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LEFSCHETZ FIXED POINT THEOREMS FOR COMPACT MORPHISMS

RAVI P. AGARWAL* AND DONAL O'REGAN**

Dedicated to Professor Ioan A. Rus on the occasion of his 70th birthday

*Department of Mathematical Sciences
Florida Institute of Technology
Melbourne, Florida 32901, USA
E-mail: agarwal@fit.edu

**Department of Mathematics
National University of Ireland, Galway, Ireland
E-mail: donal.oregan@nuigalway.ie

Abstract. The Lefschetz fixed point theorem is extended to compact morphisms defined on admissible subsets of a Hausdorff topological space. Also using the projective limit approach we present new Lefschetz fixed point theorems for compact morphisms defined on PRLF's or CPRLF's.
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1. Introduction

This paper has two main sections. In Section 2 we present new Lefschetz fixed point theorems for compact morphisms defined on admissible (to be defined later) subsets of a Hausdorff topological space. In Section 3 we present some other Lefschetz fixed point theorems for compact morphisms between Fréchet spaces. Our maps will be defined on PRLF's or CPRLF's. These sets are natural in applications in the Fréchet space setting since they include pseudo-open sets. The theory in Section 3 is based on results in Section 2 and on viewing a Fréchet space as a projective limit of a sequence of Banach spaces $\{E_n\}_{n\in N}$ (here $N = \{1, 2, \dots\}$).

Let X, Y and Γ be Hausdorff topological spaces. A continuous single valued map $p : \Gamma \to X$ is called a Vietoris map (written $p : \Gamma \Rightarrow X$) if the following two conditions are satisfied:
(i). for each $x \in X$, the set $p^{-1}(x)$ is acyclic
(ii). p is a proper map i.e. for every compact $A \subseteq X$ we have that $p^{-1}(A)$ is compact.

Let $D(X,Y)$ be the set of all pairs $X \overset{p}{\Leftarrow} \Gamma \overset{q}{\to} Y$ where p is a Vietoris map and q is continuous. We will denote every such diagram by (p,q). Given two diagrams (p,q) and (p',q'), where $X \overset{p'}{\Leftarrow} \Gamma' \overset{q'}{\to} Y$, we write $(p,q) \sim (p',q')$ if there are maps $f : \Gamma \to \Gamma'$ and $g : \Gamma' \to \Gamma$ such that $q' \circ f = q$, $p' \circ f = p$, $q \circ g = q'$ and $p \circ g = p'$. The equivalence class of a diagram $(p,q) \in D(X,Y)$ with respect to $\sim$ is denoted by

$$\phi = \{X \overset{p}{\Leftarrow} \Gamma \overset{q}{\to} Y\} : X \to Y$$

or $\phi = [(p,q)]$ and is called a morphism from X to Y. We let $M(X,Y)$ be the set of all such morphisms. For any $\phi \in M(X,Y)$ a set $\phi(x) = q\,p^{-1}(x)$ where $\phi = [(p,q)]$ is called an image of x under a morphism ϕ. Let $X \subseteq Y$. A point $x \in X$ is called a fixed point of a morphism $\phi \in M(X,Y)$ if $x \in \phi(x)$.

Consider vector spaces over a field K. Let E be a vector space and $f : E \to E$ an endomorphism. Now let $N(f) = \{x \in E :\ f^{(n)}(x) = 0$ for some $n\}$ where $f^{(n)}$ is the n^{th} iterate of f, and let $\tilde{E} = E \setminus N(f)$. Since $f(N(f)) \subseteq N(f)$ we have the induced endomorphism $\tilde{f} : \tilde{E} \to \tilde{E}$. We call f admissible if $dim\, \tilde{E} < \infty$; for such f we define the generalized trace $Tr(f)$ of f by putting $Tr(f) = tr(\tilde{f})$ where tr stands for the ordinary trace.

Let $f = \{f_q\} : E \to E$ be an endomorphism of degree zero of a graded vector space $E = \{E_q\}$. We call f a Leray endomorphism if (i). all f_q are admissible and (ii). almost all $\tilde{E}_q$ are trivial. For such f we define the generalized Lefschetz number $\Lambda(f)$ by

$$\Lambda(f) = \sum_q (-1)^q \, Tr\,(f_q).$$

Let H be the Čech homology functor with compact carriers and coefficients in the field of rational numbers K from the category of Hausdorff topological spaces and continuous maps to the category of graded vector spaces and linear maps of degree zero. Thus $H(X) = \{H_q(X)\}$ is a graded vector space, $H_q(X)$ being the q-dimensional Čech homology group with compact carriers of X. For a continuous map $f : X \to X$, $H(f)$ is the induced linear map $f_\star = \{f_{\star q}\}$ where $f_{\star q} : H_q(X) \to H_q(X)$.

With Čech homology functor extended to a category of morphisms (see [7 pp. 364]) we have the following well known definitions (note the homology functor H extends over this category i.e. for a morphism

$$\phi = \{X \overset{p}{\Leftarrow} \Gamma \overset{q}{\to} Y\} : X \to Y$$

we define the induced map

$$H\,(\phi) = \phi_\star : H(X) \to H(Y)$$

by putting $\phi_\star = q_\star \circ p_\star^{-1}$).

Definition 1.1. The morphism $\phi \in M(X, X)$ is called a Lefschetz morphism if the map $\phi_\star : H(X) \to H(X)$ is a Leray endomorphism. For a Lefschetz morphism ϕ we define the Lefschetz number $\Lambda(\phi)$ (or $\Lambda_X(\phi)$) by $\Lambda(\phi) = \Lambda(\phi_\star)$.

