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THE DATA DEPENDENCE FOR THE SOLUTIONS OF DARBOUX-IONESCU PROBLEM FOR A HYPERBOLIC INCLUSION OF THIRD ORDER

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Abstract. In this paper we consider the Darboux-Ionescu Problem for a third order hyperbolic inclusion with modified argument of the form

$$u_{xyz} \in F(x, y, z, u(f(x, y, z), g(x, y, z), h(x, y, z))).$$

We prove three existence theorems and, as corollaries, we obtain the data dependence results for the solutions of the considered problem.
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1. Introduction

On the 7$^{\text{th}}$ of July 1927, D. V. Ionescu (1901–1984) brilliantly defended his Ph. D. Thesis in mathematics with the topic "Sur une classe d' équations fonctionnelles". In this dissertation he generalized the results of Darboux, Cauchy, Picard and Goursat for partial differential equations of hyperbolic type. The essential results of his thesis were published in Comptes Rendus de l' Académie des Sciences de Paris, T. 184 (1927), presented by E. Goursat and J. Hadamard [35].

In his Ph. D. Thesis [29], [30], D.V. Ionescu studied for the first time in the mathematical literature, boundary value problems of Darboux, Cauchy, Picard and Goursat types for second order hyperbolic equations with modified argument.

More recently, a series of authors studied the same problems for second order hyperbolic equations with modified argument of various forms [1]-[10], [16]-[21], [31], [36]-[38], [41]-[43], [54], and the Cauchy-Ionescu Problem for systems of hyperbolic type equations of first order with modified argument [19], [21], [22].

The Darboux-Ionescu Problem for third order hyperbolic equations with modified argument is studied in [19], [21], [23], [24]. The Darboux-Ionescu, Cauchy-Ionescu, Picard-Ionescu, Goursat-Ionescu Problems for hyperbolic inclusions of second order with modified argument is studied in [45]-[48]. The Darboux Problem and the Darboux-Ionescu Problem for third order hyperbolic inclusions with modified argument is studied in [49], [51].

The present paper is an extension of [50]-[53]. We consider the Darboux-Ionescu Problem associated with hyperbolic inclusion of third order with modified argument

$$\begin{gathered}\frac{\partial^3 u(x,y,z)}{\partial x\,\partial y\,\partial z} \in F\left(x,y,z,u\left(f(x,y,z),g(x,y,z),h(x,y,z)\right)\right), \\ (x,y,z)\in D=[0,a]\times[0,b]\times[0,c],\quad u\in\mathbb{R}^n,\end{gathered} \tag{1.1}$$

with initial values

$$\begin{cases} u(x,y,0)=\varphi(x,y), & (x,y)\in D_1=[0,a]\times[0,b],\\ u(0,y,z)=\psi(y,z), & (y,z)\in D_2=[0,b]\times[0,c],\\ u(x,0,z)=\chi(x,z), & (x,z)\in D_3=[0,a]\times[0,c],\end{cases} \tag{1.2}$$

where $F : D\times\mathbb{R}^n \to 2^{\mathbb{R}^n}$ is a Lipschitzian multifunction with respect to u, $F(x,y,z,u)$ is a nonconvex set, $f\in C(D;[0,a])$, $g\in C(D;[0,b])$, $h\in C(D;[0,c])$, φ, ψ, χ are absolutely continuous in Carathéodory's sense [11, §565 - §570], $\varphi\in C^*(D_1;\mathbb{R}^n)$, $\psi\in C^*(D_2;\mathbb{R}^n)$, $\chi\in C^*(D_3;\mathbb{R}^n)$ (see Definition 2.11 for $C^*(\Delta;\mathbb{R}^n)$, $\Delta\subset\mathbb{R}^2$) and they satisfy the conditions

$$\begin{cases} u(x,0,0)=\varphi(x,0)=\chi(x,0)=v^1(x), & x\in[0,a],\\ u(0,y,0)=\varphi(0,y)=\psi(y,0)=v^2(y), & y\in[0,b],\\ u(0,0,z)=\psi(0,z)=\chi(0,z)=v^3(z), & z\in[0,c],\\ u(0,0,0)=v^1(0)=v^2(0)=v^3(0)=v^0. \end{cases} \tag{1.3}$$

Under suitable assumptions we prove three existence theorems. The results are obtained by the successive approximations method, using two selection theorems [27], [28]. This method was applied by A. F. Filippov [26], Henry Hermes [27] and C. J. Himmelberg and F. S. Van Vleck [28] for the inclusion $\dot{x} \in R(t, x)$. Our results are similar to those reported in [26]-[28], [50], [52], [53]. As corollaries we obtain the data dependence results for the solutions of the considered problem.

2. Preliminaries

The definitions and Theorems in this section are taken from [11]-[15], [25]-[28], [32]-[34], [39]-[40], [44]-[54].

Definition 2.1. Let X and Y be two non-empty sets. *A multifunction* $\Phi : X \to 2^Y$ is a function from X into the family of all non-empty subsets of Y.

To each $x \in X$, a subset $\Phi(x)$ of Y is associated by the multifunction Φ. The set $\bigcup_{x \in X} \Phi(x)$ is the *range* of Φ.

