus, we infer that $\{h(\widehat{h}(y_n))\} \to 0 = h(\widehat{h}(y))$, and since $h \circ \widehat{h}$ is a linear injection, we obtain $y = 0$.

According to the definition of $\widehat{h}$, we arrive at

$$\langle y_n, x_n \rangle = -n \langle y_k, h(\widehat{h}(y_k)) \rangle = -n \|\widehat{h}(y_n)\|_F^2 \leq 0.$$

By means of condition (ii), we get that $\{x_n\} \to x$ and, from demicontinuity of (p,q), we have $0 \in q(p^{-1}(x))$. $\square$

Now, we are ready to define the topological degree for mappings in the class $\mathcal{A}_{\partial U}(\overline{U}, E^*)$.

Lemma 4.3. *Assume that $(p,q) \in \mathcal{A}_{\partial U}(\overline{U}, E^*)$. Then there exists $\varepsilon^* > 0$ such that, for every $0 < \varepsilon < \varepsilon^*$, we have $0 \notin \varphi_\varepsilon(\partial U)$, where φ_ε is defined in Proposition 4.2(i).*

Proof. If not, we get a contradiction with condition (ii) in Proposition 4.2. $\square$

We put still:

$$\widetilde{\mathcal{A}}_{\partial U}(\overline{U}, E) := \{\varphi \colon \overline{U} \multimap E \mid \varphi \text{ is a compact admissible vector-field, and } 0 \notin \varphi(\partial U)\}.$$

In view of Lemma 4.3 and Proposition 4.2, we can state:

Proposition 4.4. *Assume $(p,q) \in \mathcal{A}_{\partial U}(\overline{U}, E^*)$, and let ε^* be chosen according to Lemma 4.3. Then $\varphi_\varepsilon \in \widetilde{\mathcal{A}}_{\partial U}(\overline{U}, E)$, for every $0 < \varepsilon < \varepsilon^*$. Moreover, for every $0 < \varepsilon_0, \varepsilon_1 < \varepsilon^*$, we have:*

$$\deg(\varphi_{\varepsilon_0}) = \deg(\varphi_{\varepsilon_1}),$$

where deg *denotes the topological degree for $\widetilde{\mathcal{A}}_{\partial U}(\overline{U}, E)$ (defined as in* [AG03, G06]*).*

Proof. From Lemma 4.3, it follows that $\varphi_\varepsilon \in \widetilde{\mathcal{A}}_{\partial U}(\overline{U}, E)$, for every $0<\varepsilon<\varepsilon^*$. Observe that if $0 < \varepsilon_0, \varepsilon_1 < \varepsilon^*$, then the formula

$$t \cdot \varphi_{\varepsilon_1} + (1-t)\varphi_{\varepsilon_0} = \mathrm{id} + \left(\frac{t}{\varepsilon_1} + \frac{1-t}{\varepsilon_0}\right) h\widehat{h}qp^{-1} = \varphi_{\varepsilon_t},$$

where

$$\frac{1}{\varepsilon_t} = t \cdot \frac{1}{\varepsilon_1} + (1-t)\frac{1}{\varepsilon_0},$$

gives us the homotopy linking φ_{ε_0} with φ_{ε_1}, and the conclusion follows from the homotopy property of the given topological degree. $\square$

We define the function

$$\mathrm{Deg}\colon \mathcal{A}_{\partial U}(\overline{U}, E^*) \to \mathbb{Z}, \tag{4}$$

where $\mathbb{Z}$ denotes the set of integers, by putting

$$\mathrm{Deg}\,((p,q)) := \deg(\varphi_\varepsilon),$$

where $0 < \varepsilon < \varepsilon^*$ and ε^* is chosen for (p,q), according to Lemma 4.3.

It follows from Proposition 4.4 that

Theorem 4.5. *The degree in (4) is correctly defined and satisfies the standard existence, additivity and homotopy properties.*

5. Concluding remarks

Some concluding remarks are in order.

• Since in finite dimensional Banach spaces $E = E^*$, the weak and strong convergences coincide, demicontinuous means continuous, and relative compactness reduces to boundedness, condition (ii) in Definition 4.1 becomes superfluous for the definition of the topological degree, as defined e.g. in [AG03, G06].

• If $(p,q) = (\mathrm{id}\,|_{\overline{U}}, q)$, where $q : \overline{U} \to E^*$ is a single-valued map, then condition (ii) in Definition 4.1 can be, without any loss of generality, replaced by the hypothesis:

if $\limsup \langle x_n - x, q(x_n)\rangle \leq 0$, for each $\{x_n\} \rightharpoonup x$, then $\{x_n\} \to x$, characterizing so the (S_+)-class of functions.

• As pointed out in [Ber86, Zei90], the following implications take place for the properties of demicontinuous maps $q : \overline{U} \to E^*$:

strong monotonicity $\Longrightarrow$ (S_+)-property $\Longrightarrow$ $\liminf \langle x_n - x, q(x_n) \rangle \geq 0$, for each $\{x_n\} \subset \overline{U}$ with $x_n \rightharpoonup x$.

The authors of [Ber86, BM86] studied also the classes of pseudomonotone and quasimonotone demicontinuous maps. The latter can be characterized by the last implied inequality, while pseudomonotone maps contain the (S_+)-class and are contained in the class of quasimonotone maps.

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Received: October 26, 2006; Accepted: November 12, 2006.