Definition 1.2. A Hausdorff topological space X is said to be a Lefschetz space provided every compact $\phi \in M(X, X)$ is a Lefschetz morphism and $\Lambda(\phi) \neq 0$ implies ϕ has a fixed point.

Recall the following result [6 pp. 227].

Theorem 1.1. *If $\phi : X \to Y$ and $\psi : Y \to Z$ are two morphisms (here X, Y and Z are Hausdorff topological spaces) then* $(\psi \circ \phi)_\star = \psi_\star \circ \phi_\star$.

Two morphisms $\phi, \psi \in M(X, Y)$ are homotopic (written $\phi \sim \psi$) provided there is a morphism $\chi \in M(X \times [0,1], Y)$ such that $\chi(x, 0) = \phi(x)$, $\chi(x, 1) = \psi(x)$ for every $x \in X$ (i.e. $\phi = \chi \circ i_0$ and $\psi = \chi \circ i_1$, where $i_0, i_1 : X \to X \times [0,1]$ are defined by $i_0(x) = (x, 0)$, $i_1(x) = (x, 1)$).

Recall the following result [6 pp. 231].

Theorem 1.2. *If $\phi \sim \psi$ then $\phi_\star = \psi_\star$.*

Now let I be a directed set with order $\leq$ and let $\{E_\alpha\}_{\alpha \in I}$ be a family of locally convex spaces. For each $\alpha \in I$, $\beta \in I$ for which $\alpha \leq \beta$ let $\pi_{\alpha,\beta}$:

$E_\beta \to E_\alpha$ be a continuous map. Then the set

$$\left\{ x = (x_\alpha) \in \prod_{\alpha \in I} E_\alpha : \; x_\alpha = \pi_{\alpha,\beta}(x_\beta) \; \forall\, \alpha,\, \beta \in I,\, \alpha \leq \beta \right\}$$

is a closed subset of $\prod_{\alpha \in I} E_\alpha$ and is called the projective limit of $\{E_\alpha\}_{\alpha \in I}$ and is denoted by $\lim_{\leftarrow} E_\alpha$ (or $\lim_{\leftarrow} \{E_\alpha, \pi_{\alpha,\beta}\}$ or the generalized intersection [8 pp. 439] $\cap_{\alpha \in I} E_\alpha$.)

2. Fixed point theory

We begin with Hausdorff topological vector spaces. Some of the ideas in this section were motivated from [1].

For our first result we assume X is a subset of a Hausdorff topological vector space E. We say X is NES admissible if for every compact subset K of X and every neighborhood V of zero there exists a continuous function $h_V : K \to E$ such that

(i). $x - h_V(x) \in V$ for all $x \in K$;

(ii). $h_V(K)$ is contained in a subset C of X with C a Lefschetz space;

and

(iii). h_V and $i : K \hookrightarrow X$ are homotopic.

Theorem 2.1. *Let E be a Hausdorff topological vector space and let $X \subseteq E$ be NES admissible. If $\phi \in M(X, X)$ is a compact morphism then*

(i). the Lefschetz number $\Lambda(\phi)$ is well defined

and

(ii). if $\Lambda(\phi) \neq 0$ then ϕ has a fixed point

i.e. X is a Lefschetz space.

Proof. Let $\phi = \{X \overset{p}{\Leftarrow} \Gamma \overset{q}{\to} X\} : X \to X$ and let K denote a compact set in which $\phi(X)$ is included. Next let $\mathcal{N}$ be a fundamental system of neighborhoods of the origin 0 in E and $V \in \mathcal{N}$. Now there exists a continuous function $h_V : K \to E$ and a $C \subseteq X$, C a Lefschetz space with $x - h_V(x) \in V$ for all $x \in K$, $h_V(K) \subseteq C$ and $h_V \sim i$ where $i : K \hookrightarrow C$.

Now let $q_V = h_V\, q : \Gamma \to X$. Notice q_V is a compact map, $q_V(\Gamma) \subseteq C$ and $q \sim q_V$; we know since $h_V \sim i$ that there exists a map $\chi : K \times [0,1] \to X$

with $\chi(x,0) = h_V(x)$ and $\chi(x,1) = i(x)$, and now let $\Phi(x,t) = \chi(q(x),t)$ for $(x,t) \in \Gamma \times [0,1]$ (note p is surjective so $p^{-1}(X) = \Gamma$ and so $q : \Gamma \to K$) and note $\Phi(x,0) = h_V\, q(x) = q_V(x)$ and $\Phi(x,1) = i(q(x)) = q(x)$.

Next we note that the following diagram commutes:

$$\begin{tikzcd} H(C) \arrow[r, "i_*"] & H(X) \\ H(C) \arrow[u, "(\overline{q_V})_*(p_V)_*^{-1}"] \arrow[r, "i_*"'] & H(X) \arrow[u, "q_* p_*^{-1}"'] \arrow[ul, "(q_V')_* p_*^{-1}"'] \end{tikzcd}$$

where

$$p_V : p^{-1}(C) \to C,\ \overline{q_V} : p^{-1}(C) \to C,\ q_V' : \Gamma \to C$$

denote contractions of the appropriate maps (see also (1.1) on [3 pp. 214]). To see that the above diagram commutes note Theorem 1.1 and Theorem 1.2 imply

$$i_\star\, (q_V')_\star\, p_\star^{-1} = (i\, q_V')_\star\, p_\star^{-1} = (q_V)_\star\, p_\star^{-1} = q_\star\, p_\star^{-1}$$