Definition 2.2. Let us consider $\Phi : X \to 2^Y$.

a) If $A \subset X$, the *image* of A by Φ is $\Phi(A) = \bigcup_{x \in A} \Phi(x)$;

b) If $B \subset Y$, the *counterimage* of B by Φ is

$$\Phi^-(B) = \{x \in X \mid \Phi(x) \cap B \neq \emptyset\};$$

c) The graph of Φ, denoted *graph* Φ, is the set

$$graph\ \Phi = \{(x, y) \in X \times Y \mid y \in \Phi(x)\}.$$

Definition 2.3. A single-valued function $\varphi : X \to Y$ is said to be a *selection* of $\Phi : X \to 2^Y$ if $\varphi(x) \in \Phi(x)$ for all $x \in X$.

Definition 2.4. Let X and Y be two topological spaces. The multifunction $\Phi : X \to 2^Y$ is *upper-semicontinuous* if, for any closed subset $B \subseteq Y$, the set $\Phi^-(B)$ is closed in X.

Definition 2.5. If X and Y are two topological spaces, the multifunction $\Phi : X \to 2^Y$ is *lower-semicontinuous* if, for every open subset $\Omega \subseteq Y$, the set $\Phi^-(\Omega)$ is open in X.

Definition 2.6. The multifunction $\Phi : X \to 2^Y$ is *continuous* if it is upper-semicontinuous and lower-semicontinuous.

Definition 2.7. If $(X, \mathcal{F})$ is a measurable space and Y is a topological space, the multifunction $\Phi : X \to 2^Y$ is *measurable* (*weakly measurable*) if $\Phi^-(B) \in \mathcal{F}$ for every closed (open) subset $B \subseteq Y$, $\mathcal{F}$ being the σ-algebra of the measurable sets of X, i.e. $\Phi^-(B)$ is measurable.

Let (X, d) be a metric space and $\mathcal{P}(X)$ the set of subsets of X. For $A, B \subset X$, we denote

$$d(x, A) = \inf_{y \in A} d(x, y), \quad d(x, \emptyset) = \infty, \quad d^*(A, B) = \sup_{y \in A} d(y, B).$$

Definition 2.8. The function $d_H : \mathcal{P}(X) \to [0, +\infty]$

$$d_H(A, B) = \max\{d^*(A, B), d^*(B, A)\} = \max\left\{\sup_{x \in A} d(x, B), \sup_{x \in B} d(x, A)\right\}$$

is the *Hausdorff-Pompeiu pseudometric.*

The function d_H defines a metric on the space $\mathcal{F}(X)$ of non-empty and closed subsets of X, called the *Hausdorff-Pompeiu metric.*

Definition 2.9. Let (X, d) be a metric space and $\mathcal{P}(X)$ the set of subsets of X.

$$N_\varepsilon(C) = \{x \in X \mid d(x, c) < \varepsilon \text{ for any } c \in C\}, \quad \varepsilon > 0, \quad C \in \mathcal{P}(X).$$

For $A, B \in \mathcal{P}(X)$,

$$h_d(A, B) = \begin{cases} \inf\{\varepsilon > 0, A \subseteq N_\varepsilon(B) \text{ and } B \subseteq N_\varepsilon(A)\}, & \text{if the infimum exists,} \\ \infty, & \text{otherwise} \end{cases}$$

is the *Hausdorff-Pompeiu generalized pseudometric.*

Definition 2.10. Let be (X, d) and (Y, ρ) two metric spaces. A multifunction $\Phi : X \to 2^Y$ with non-empty values is said to be *Lipschitzian* (L-Lipschitz) if there exists a real number $L > 0$ (*Lipschitz constant*) such that

$$\rho_H(\Phi(x), \Phi(y)) \leq L \cdot d(x, y)$$

for all $x, y \in X$. If $L < 1$, we say that Φ is a *multi-valued contraction.*

Notice that any Lipschitzian multifunction is lower-semicontinuous.

Definition 2.11 [11]-[15], [44]. The function $u : \Delta \to \mathbb{R}^n$, $\Delta \subset \mathbb{R}^2$, is *absolutely continuous in Carathéodory' s sense* [11, §565-§570] if $u(x, y)$ is continuous on Δ, absolutely continuous in x (for any y), absolutely continuous in y (for any x), $u_x(x,y)$ is (possibly after a suitable definition on a two-dimensional set of zero measure) absolutely continuous in y (for any x) and u_{xy} is Lebesgue integrable on Δ.

Theorem 2.1 [11]-[15], [44]. The function $u : \Delta \to \mathbb{R}^n$, $\Delta \subset \mathbb{R}^2$, is absolutely continuous in Carathéodory's sense on Δ if and only if there exist $f \in L^1(\Delta;\mathbb{R}^n)$, $g \in L^1([0,a];\mathbb{R}^n)$, $h \in L^1([0,b];\mathbb{R}^n)$ such that

$$u(x,y) = \int_0^x \int_0^y f(s,t)\,ds\,dt + \int_0^x g(s)\,ds + \int_0^y h(t)\,dt + u(0,0).$$

We denote the class of absolutely continuous functions in Carathéodory's sense by $C^*(\Delta;\mathbb{R}^n)$ [13], [14], [15]. In [12], this space is denoted by $AC(\Delta;\mathbb{R}^n)$.

Theorem 2.2 [12]. The space $C^*(\Delta;\mathbb{R}^n)$ endowed with the norm

$$\|u(\cdot,\cdot)\| = \int_0^a \int_0^b \|u_{xy}(s,t)\|\,ds\,dt + \int_0^a \|u_x(s,0)\|\,ds +$$

$$+ \int_0^b \|u_y(0,t)\|\,dt + \|u(0,0)\|,$$

where $\|\cdot\|$ is the Euclidean norm, is a Banach space.