since $q \sim q_V$. Also it is easy to see that $(q_V')_\star\, p_\star^{-1}\, i_\star = (\overline{q_V})_\star\, (p_V)_\star^{-1}$. Now since q_V is a compact map then the morphism $[(p_V, \overline{q_V})]$ is compact. Also since C is a Lefschetz space then $(\overline{q_V})_\star\, (p_V)_\star^{-1}$ is a Leray endomorphism and $\Lambda((\overline{q_V})_\star\, (p_V)_\star^{-1}) \neq 0$ implies the morphism $[(p_V, \overline{q_V})]$ has a fixed point. Since the diagram commutes then [3 pp. 214] guarantees that ϕ is a Lefschetz morphism and $\Lambda(q_\star\, p_\star^{-1}) = \Lambda((\overline{q_V})_\star\, (p_V)_\star^{-1})$.

Next assume $\Lambda(q_\star\, p_\star^{-1}) \neq 0$. Then $\Lambda((\overline{q_V})_\star\, (p_V)_\star^{-1}) \neq 0$ so since C is a Lefschetz space there exists $x_V \in C$ with $x_V \in \overline{q_V}\, p_V^{-1}\, (x_V)$. Now $x_V = h_V(y_V)$ for some $y_V \in q\, p^{-1}\, (x_V)$. Now since $q\, p^{-1}\, (x_V) \in K$ (note $q : \Gamma \to K$) from (i) in the definition above we have $y_V - h_V(y_V) \in V$. Thus $y_V - x_V \in V$. Now since K is compact we may assume without loss of generality that there exists x with $y_V \to x$. Also since $y_V - x_V \in V$ we have $x_V \to x$. This together with $y_V \in q\, p^{-1}\, (x_V)$ and the upper semicontinuity of $q\, p^{-1}$ (see [5 pp.26]) implies $x \in q\, p^{-1}\, (x)$ and the proof is complete. $\square$

Let X be a subset of a Hausdorff topological vector space E. Let V be a neighborhood of the origin 0 in E. X is said to be NES admissible V-dominated if there exists a NES admissible space X_V and two continuous maps $r_V : X_V \to X$, $s_V : X \to X_V$ such that $x - r_V\, s_V(x) \in V$ for all $x \in X$ and also that $r_V\, s_V \sim Id_X$. X is said to be almost NES admissible dominated if X is NES admissible V-dominated for every neighborhood V of the origin 0 in E.

Theorem 2.2. *Let X be a subset of a Hausdorff topological vector space E. Also assume X is almost NES admissible dominated. If $\phi \in M(X,X)$ is a compact morphism then*

(i). the Lefschetz number $\Lambda(\phi)$ is well defined

and

(ii). if $\Lambda(\phi) \neq 0$ then ϕ has a fixed point

i.e. X is a Lefschetz space.

Proof. Let $\phi = \{X \overset{p}{\Leftarrow} \Gamma \overset{q}{\to} X\} : X \to X$ and let K denote a compact set in which $\phi(X)$ is included. Next let $\mathcal{N}$ be a fundamental system of neighborhoods of the origin 0 in E and $V \in \mathcal{N}$. Now there exists a NES admissible space X_V and two continuous maps $r_V : X_V \to X$, $s_V : X \to X_V$ such that $x - r_V\, s_V(x) \in V$ for all $x \in X$ and $r_V\, s_V \sim Id_X$. Let $\phi_V = s_V\, \phi\, r_V$ and note $\phi_V : X_V \to X_V$ is a compact morphism [6 pp. 225] so let $\phi_V = \{X_V \overset{p_V}{\Leftarrow} \Gamma' \overset{q_V}{\to} X_V\} : X_V \to X_V$. Now since $r_V\, s_V \sim Id_X$ then Theorem 1.2 guarantees that $(r_V)_\star (s_V)_\star = Id_{H(X)}$. Next we note that the following diagram commutes:

$$\begin{tikzcd}
H(X) \arrow[r, "(s_V)_*"] & H(X_V) \\
H(X) \arrow[u, "q_*p_*^{-1}"] \arrow[r, "(s_V)_*"'] & H(X_V) \arrow[u, "(q_V)_*(p_V)_*^{-1}"'] \arrow[ul, "q_*p_*^{-1}(r_V)_*"']
\end{tikzcd}$$

To see that the diagram commutes note $(r_V)_\star\,(s_V)_\star = Id_{H(X)}$ guarantees that

$$q_\star\, p_\star^{-1}\,(r_V)_\star\,(s_V)_\star = q_\star\, p_\star^{-1}$$

and Theorem 1.1 implies

$$(s_V)_\star\, q_\star\, p_\star^{-1}\,(r_V)_\star = (s_V)_\star\,\phi_\star\,(r_V)_\star = (s_V\,\phi\, r_V)_\star = (\phi_V)_\star = (q_V)_\star\,(p_V)_\star^{-1}.$$

Now since X_V is NES admissible (so a Lefschetz space from Theorem 2.1) then $(q_V)_\star\,(p_V)_\star^{-1}$ is a Leray endomorphism and $\Lambda((q_V)_\star\,(p_V)_\star^{-1}) \neq 0$ implies the morphism $[(p_V, q_V)]$ has a fixed point. Since the diagram commutes then [3 pp. 214] guarantees that ϕ is a Lefschetz morphism and $\Lambda(q_\star\, p_\star^{-1}) = \Lambda((q_V)_\star\,(p_V)_\star^{-1})$.