Definition 2.12 [11], [15]. The function $u : D \to \mathbb{R}^n$, $D \subset \mathbb{R}^3$, is *absolutely continuous in Carathéodory's sense* if $u(x,y,z)$ is continuous on D, is absolutely continuous in each variable (for any pair of the other two variables), and similarly – $u_x(x,y,z)$, $u_y(x,y,z)$, $u_z(x,y,z)$, $u_{xy}(x,y,z)$, $u_{yz}(x,y,z)$, $u_{xz}(x,y,z)$, and u_{xyz} is Lebesgue integrable.

We denote the class of absolutely continuous functions in Carathéodory's sense by $C^*(D;\mathbb{R}^n)$.

Theorem 2.3 [15]. The space $C^*(D;\mathbb{R}^n)$ endowed with the norm

$$\|u(\cdot,\cdot,\cdot)\| = \int_0^a \int_0^b \int_0^c \|u_{xyz}(r,s,t)\|\,dr\,ds\,dt + \int_0^a \int_0^b \|u_{xy}(r,s,0)\|\,dr\,ds +$$

$$+ \int_0^a \int_0^c \|u_{xz}(r,0,t)\|\,dr\,dt + \int_0^b \int_0^c \|u_{yz}(0,s,t)\|\,ds\,dt +$$

$$+ \int_0^a \|u_x(r,0,0)\|\,dr + \int_0^b \|u_y(0,s,0)\| + \int_0^c \|u_z(0,0,t)\|\,dt + \|u(0,0,0)\|$$

where $\|\cdot\|$ is the Euclidean norm, is a Banach space.

Definition 2.13 [25]. The sequence of multifunctions $\{F_i\}_{i\in\mathbb{N}}$, $F_i : D \to 2^{\mathbb{R}^n}$, with values in open set $\Omega \subset \mathbb{R}^n$, converge to $F : D \to 2^{\mathbb{R}^n}$ if for every

$\varepsilon > 0$ and every $S \in comp(\Omega)$ there exist a number N such that for every $n > N$,

$$d_H\left(\tilde{F}_n, \tilde{F}\right) < \varepsilon,$$

where d is a metric on $\mathbb{R}^n$, d_H is the Hausdorff-Pompeiu metric on $\mathcal{F}(\mathbb{R}^n)$, $\tilde{F}_n$ and $\tilde{F}$ denote the graphs of restrictions of multifunctions F_n and F on S.

$$\tilde{F}_n = graph\, F_n\big|_S, \quad \tilde{F} = graph\, F\big|_S.$$

Given a sequence $F, F_1, F_2, \ldots$ of measurable multifunctions, with complete values, from a measurable space $(X, \mathcal{F})$ into a separable metric space (Y, d), the following two theorems characterize the possibility of pointwise approximating every measurable selection f of F by means of a sequence of functions $\{f_n\}$, each f_n being a measurable selection of F_n [39].

Theorem 2.4 [39]. The following assertions are equivalent:

(a) For every measurable selection f of F there exists a sequence of functions $\{f_n\}$ such that:

(i) $\lim\limits_{n\to\infty} f_n(x) = f(x)$ for each $x \in X$;

(ii) f_n is a measurable selection of F_n for any $n \in \mathbb{N}$.

(b) For any $x \in X$, $y \in F(x)$, $\lim\limits_{n\to\infty} d(y, F_n(x)) = 0$.

Theorem 2.5 [39]. Suppose that μ is a σ-finite, non-negative measure on $\mathcal{F}$ and the values of F are compacts. Then the following assertions are equivalent:

$a)$ There exists a sequence $\{X_k\}$ in $\mathcal{F}$, with $\mu\left(X - \bigcup\limits_{k=1}^{\infty} X_k\right) = 0$, such that for each measurable selection f of F there exists a sequence of functions $\{f_n\}$ such that:

(i) $\lim\limits_{n\to\infty} f_n(x) = f(x)$ uniformly in X_k for every $k \in \mathbb{N}$;

(ii) f_n is a measurable selection of F_n for every $n \in \mathbb{N}$.

$b)$ There exists $X^* \in \mathcal{F}$, with $\mu(X^*) = 0$, such that for each $x \in X \setminus X^*$, $y \in F(x)$, $\lim\limits_{n\to\infty} d(y, F_n(x)) = 0$.

Theorem 2.6 [40]. Let X and Y be two metric spaces, Y compact and $\Phi : X \to 2^Y$ a multifunction with the property that $\Phi(x)$ is a closed subset of Y for any $x \in X$. The following assertions are equivalent:

(i) the multifunction Φ is upper-semicontinuous;

(ii) the $graph\, \Phi$ is a closed subset of $X \times Y$;

(iii) any would be the sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$, from $x_n \to x$, $y_n \in \Phi(x_n)$, $y_n \to y$, it follows that $y \in \Phi(x)$.

The set of all non-empty compact subsets of $\mathbb{R}^n$, with the topology induced by the Hausdorff-Pompeiu metric, is a complete metric space denoted by Ω^n. A function R, defined on a real interval I with values in Ω^n is *measurable* (in the sense of Lebesgue) if, for every closed subset $D \subset \mathbb{R}^n$, $\{t \in I \mid R(t) \cap D \neq \emptyset\}$ is measurable [27].

Lemma 1.2 [27]. Let $R : I \to \Omega^n$ be a measurable multifunction, with values in a ball of radius ρ centered at origin, and let $w : I \to \mathbb{R}^n$ be a measurable point-valued function. Then there exists a measurable function r with values $r(t) \in R(t)$ for almost t and such that $\|w(t) - r(t)\| = \rho(w(t), R(t))$.

The following Proposition is a slight extension of a lemmas used by Filippov [26] and Hermes [27].

Proposition 1 [28]. Let $(T, \mathcal{A})$, $T = [t_0, t_1]$ be a measurable space, let $F : T \to 2^{\mathbb{R}^n}$ be a measurable multifunction with closed values and let $w : T \to \mathbb{R}^n$ be a measurable function. Then there exists a measurable function $\nu : T \to \mathbb{R}^n$ such that $\nu(t) \in F(t)$ and $\|\nu(t) - w(t)\| = d(w(t), F(t))$ for $t \in T$.

Let now T be a compact Hausdorff space with the positive Radon measure μ, let X be a Polish space (i.e. a metrizable separable space with complete metric), $F : T \times X \to 2^{\mathbb{R}^n}$ (X – a metric space with metric ρ) a multifunction such that $F(t, x)$ is measurable in t for any x and continuous (with respect to the pseudometric Hausdorff-Pompeiu on $S(\mathbb{R}^n)$ – the set of non-empty subsets of $\mathbb{R}^n$) at x for any t (conditions of Carathéodory type) (Theorem 1 [28]).

Corollary 1 [28]. Let F, T, X be as in Theorem 1 [28]. Let $x : T \to X$ be a measurable function and let us define the multifunction $G : T \to 2^{\mathbb{R}^n}$ by $G(t) = F(t, x(t))$. Then F and G are weakly measurable. If F has closed values then F and G are measurable.

3. Results

Let $F : D \times \mathbb{R}^n \to 2^{\mathbb{R}^n}$ be a multifunction satisfying the hypotheses:

(H_1) The values of F are contained in the ball of radius r centered at the origin of $\mathbb{R}^n$;

(H_2) $F(x, y, z, u)$ is a compact set for every $(x, y, z, u) \in D \times \mathbb{R}^n$;

(H_3) F is a continuous multifunction;

(H_4) F is Lipschitzian with respect to u, there exists a function $k : D \to \mathbb{R}_+$, $k \in L^1(D)$ such that

$$d_H\left(F(x,y,z,u), F\left(x,y,z,u'\right)\right) \le k(x,y,z)\left\|u-u'\right\|, \qquad (3.1)$$
$$(x,y,z) \in D, \quad u,u' \in \mathbb{R}^n$$

where $d(u,u') = \|u-u'\|$, $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^n$ and d_H is the Hausdorff-Pompeiu metric.

(H_5) The functions $\varphi \in C^*(D_1;\mathbb{R}^n)$, $\psi \in C^*(D_2;\mathbb{R}^n)$, $\chi \in C^*(D_3;\mathbb{R}^n)$ satisfy the conditions (1.3).

(H_6) There exists an absolutely continuous in Carathéodory's sense function [11, §565-§570], $\lambda : D \to \mathbb{R}^n$, $\lambda \in C^*(D;\mathbb{R}^n)$, such that

$$\sup_{(x,y,z)\in D} d\left(\frac{\partial^3\lambda(x,y,z)}{\partial x\,\partial y\,\partial z}, F(x,y,z,\lambda(x,y,z))\right) \le M < +\infty, \qquad (3.2)$$

for some $M > 0$;

(H_7) $f \in C(D;[0,a])$, $g \in C(D;[0,b])$, $h \in C(D;[0,c])$, $0 \le f(x,y,z) \le x \le a$, $0 \le g(x,y,z) \le y \le b$, $0 \le h(x,y,z) \le z \le c$;

(H_8) The functions $\alpha : D \to \mathbb{R}^n$, $\alpha_0 : D \to \mathbb{R}^n$ defined by

$$\begin{aligned}\alpha(x,y,z) = {}& \varphi(x,y) + \psi(y,z) + \chi(x,z) - \varphi(x,0) - \\ & - \varphi(0,y) - \psi(0,z) + \psi(0,0) = \\ & = \varphi(x,y) + \psi(y,z) + \chi(x,z) - v^1(x) - v^2(y) - v^3(z) + v^0,\end{aligned} \qquad (3.3)$$

$$\begin{aligned}\alpha_0(x,y,z) = {}& \lambda(x,y,0) + \lambda(0,y,z) + \lambda(x,0,z) - \\ & - \lambda(x,0,0) - \lambda(0,y,0) - \lambda(0,0,z) + \lambda(0,0,0),\end{aligned} \qquad (3.4)$$

satisfy the condition

$$\|\alpha(x,y,z) - \alpha_0(x,y,z)\| \le M_1, \qquad (3.5)$$

$M_1 > 0$ is a constant.

Remark 3.1. Such function λ in (H_6) exist; for example, if λ is constant, M may be taken $= r$.

Remark 3.2. The functions α and α_0 are absolutely continuous in Carathéodory's sense on D [11, §565-§570], $\alpha, \alpha_0 \in C^*(D;\mathbb{R}^n)$.

Definition 3.1 [52]. The *Darboux-Ionescu Problem* for third order hyperbolic inclusion with modified argument (1.1) consists in determining of *solution* of (1.1) which satisfies the initial conditions (1.2).

Definition 3.2 [52]. A *solution of Darboux-Ionescu Problem* (1.1) + (1.2) is a function $u : D \to \mathbb{R}^n$, absolutely continuous in Carathéodory's sense [11, §565-§570], $u \in C^*(D;\mathbb{R}^n)$, which satisfies a.e. for $(x,y,z) \in D$ the inclusion (1.1) and also initial conditions (1.2) for all $(x,y) \in D_1$, $(y,z) \in D_2$, $(x,z) \in D_3$.