Next assume $\Lambda(q_\star\, p_\star^{-1}) \neq 0$. Then $\Lambda((q_V)_\star\,(p_V)_\star^{-1}) \neq 0$ so since X_V is a Lefschetz space there exists $x_V \in X_V$ with $x_V \in q_V\, p_V^{-1}\,(x_V)$ i.e. $x_V \in \phi_V\,(x_V)$ i.e. $x_V \in s_V\,\phi\, r_V\,(x_V)$. Let $y_V = r_V(x_V)$. Now $x_V = s_V(w_V)$ for some $w_V \in \phi\,(y_V)$. Notice $y_V = r_V\, s_V\,(w_V)$. Now since $x - r_V\, s_V\,(x) \in V$ for all $x \in X$ we have $w_V - r_V\, s_V\,(w_V) \in V$ i.e. $w_V - y_V \in V$. Now $w_V \in K$ and K compact guarantees we may assume without loss of generality that there exists x with $w_V \to x$. Also since $w_V - y_V \in V$ we have $y_V \to x$. This together with $w_V \in \phi\,(y_V)$ and the upper semicontinuity of ϕ (see [5 pp.26]) implies $x \in \phi\,(x)$ and the proof is complete. □

Next we extend Theorem's 2.1 and 2.2 to the case of Hausdorff topological spaces. First we gather together some well known preliminaries. For a subset K of a topological space X, we denote by $Cov_X\,(K)$ the set of all coverings of K by open sets of X (usually we write $Cov\,(K) = Cov_X\,(K)$). Given two maps $f, g : X \to Y$ and $\alpha \in Cov\,(Y)$, f and g are said to be α-close if for any $x \in X$ there exists $U_x \in \alpha$, $f(x) \in U_x$ and $g(x) \in U_x$. Given two morphisms $\phi, \psi \in M(X,Y)$ and $\alpha \in Cov\,(Y)$, ϕ and ψ are said to be α-close if for any $x \in X$ there exists $U_x \in \alpha$ such that $\phi(x) \cap U_x \neq \emptyset$ and $\psi(x) \cap U_x \neq \emptyset$. Given a $\phi \in M(X,X)$, and $\alpha \in Cov\,(X)$, a point $x \in X$ is said to be an α-fixed point of the morphism ϕ if there exists a member $U \in \alpha$ such that $x \in U$ and $\phi(x) \cap U \neq \emptyset$.

The following result can be found in [2 pp. 297].

Theorem 2.3. *Let X be a topological space and $\Phi : X \to C(X)$ a upper semicontinuous map (here $C(X)$ denotes the family of nonempty* *closed subsets of X). Suppose there exists a cofinal family of coverings $\theta \subseteq Cov_X\left(\overline{\Phi(X)}\right)$ such that Φ has an α-fixed point for every $\alpha \in \theta$. Then Φ has a fixed point.*

Remark 2.1. From Theorem 2.3 in proving the existence of fixed points in uniform spaces for continuous compact maps it suffices [2 pp. 298] to prove the existence of approximate fixed points (since open covers of a compact set A admit refinements of the form $\{U[x] : x \in A\}$ where U is a member of the uniformity [9 pp. 199] so such refinements form a cofinal family of open covers). For convenience in this paper we will apply Theorem 2.3 only when the space is uniform.

Let X be a subset of a Hausdorff topological space and let X be a uniform space. Then X is said to be Schauder NES admissible if for every compact subset K of X and every open covering $\alpha \in Cov_X(K)$ there exists a continuous function $\pi_\alpha : K \to E$ such that

(i). π_α and $i : K \hookrightarrow X$ are α-close;

(ii). $\pi_\alpha(K)$ is contained in a subset C of X with C a Lefschetz space;

and

(iii). π_α and $i : K \hookrightarrow X$ are homotopic.

Theorem 2.4. *Let X be a subset of a Hausdorff topological space and let X be a uniform space. Also suppose X is Schauder NES admissible. If $\phi \in M(X, X)$ is a compact morphism then*

(i). the Lefschetz number $\Lambda(\phi)$ is well defined

and

(ii). if $\Lambda(\phi) \neq 0$ then ϕ has a fixed point

i.e. X is a Lefschetz space.

Proof. Let

$$\phi = \{X \overset{p}{\Leftarrow} \Gamma \overset{q}{\rightarrow} X\} : X \to X$$

and let K denote a compact set in which $\phi(X)$ is included. Also let $\alpha \in Cov_X(K)$. Then there exists a continuous function $\pi_\alpha : K \to E$, a subset C of X, C a Lefschetz space, $\pi_\alpha(K) \subseteq C$, π_α and $i : K \hookrightarrow X$ are α-close and $\pi_\alpha \sim i$. Let $q_\alpha = \pi_\alpha\, q : \Gamma \to X$. Notice as in Theorem 2.1, q_α is a compact map, $q_\alpha(\Gamma) \subseteq C$ and $q \sim q_\alpha$. Let

$$p_\alpha : p^{-1}(C) \to C,\ \overline{q_\alpha} : p^{-1}(C) \to C,\ q'_\alpha : \Gamma \to C$$

denote contractions of the appropriate maps and as in Theorem 2.1 the following diagram commutes:

$$\begin{tikzcd}
H(C) \arrow[r, "i_*"] & H(X) \\
H(C) \arrow[u, "(\overline{q_\alpha})_*(p_\alpha)_*^{-1}"] \arrow[r, "i_*"'] & H(X) \arrow[u, "q_*p_*^{-1}"'] \arrow[ul, "(q'_\alpha)_*p_*^{-1}"']
\end{tikzcd}$$

Now since q_α is a compact map then the morphism $[(p_\alpha, \overline{q_\alpha})]$ is compact. Also since C is a Lefschetz space then $(\overline{q_\alpha})_\star\,(p_\alpha)_\star^{-1}$ is a Leray endomorphism and $\Lambda((\overline{q_\alpha})_\star\,(p_\alpha)_\star^{-1}) \neq 0$ implies the morphism $[(p_\alpha, \overline{q_\alpha})]$ has a fixed point. Since the diagram commutes then ϕ is a Lefschetz morphism and $\Lambda(q_\star\, p_\star^{-1}) = \Lambda((\overline{q_\alpha})_\star\,(p_\alpha)_\star^{-1})$.