Theorem 3.1. If the hypotheses (H_1) - (H_8) are satisfied, the Darboux-Ionescu Problem (1.1) + (1.2) has a solution in D.

Proof. We define the sequence of successive approximations $\{u_i\}$, $i \in \mathbb{N}$;

$$u_0(x,y,z) = u_0(f(x,y,z), g(x,y,z), h(x,y,z)) = \lambda(x,y,z), \quad (x,y,z) \in D; \tag{3.6$_0$}$$

According to Lemma 1.2 [27] there exists a measurable function $v_0 : D \to \mathbb{R}^n$ such that

$$\begin{aligned} v_0(x,y,z) = v_0(f(x,y,z), g(x,y,z), h(x,y,z)) \in \\ \in F(x,y,z,u_0(f(x,y,z), g(x,y,z), h(x,y,z))), \\ \text{a.e. for } (x,y,z) \in D, \end{aligned} \tag{3.7$_0$}$$

and

$$\begin{aligned} d\left(\frac{\partial^3 u_0(x,y,z)}{\partial x\, \partial y\, \partial z}, F(x,y,z,u_0(f(x,y,z), g(x,y,z), h(x,y,z)))\right) = \\ = \left\| v_0(x,y,z) - \frac{\partial^3 u_0(x,y,z)}{\partial x\, \partial y\, \partial z} \right\|, \quad (x,y,z) \in D. \end{aligned} \tag{3.8$_0$}$$

We define the second approximation by

$$\begin{aligned} u_1(x,y,z) = u_1(f(x,y,z), g(x,y,z), h(x,y,z)) = \\ = \alpha(x,y,z) + \int_0^x \int_0^y \int_0^z v_0(\xi,\eta,\zeta)\, d\xi\, d\eta\, d\zeta, \quad (x,y,z) \in D \end{aligned} \tag{3.9$_1$}$$

from which it follows

$$\frac{\partial^3 u_1(x,y,z)}{\partial x\,\partial y\,\partial z} = v_0(x,y,z) \in \qquad (3.10_1)$$
$$\in F(x,y,z,u_0(f(x,y,z),g(x,y,z),h(x,y,z))),$$
$$\text{a.e. for } (x,y,z)\in D.$$

Again applying the cited Lemma, it follows the existence of a measurable function $v_1 : D \to \mathbb{R}^n$ having the properties

$$v_1(x,y,z) = v_1(f(x,y,z),g(x,y,z),h(x,y,z)) \in \qquad (3.7_1)$$
$$\in F(x,y,z,u_1(f(x,y,z),g(x,y,z),h(x,y,z))),$$
$$\text{a.e. for } (x,y,z)\in D,$$

and

$$d\left(\frac{\partial^3 u_1(x,y,z)}{\partial x\,\partial y\,\partial z}, F(x,y,z,u_1(f(x,y,z),g(x,y,z),h(x,y,z)))\right) = \qquad (3.8_1)$$
$$= \left\| v_1(x,y,z) - \frac{\partial^3 u_1(x,y,z)}{\partial x\,\partial y\,\partial z}\right\|, \quad (x,y,z)\in D.$$

The third approximation is given by

$$u_2(x,y,z) = u_2(f(x,y,z),g(x,y,z),h(x,y,z)) = \qquad (3.9_2)$$
$$= \alpha(x,y,z) + \int_0^x\int_0^y\int_0^z v_1(\xi,\eta,\zeta)\,d\xi\,d\eta\,d\zeta, \quad (x,y,z)\in D$$

which implies

$$\frac{\partial^3 u_2(x,y,z)}{\partial x\,\partial y\,\partial z} = v_1(x,y,z) \in \qquad (3.10_2)$$
$$\in F(x,y,z,u_1(f(x,y,z),g(x,y,z),h(x,y,z))),$$
$$\text{a.e. for } (x,y,z)\in D.$$

In this way we obtain the function sequences $\{u_i\}_{i\in\mathbb{N}}$, $\{v_i\}_{i\in\mathbb{N}}$, $u_i, v_i : D \to \mathbb{R}^n$, which satisfy

$$v_i(x,y,z) \in F(x,y,z,u_i(f(x,y,z),g(x,y,z),h(x,y,z))), \qquad (3.7_i)$$
$$\text{a.e. for } (x,y,z)\in D,\ i=0,1,2,\ldots$$

and

$$d\left(\frac{\partial^3 u_i(x,y,z)}{\partial x\,\partial y\,\partial z}, F(x,y,z,u_i(f(x,y,z),g(x,y,z),h(x,y,z)))\right) = \tag{3.8$_i$}$$
$$= \left\|v_i(x,y,z) - \frac{\partial^3 u_i(x,y,z)}{\partial x\,\partial y\,\partial z}\right\|, \quad (x,y,z)\in D,\ i=0,1,2,\ldots$$

where

$$u_i(x,y,z) = \alpha(x,y,z) + \int_0^x \int_0^y \int_0^z v_{i-1}(\xi,\eta,\zeta)\,d\xi\,d\eta\,d\zeta, \tag{3.9$_i$}$$
$$(x,y,z)\in D,\ i=1,2,\ldots$$

From the preceding relation it follows

$$\frac{\partial^3 u_i(x,y,z)}{\partial x\,\partial y\,\partial z} = v_{i-1}(x,y,z) \in \tag{3.10$_i$}$$
$$\in F(x,y,z,u_{i-1}(f(x,y,z),g(x,y,z),h(x,y,z))),$$
$$\text{a.e. for } (x,y,z)\in D,\ i=1,2,\ldots$$