Next assume $\Lambda(q_\star\, p_\star^{-1}) \neq 0$. Then $\Lambda((\overline{q_\alpha})_\star\,(p_\alpha)_\star^{-1}) \neq 0$ so since C is a Lefschetz space there exists $x_\alpha \in C$ with $x_\alpha \in \overline{q_\alpha}\, p_\alpha^{-1}\,(x_\alpha)$. Now since π_α and i are α-close we have that x_α is an α-fixed point of the morphism ϕ (note $x_\alpha = \pi_\alpha\,(y_\alpha)$ and $y_\alpha \in q\, p^{-1}\,(x_\alpha) = \phi(x_\alpha)$ so there exists $U_\alpha \in \alpha$ with $x_\alpha = \pi_\alpha\,(y_\alpha) \in U_\alpha$ and $y_\alpha \in U_\alpha$ i.e. $x_\alpha \in U_\alpha$ and $y_\alpha \in U_\alpha$ i.e. $x_\alpha \in U_\alpha$ and $\phi(x_\alpha) \cap U_\alpha \neq \emptyset$ since $y_\alpha \in U_\alpha$ and $y_\alpha \in \phi(x_\alpha)$). The result now follows from Theorem 2.3 (with Remark 2.1). $\square$

Let X be a Hausdorff topological space and let $\alpha \in Cov(X)$. X is said to be Schauder NES admissible α-dominated if there exists a Schauder NES admissible space X_α and two continuous functions $r_\alpha : X_\alpha \to X,\ s_\alpha : X \to X_\alpha$ such that $r_\alpha\, s_\alpha : X \to X$ and $i : X \to X$ are α-close and also that $r_\alpha\, s_\alpha \sim Id_X$. X is said to be almost Schauder NES admissible dominated if X is Schauder NES admissible α-dominated for every $\alpha \in Cov(X)$.

Theorem 2.5. *Let X be a uniform space and let X be almost Schauder NES admissible dominated. If $\phi \in M(X,X)$ is a compact morphism then*

(i). the Lefschetz number $\Lambda(\phi)$ is well defined

and

(ii). if $\Lambda(\phi) \neq 0$ then ϕ has a fixed point

i.e. X is a Lefschetz space.

Proof. Let $\phi = \{X \overset{p}{\Leftarrow} \Gamma \overset{q}{\rightarrow} X\} : X \to X$ and let K denote a compact set in which $\phi(X)$ is included. Also let $\alpha \in Cov_X(K)$. Now there exists a Schauder NES admissible space X_α and two continuous functions $r_\alpha : X_\alpha \to X$, $s_\alpha : X \to X_\alpha$ such that $r_\alpha\, s_\alpha : X \to X$ and $i : X \to X$ are α-close and also that $r_\alpha\, s_\alpha \sim Id_X$. Let $\phi_\alpha = s_\alpha\, \phi\, r_\alpha$. Note $\phi_\alpha : X_\alpha \to X_\alpha$ is a compact morphism so let $\phi_\alpha = \{X_\alpha \overset{p_\alpha}{\Leftarrow} \Gamma' \overset{q_\alpha}{\rightarrow} X_\alpha\} : X_\alpha \to X_\alpha$. Now as in Theorem 2.2 the following diagram commutes:

$$\begin{tikzcd}
H(X) \arrow[r, "(s_\alpha)_*"] & H(X_\alpha) \\
H(X) \arrow[u, "q_* p_*^{-1}"] \arrow[r, "(s_\alpha)_*"'] & H(X_\alpha) \arrow[u, "(q_\alpha)_*(p_\alpha)_*^{-1}"'] \arrow[ul, "q_* p_*^{-1}(r_\alpha)_*"']
\end{tikzcd}$$

Also since X_α is Schauder NES admissible (so a Lefschetz space) then $(q_\alpha)_\star\,(p_\alpha)_\star^{-1}$ is a Leray endomorphism and $\Lambda((q_\alpha)_\star\,(p_\alpha)_\star^{-1}) \neq 0$ implies the morphism $[(p_\alpha, q_\alpha)]$ has a fixed point. Since the diagram commutes then ϕ is a Lefschetz morphism and $\Lambda(q_\star\, p_\star^{-1}) = \Lambda((q_\alpha)_\star\,(p_\alpha)_\star^{-1})$.

Next assume $\Lambda(q_\star\, p_\star^{-1}) \neq 0$. Then $\Lambda((q_\alpha)_\star\,(p_\alpha)_\star^{-1}) \neq 0$ so since X_α is a Lefschetz space there exists $x_\alpha \in X_\alpha$ with $x_\alpha \in q_\alpha\, p_\alpha^{-1}\,(x_\alpha)$ i.e. $x_\alpha \in s_\alpha\, \phi\, r_\alpha\,(x_\alpha)$. Let $y_\alpha = r_\alpha(x_\alpha)$. Now $x_\alpha = s_\alpha(w_\alpha)$ for some $w_\alpha \in \phi\,(y_\alpha)$. Notice $y_\alpha = r_\alpha\, s_\alpha\,(w_\alpha)$. Now since i and $r_\alpha\, s_\alpha$ are α-close we have that ϕ has an α-fixed point (note there exists $U_\alpha \in \alpha$ with $(y_\alpha =)\, r_\alpha\, s_\alpha\,(w_\alpha) \in U_\alpha$

and $w_\alpha \in U_\alpha$ i.e. $y_\alpha \in U_\alpha$ and $\phi(y_\alpha) \cap U_\alpha \neq \emptyset$ since $w_\alpha \in \phi(y_\alpha)$ and $w_\alpha \in U_\alpha$). The result now follows from Theorem 2.3 (with Remark 2.1). $\square$

3. Fixed point theory in Fréchet spaces

Let $E = (E, \{|\cdot|_n\}_{n\in N})$ be a Fréchet space with the topology generated by a family of seminorms $\{|\cdot|_n : n \in N\}$. We assume that the family of seminorms satisfies

$$|x|_1 \leq |x|_2 \leq |x|_3 \leq \dots \quad \text{for every} \quad x \in E. \tag{3.1}$$

A subset X of E is bounded if for every $n \in N$ there exists $r_n > 0$ such that $|x|_n \leq r_n$ for all $x \in X$. To E we associate a sequence of Banach spaces $\{(\mathbf{E}_n, |\cdot|_n)\}$ described as follows. For every $n \in N$ we consider the equivalence relation $\sim_n$ defined by

$$x \sim_n y \quad \text{iff} \quad |x-y|_n = 0. \tag{3.2}$$

We denote by $\mathbf{E}^n = (E\,/\sim_n, |\cdot|_n)$ the quotient space, and by $(\mathbf{E}_n, |\cdot|_n)$ the completion of $\mathbf{E}^n$ with respect to $|\cdot|_n$ (the norm on $\mathbf{E}^n$ induced by $|\cdot|_n$ and its extension to $\mathbf{E}_n$ are still denoted by $|\cdot|_n$). This construction defines a continuous map $\mu_n : E \to \mathbf{E}_n$. Now since (3.1) is satisfied the seminorm $|\cdot|_n$ induces a seminorm on $\mathbf{E}_m$ for every $m \geq n$ (again this seminorm is denoted by $|\cdot|_n$). Also (3.2) defines an equivalence relation on $\mathbf{E}_m$ from which we obtain a continuous map $\mu_{n,m} : \mathbf{E}_m \to \mathbf{E}_n$ since $\mathbf{E}_m\,/\sim_n$ can be regarded as a subset of $\mathbf{E}_n$. We now assume the following condition holds:

$$\begin{cases} \text{for each } n \in N, \quad \text{there exists a Banach space } (E_n, |\cdot|_n) \\ \text{and an isomorphism (between normed spaces) } j_n : \mathbf{E}_n \to E_n. \end{cases} \tag{3.3}$$

Remark 3.1. (i). For convenience the norm on E_n is denoted by $|\cdot|_n$.
(ii). Usually in applications $\mathbf{E}_n = \mathbf{E}^n$ for each $n \in N$.
(iii). Note if $x \in \mathbf{E}_n$ (or $\mathbf{E}^n$) then $x \in E$. However if $x \in E_n$ then x is not necessaily in E and in fact E_n is easier to use in applications (even though E_n is isomorphic to $\mathbf{E}_n$). For example if $E = C[0,\infty)$, then $\mathbf{E}^n$ consists of the class of functions in E which coincide on the interval $[0,n]$ and $E_n = C[0,n]$.

Finally we assume

$$E_1 \supseteq E_2 \supseteq \dots \text{ and for each } n \in N,\ |x|_n \le |x|_{n+1}\ \forall\, x \in E_{n+1}. \tag{3.4}$$

Let $\lim_{\leftarrow} E_n$ (or $\cap_1^\infty E_n$ where $\cap_1^\infty$ is the generalized intersection [8]) denote the projective limit of $\{E_n\}_{n\in N}$ (note $\pi_{n,m} = j_n\, \mu_{n,m}\, j_m^{-1} : E_m \to E_n$ for $m \ge n$) and note $\lim_{\leftarrow} E_n \cong E$, so for convenience we write $E \lim_{\leftarrow} E_n$.

For each $X \subseteq E$ and each $n \in N$ we set $X_n = j_n\, \mu_n(X)$, and we let $\overline{X_n}$ and ∂X_n denote respectively the closure and the boundary of X_n with respect to $|\cdot|_n$ in E_n. Also the pseudo-interior of X is defined by [4]

$$pseudo - int\,(X) = \{x \in X :\ j_n\, \mu_n(x) \in \overline{X_n} \setminus \partial X_n \text{ for every } n \in N\}.$$

The set X is pseudo-open if $X = pseudo - int\,(X)$.

Let E and E_n be as described above. Some of the ideas in this section were motivated from [10].

Definition 3.1. A set $A \subseteq E$ is said to be PRLS if for each $n \in N$, $A_n \equiv j_n\, \mu_n\,(A)$ is a Lefschetz space.

Definition 3.2. A set $A \subseteq E$ is said to be CPRLS if for each $n \in N$, $\overline{A_n}$ is a Lefschetz space.

Example 3.1. Let A be pseudo-open. Then A is a PRLS.