The functions $\{v_i\}_{i\in\mathbb{N}}$ are integrable in view of (3.10$_i$), (H_4) and (H_6). Moreover, from (3.1) we have

$$d\left(\frac{\partial^3 u_i(x,y,z)}{\partial x\,\partial y\,\partial z}, F(x,y,z,u_i(f(x,y,z),g(x,y,z),h(x,y,z)))\right) = \tag{3.11$_i$}$$
$$= d(v_{i-1}(x,y,z), F(x,y,z,u_i(f(x,y,z),g(x,y,z),h(x,y,z)))) \le$$
$$\le d(v_{i-1}(x,y,z), F(x,y,z,u_{i-1}(f(x,y,z),g(x,y,z),h(x,y,z)))) +$$
$$+ d_H\Big(F(x,y,z,u_{i-1}(f(x,y,z),g(x,y,z),h(x,y,z))),$$
$$F(x,y,z,u_i(f(x,y,z),g(x,y,z),h(x,y,z)))\Big) \le$$
$$\le k(x,y,z)\Big\|u_{i-1}(f(x,y,z),g(x,y,z),h(x,y,z)) -$$
$$-u_i(f(x,y,z),g(x,y,z),h(x,y,z))\Big\| =$$
$$= k(x,y,z)\left\|u_{i-1}(x,y,z) - u_i(x,y,z)\right\|, \quad (x,y,z)\in D,\ i=2,3,\ldots$$

and for $i=1$ holds (3.2) and (3.6$_0$).

After some standard calculation, using the inequality

$$\int_0^x \int_0^y \int_0^z k(r,s,t) \int_0^r \int_0^s \int_0^t k(r_1,s_1,t_1) \int_0^{r_1} \int_0^{s_1} \int_0^{t_1} k(r_2,s_2,t_2) \ldots \tag{3.12}$$

$$\int_0^{r_{n-1}} \int_0^{s_{n-1}} \int_0^{t_{n-1}} k(r_n,s_n,t_n)\, dr_n\, ds_n\, dt_n \ldots dr_1\, ds_1\, dt_1\, dr\, ds\, dt \leq$$

$$\leq \frac{1}{(n+1)!} \left[\int_0^x \int_0^y \int_0^z k(u,v,w)\, du\, dv\, dw\right]^{n+1}, \quad (x,y,z) \in D,$$

we obtain the following basic estimations:

$$\left\| \frac{\partial^3 u_{i+1}(x,y,z)}{\partial x\, \partial y\, \partial z} - \frac{\partial^3 u_i(x,y,z)}{\partial x\, \partial y\, \partial z} \right\| = \| v_i(x,y,z) - v_{i-1}(x,y,z) \| \leq \quad (3.13_{i+1})$$

$$\leq k(x,y,z) \frac{M_1 + Mabc}{(i-1)!} \left[\int_0^x \int_0^y \int_0^z k(\xi,\eta,\zeta)\, d\xi\, d\eta\, d\zeta\right]^{i-1}, \quad i = 1,2,\ldots$$

and

$$\| u_{i+1}(x,y,z) - u_i(x,y,z) \| \leq \quad (3.14_{i+1})$$

$$\leq \frac{M_1 + Mabc}{i!} \left[\int_0^x \int_0^y \int_0^z k(\xi,\eta,\zeta)\, d\xi\, d\eta, d\zeta\right]^{i}, \quad i = 0,1,2,\ldots$$

From (3.13_{i+1}) we conclude that $\{v_i(x,y,z)\}_{i\in\mathbb{N}}$ converges to $v : D \to \mathbb{R}^n$ in $L^n_\infty(D)$ and from (3.14_{i+1}) the sequence $\{u_i(x,y,z)\}_{i\in\mathbb{N}}$ is uniformly convergent to $u : D \to \mathbb{R}^n$. Letting $i \to \infty$ in (3.8_i), (3.9_i), (3.10_i) and using the hypotheses (H_2), (H_3) it follows that the limit u is absolutely continuous function in Carathéodory's sense, $u \in C^*(D;\mathbb{R}^n)$ and satisfies the Darboux-Ionescu Problem (1.1) + (1.2). We obtain

$$d\left(\frac{\partial^3 u(x,y,z)}{\partial x\, \partial y\, \partial z}, F(x,y,z,u(f(x,y,z), g(x,y,z), h(x,y,z)))\right) =$$

$$= \left\| v(x,y,z) - \frac{\partial^3 u(x,y,z)}{\partial x\, \partial y\, \partial z} \right\|, \quad (x,y,z) \in D, \tag{3.8}$$

$$u(x,y,z) = \alpha(x,y,z) + \int_0^x \int_0^y \int_0^z v(\xi,\eta,\zeta)\, d\xi\, d\eta\, d\zeta, \tag{3.9}$$

$$(x,y,z) \in D,$$

$$\frac{\partial^3 u(x,y,z)}{\partial x\,\partial y\,\partial z} = v(x,y,z) \in \tag{3.10}$$
$$\in F\left(x,y,z,u\left(f\left(x,y,z\right),g\left(x,y,z\right),h\left(x,y,z\right)\right)\right),$$
$$\text{a.e. for } (x,y,z) \in D.$$

Taking into account (3.9) and (3.10), the function $u(x,y,z)$ given by (3.9) satisfies the Darboux-Ionescu Problem (1.1) + (1.2).