To see this fix $n \in N$. We now show

$$A_n \text{ is a open subset of } E_n.$$

First notice $A_n \subseteq \overline{A_n} \setminus \partial A_n$ since if $y \in A_n$ then there exists $x \in A$ with $y = j_n \mu_n(x)$ and this together with $A = pseudo - int\, A$ yields $j_n \mu_n(x) \in \overline{A_n} \setminus \partial A_n$ i.e. $y \in \overline{A_n} \setminus \partial A_n$. In addition notice

$$\overline{A_n} \setminus \partial A_n = (int\, A_n \cup \partial A_n) \setminus \partial A_n = int\, A_n \setminus \partial A_n = int\, A_n$$

since $int\, A_n \cap \partial A_n = \emptyset$. Consequently

$$A_n \subseteq \overline{A_n} \setminus \partial A_n = int\, A_n, \quad \text{so} \quad A_n = int\, A_n.$$

As a result A_n is open in E_n. Thus A_n is a Lefschetz space [7 pp. 368], so A is a PRLS.

Our first result is for Volterra type operators.

Theorem 3.1. *Let E and E_n be as described above, $C \subseteq E$ is an PRLS and $F : C \to 2^E$ and for each $n \in N$ assume $F : C_n \to 2^{E_n}$. Suppose the following conditions are satisfied:*

$$\text{for each } n \in N,\ F \in M(C_n, C_n) \text{ is a compact morphism} \tag{3.5}$$

$$\text{for each } n \in N,\ \Lambda_{C_n}(F) \neq 0 \tag{3.6}$$

and

$$\begin{cases} \text{for each } n \in \{2,3,\dots\} \text{ if } y \in C_n \text{ solves } y \in F\,y \text{ in } E_n \\ \text{then } y \in C_k \text{ for } k \in \{1,\dots,n-1\}. \end{cases} \tag{3.7}$$

Then F has a fixed point in E.

Proof. Fix $n \in N$. Now there exists $y_n \in C_n$ with $y_n \in F\,y_n$. Lets look at $\{y_n\}_{n\in N}$. Notice $y_1 \in C_1$ and $y_k \in C_1$ for $k \in N\backslash\{1\}$ from (3.7). As a result $y_n \in C_1$ for $n \in N$, $y_n \in F\,y_n$ in E_n together with (3.5) implies there is a subsequence $N_1^\star$ of N and a $z_1 \in C_1$ with $y_n \to z_1$ in E_1 as $n \to \infty$ in $N_1^\star$. Let $N_1 = N_1^\star \setminus \{1\}$. Now $y_n \in C_2$ for $n \in N_1$ together with (3.5) guarantees that there exists a subsequence $N_2^\star$ of N_1 and a $z_2 \in C_2$ with $y_n \to z_2$ in E_2 as $n \to \infty$ in $N_2^\star$. Note from (3.4) that $z_2 = z_1$ in E_1 since $N_2^\star \subseteq N_1$. Let $N_2 = N_2^\star \setminus \{2\}$. Proceed inductively to obtain subsequences of integers

$$N_1^\star \supseteq N_2^\star \supseteq \dots, \quad N_k^\star \subseteq \{k, k+1, \dots\}$$

and $z_k \in C_k$ with $y_n \to z_k$ in E_k as $n \to \infty$ in $N_k^\star$. Note $z_{k+1} = z_k$ in E_k for $k \in \{1, 2, \dots\}$. Also let $N_k = N_k^\star \setminus \{k\}$.

Fix $k \in N$. Let $y = z_k$ in E_k. Notice y is well defined and $y \in \lim_{\leftarrow} E_n = E$. Now $y_n \in F\,y_n$ in E_n for $n \in N_k$ and $y_n \to y$ in E_k as $n \to \infty$ in N_k (since $y = z_k$ in E_k) together with the fact that $F : C_k \to 2_k^E$ is upper semicontinuous (note $y_n \in C_k$ for $n \in N_k$) implies $y \in F\,y$ in E_k. We can do this for each $k \in N$ so as a result we have $y \in F\,y$ in E. $\square$

Essentially the same reasoning as in Theorem 3.1 yields the following result.

Theorem 3.2. *Let E and E_n be as described above, $C \subseteq E$ is an CPRLS and $F : C \to 2^E$ and for each $n \in N$ assume $F : \overline{C_n} \to 2^{E_n}$. Suppose the following conditions are satisfied:*

$$\text{for each } n \in N,\ F \in M(\overline{C_n}, \overline{C_n}) \text{ is a compact morphism} \tag{3.8}$$

$$\text{for each } \; n \in N, \; \Lambda_{\overline{C_n}}(F) \neq 0 \tag{3.9}$$

and

$$\begin{cases} \text{for each } \; n \in \{2, 3, \dots\} \; \text{ if } \; y \in \overline{C_n} \; \text{ solves } \; y \in F\, y \; \text{ in } \; E_n \\ \text{then } \; y \in \overline{C_k} \; \text{ for } \; k \in \{1, \dots, n-1\}. \end{cases} \tag{3.10}$$

Then F *has a fixed point in* E.

Our next result was motivated by Urysohn type operators. In this case the map F_n will be related to F by the closure property (3.15).

Theorem 3.3. *Let* E *and* E_n *be as described in the beginning of Section 3,* $C \subseteq E$ *is an PRLS and* $F : C \to 2^E$. *Also for each* $n \in N$ *assume there exists* $F_n : C_n \to 2^{E_n}$. *Suppose the following conditions are satisfied:*

$$C_1 \supseteq C_2 \supseteq \dots \tag{3.11}$$

$$\text{for each } \; n \in N, \; F_n \in M(C_n, C_n) \; \text{ is a compact morphism} \tag{3.12}$$

$$\text{for each } \; n \in N, \; \Lambda_{C_n}(F) \neq 0 \tag{3.13}$$

$$\begin{cases} \text{for each } \; n \in N, \;\; \text{the map } \; \mathcal{K}_n : C_n \to 2^{E_n}, \;\; \text{given by} \\ \mathcal{K}_n(y) = \cup_{m=n}^{\infty} F_m(y) \;\; \text{(see Remark 3.2), is compact} \end{cases} \tag{3.14}$$

and

$$\begin{cases} \text{if there exists a } \; w \in E \; \text{ and a sequence } \; \{y_n\}_{n \in N} \\ \text{with } \; y_n \in C_n \; \text{ and } \; y_n \in F_n\, y_n \; \text{ in } \; E_n \; \text{ such that} \\ \text{for every } \; k \in N \; \text{ there exists a subsequence} \\ S \subseteq \{k+1, k+2, \dots\} \; \text{ of } \; N \; \text{ with } \; y_n \to w \; \text{ in } \; E_k \\ \text{as } \; n \to \infty \; \text{ in } \; S, \; \text{ then } \; w \in F\, w \; \text{ in } \; E. \end{cases} \tag{3.15}$$

Then F *has a fixed point in* E.

Remark 3.2. The definition of $\mathcal{K}_n$ is as follows. If $y \in C_n$ and $y \notin C_{n+1}$ then $\mathcal{K}_n(y) = F_n(y)$, whereas if $y \in C_{n+1}$ and $y \notin C_{n+2}$ then $\mathcal{K}_n(y) = F_n(y) \cup F_{n+1}(y)$, and so on.

Proof. Fix $n \in N$. Now there exists $y_n \in C_n$ with $y_n \in F_n\, y_n$ in E_n. Lets look at $\{y_n\}_{n \in N}$. Note from (3.11) and the fact that $|x|_1 \leq |x|_n$ for all $x \in E_n$ that $y \in C_1$ and $y_n \in \mathcal{K}_1(y_n)$ in E_1 for each $n \in N$. Now $\mathcal{K}_1 : C_1 \to 2^{E_1}$ compact guarantees that there exists a subsequence $N_1^\star$ of N and a $z_1 \in E_1$ with $y_n \to z_1$ in E_1 as $n \to \infty$ in $N_1^\star$. Let $N_1 = N_1^\star \setminus \{1\}$. Look at $\{y_n\}_{n \in N_1}$. Also there exists a subsequence $N_2^\star$ of N_1 and a $z_2 \in E_2$

with $y_n \to z_2$ in E_2 as $n \to \infty$ in $N_2^\star$. Note $z_2 = z_1$ in E_1 since $N_2^\star \subseteq N_1^\star$. Let $N_2 = N_2^\star \setminus \{2\}$. Proceed inductively to obtain subsequences of integers

$$N_1^\star \supseteq N_2^\star \supseteq \dots, \quad N_k^\star \subseteq \{k, k+1, \dots\}$$

and $z_k \in E_k$ with $y_n \to z_k$ in E_k as $n \to \infty$ in $N_k^\star$. Note $z_{k+1} = z_k$ in E_k for $k \in N$. Also let $N_k = N_k^\star \setminus \{k\}$.

Fix $k \in N$. Let $y = z_k$ in E_k. Notice y is well defined and $y \in \lim_{\leftarrow} E_n = E$. Now $y_n \in F_n\, y_n$ in E_n for $n \in N_k$ and $y_n \to y$ in E_k as $n \to \infty$ in N_k (since $y = z_k$ in E_k) together with (3.15) implies $y \in F\, y$ in E. □

Similarly we have the following result.

Theorem 3.4. *Let E and E_n be as described in the beginning of Section 3, $C \subseteq E$ is an CPRLS and $F : C \to 2^E$. Also for each $n \in N$ assume there exists $F_n : \overline{C_n} \to 2^{E_n}$. Suppose the following conditions are satisfied:*

$$\overline{C_1} \supseteq \overline{C_2} \supseteq \dots \tag{3.16}$$

$$\textit{for each } n \in N,\ F_n \in M(\overline{C_n}, \overline{C_n}) \ \textit{ is a compact morphism} \tag{3.17}$$

$$\textit{for each } n \in N,\ \Lambda_{\overline{C_n}}(F) \neq 0 \tag{3.18}$$

$$\begin{cases} \textit{for each } n \in N, \ \textit{the map } \mathcal{K}_n : \overline{C_n} \to 2^{E_n}, \ \textit{given by} \\ \mathcal{K}_n(y) = \cup_{m=n}^{\infty} F_m(y) \ \textit{ is compact} \end{cases} \tag{3.19}$$

and

$$\begin{cases} \textit{if there exists a } w \in E \textit{ and a sequence } \{y_n\}_{n \in N} \\ \textit{with } y_n \in \overline{C_n} \textit{ and } y_n \in F_n\, y_n \textit{ in } E_n \textit{ such that} \\ \textit{for every } k \in N \textit{ there exists a subsequence} \\ S \subseteq \{k+1, k+2, \dots\} \textit{ of } N \textit{ with } y_n \to w \textit{ in } E_k \\ \textit{as } n \to \infty \textit{ in } S, \textit{ then } w \in F\, w \textit{ in } E. \end{cases} \tag{3.20}$$

Then F has a fixed point in E.

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