Theorem 3.2. We suppose that F satisfies the hypotheses $(H_1) - (H_5)$, (H_7), (H_8) and

(H_6') There exists an absolutely continuous function in Carathéodory's sense [11, §565-§570], $\lambda : D \to \mathbb{R}^n$, $\lambda \in C^*(D;\mathbb{R}^n)$ such that

$$d\left(\frac{\partial^3 \lambda(x,y,z)}{\partial x\,\partial y\,\partial z}, F\left(x,y,z,\lambda(x,y,z)\right)\right) < \varepsilon, \tag{3.2'}$$
$$(x,y,z) \in D \text{ for some } \varepsilon > 0.$$

Then there exists a solution $u \in C^*(D;\mathbb{R}^n)$ of the Darboux-Ionescu Problem (1.1) + (1.2) satisfying

$$\|u(x,y,z) - \lambda(x,y,z)\| \leq \tag{3.15}$$
$$\leq (M_1 + \varepsilon abc)\exp\left[\int_0^x\int_0^y\int_0^z k(\xi,\eta,\zeta)\,d\xi\,d\eta\,d\zeta\right], \quad (x,y,z) \in D.$$

The proof is similar to that one of Theorem 3.1; we obtain

$$\|u_{i+1}(x,y,z) - u_i(x,y,z)\| \leq \tag{$3.14'_{i+1}$}$$
$$\leq \frac{M_1 + \varepsilon abc}{i!}\left[\int_0^x\int_0^y\int_0^z k(\xi,\eta,\zeta)\,d\xi\,d\eta,d\zeta\right]^i, \quad i = 0,1,2,\ldots$$

and using the elementary inequality

$$\sum_{n=0}^{j}\frac{w^n}{n!} \leq e^w, \quad w \geq 0,$$

it follows for every $j = 0,1,2,\ldots$

$$\|u_j(x,y,z) - \lambda(x,y,z)\| \leq \tag{$3.14''_j$}$$
$$\leq (M_1 + \varepsilon abc)\exp\left[\int_0^x\int_0^y\int_0^z k(\xi,\eta,\zeta)\,d\xi\,d\eta\,d\zeta\right], \quad (x,y,z) \in D.$$

The conclusion follows letting $j \to \infty$ in the preceding relation, $u_j(x,y,z) \to u(x,y,z)$, $u : D \to \mathbb{R}^n$, uniformly for $j \to \infty$.

Theorem 3.3. Let $F : D \times \mathbb{R}^n \to 2^{\mathbb{R}^n}$ be a multifunction satisfying the hypotheses:

a) $F(x,y,z,u)$ is closed for every $(x,y,z,u) \in D \times \mathbb{R}^n$;

b) $F(\cdot,\cdot,\cdot,u)$ is measurable for each $u \in \mathbb{R}^n$, with respect to the Lebesque measure on D;

c) $F(x,y,z,\cdot)$ is Lipschitzian with respect to u; there exists a function $k : D \to \mathbb{R}^n$, $k \in L^1(D)$, such that

$$h_d\left(F(x,y,z,u), F\left(x,y,z,u'\right)\right) \le k(x,y,z)\left\|u-u'\right\|, \qquad (x,y,z) \in D, \quad u,u' \in \mathbb{R}^n, \tag{3.1'}$$

where h_d is the Hausdorff-Pompeiu generalized pseudometric, and $(H_5)-(H_8)$. Then, the Darboux-Ionescu Problem (1.1) + (1.2) has a solution in D.

The proof is similar to that one of Theorem 3.1, and use the Proposition 1 [28] and Corollary 1 [28] to ensure the existence of measurable functions $\nu_i : D \to \mathbb{R}^n$, $i = 0,1,2,\dots$ with

$$\nu_i(x,y,z) \in F(x,y,z,u_i(f(x,y,z), g(x,y,z), h(x,y,z))), \quad \text{a.e. for } (x,y,z) \in D, \tag{$3.7'_i$}$$

$$d\left(\frac{\partial^3 u_i(x,y,z)}{\partial x\,\partial y\,\partial z}, F(x,y,z,u_i(f(x,y,z), g(x,y,z), h(x,y,z)))\right) = \left\|\nu_i(x,y,z) - \frac{\partial^3 u_i(x,y,z)}{\partial x\,\partial y\,\partial z}\right\|, \quad (x,y,z) \in D,\ i = 0,1,2,\dots \tag{$3.8'_i$}$$

$$u_i(x,y,z) = \alpha(x,y,z) + \int_0^x \int_0^y \int_0^z \nu_{i-1}(\xi,\eta,\zeta)\,d\xi\,d\eta\,d\zeta, \quad (x,y,z) \in D,\ i = 0,1,2,\dots \tag{$3.9'_i$}$$

$$\frac{\partial^3 u_{i+1}(x,y,z)}{\partial x\,\partial y\,\partial z} = \nu_i(x,y,z) \in F(x,y,z,u_i(f(x,y,z), g(x,y,z), h(x,y,z))), \quad \text{a.e. for } (x,y,z) \in D,\ i = 0,1,2,\dots \tag{$3.10'_i$}$$

The sequence $\{u_i(x,y,z)\}_{i\in\mathbb{N}}$ is uniformly convergent to $u : D \to \mathbb{R}^n$, and the sequence $\{\nu_i(x,y,z)\}_{i\in\mathbb{N}}$ converges in $L^n_\infty(D)$ to $\nu : D \to \mathbb{R}^n$.

Letting $i \to \infty$ in $(3.9'_i)$ and $(3.10'_i)$, we obtain

$$u(x,y,z) = \alpha(x,y,z) + \int_0^x \int_0^y \int_0^z \nu(\xi,\eta,\zeta)\, d\xi\, d\eta\, d\zeta, \qquad (x,y,z) \in D, \tag{3.9'}$$

$$\frac{\partial^3 u(x,y,z)}{\partial x\, \partial y\, \partial z} = \nu(x,y,z) \in \in F(x,y,z,u(f(x,y,z), g(x,y,z), h(x,y,z))), \quad \text{a.e. for } (x,y,z) \in D. \tag{3.10'}$$

To obtain the relation $(3.10')$ we make use of the fact the *graph* F is closed, according to the hypotheses $a)$, $c)$. The function u given by $(3.9')$ satisfies (1.2), is an absolutely continuous in Carathéodory's sense function [11, §565-§570], $u \in C^*(D;\mathbb{R}^n)$ and from $(3.10')$, u is a solution of the Darboux-Ionescu Problem $(1.1)+(1.2)$.

We now study the data dependence of solutions of Darboux-Ionescu Problem $(1.1)+(1.2)$.

Corollary 3.1.

(i) In the Darboux-Ionescu Problem $(1.1)+(1.2)$ the functions satisfy the same hypotheses as in Theorem 3.1 and let $u^* \in C^*(D;\mathbb{R}^n)$ be a solution of this problem;

(ii) Let $F_n : D \times \mathbb{R}^n \to 2^{\mathbb{R}^n}$ be a sequence of multifunctions convergent to F [25];

(iii) Let $u_n^* \in C^*(D;\mathbb{R}^n)$ a solution of the Darboux-Ionescu Problem $(3.15)+(1.2)$,

$$\frac{\partial^3 u(x,y,z)}{\partial x\, \partial y\, \partial z} \in F_n(x,y,z,u(f(x,y,z), g(x,y,z), h(x,y,z))), \qquad (x,y,z) \in D,\ u \in \mathbb{R}^n \tag{3.15}$$

where F_n and all functions satisfy the same hypotheses as in Theorem 3.1. Then $u_n^* \to u^*$ in $C^*(D;\mathbb{R}^n)$.

Corollary 3.2.

(i) In the Darboux-Ionescu Problem $(1.1)+(1.2)$ the functions satisfy the same hypotheses as in Theorem 3.1 and let $u^* \in C^*(D;\mathbb{R}^n)$ be a solution of this problem;

(*ii*) Let $f_n \in C(D;[0,a])$, $g_n \in C(D;[0,b])$, $h_n \in C(D;[0,c])$, $0 \leq f_n(x,y,z) \leq x \leq a$, $0 \leq g_n(x,y,z) \leq y \leq b$, $0 \leq h_n(x,y,z) \leq z \leq c$, be a functions, $f_n \to f$, $g_n \to g$, $h_n \to h$, $f \in C(D;[0,a])$, $g \in C(D;[0,b])$, $h \in C(D;[0,c])$, $0 \leq f(x,y,z) \leq x \leq a$, $0 \leq g(x,y,z) \leq y \leq b$, $0 \leq h(x,y,z) \leq z \leq c$;

(*iii*) Let $u_n^* \in C^*(D;\mathbb{R}^n)$ a solution of the Darboux-Ionescu Problem (3.16) + (1.2)

$$\frac{\partial^3 u(x,y,z)}{\partial x\,\partial y\,\partial z} \in F(x,y,z,u(f_n(x,y,z),g_n(x,y,z),h_n(x,y,z))), \qquad (3.16)$$

$$(x,y,z) \in D,\ u \in \mathbb{R}^n.$$

Then $u_n^* \to u^*$ in $C^*(D;\mathbb{R}^n)$.

Corollary 3.3.

(*i*) In the Darboux-Ionescu Problem (1.1) + (1.2) the functions satisfy the same hypotheses as in Theorem 3.1 and let $u^* \in C^*(D;\mathbb{R}^n)$ be a solution of this problem;

(*ii*) The functions $\varphi_n \in C^*(D_1;\mathbb{R}^n)$, $\psi_n \in C^*(D_2;\mathbb{R}^n)$, $\chi_n \in C^*(D_3;\mathbb{R}^n)$ satisfy (1.3_n) and $\varphi_n \to \varphi$, $\psi_n \to \psi$, $\chi_n \to \chi$,

$$\begin{cases} u(x,y,0) = \varphi_n(x,y),\ (x,y) \in D_1 = [0,a] \times [0,b] \\ u(0,y,z) = \psi_n(y,z),\ (y,z) \in D_2 = [0,b] \times [0,c] \\ u(x,0,z) = \chi_n(x,z),\ (x,z) \in D_3 = [0,a] \times [0,c] \end{cases} \qquad (1.2_n)$$

and

$$\begin{cases} u(x,0,0) = \varphi_n(x,0) = \chi_n(x,0) = v_n^1(x), \quad x \in [0,a] \\ u(0,y,0) = \varphi_n(0,y) = \psi_n(y,0) = v_n^2(y), \quad y \in [0,b] \\ u(0,0,z) = \psi_n(0,z) = \chi_n(0,z) = v_n^3(z), \quad z \in [0,c] \\ u(0,0,0) = v_n^1(0) = v_n^2(0) = v_n^3(0) = v_n^0. \end{cases} \qquad (1.3_n)$$

(*iii*) Let $u_n^* \in C^*(D;\mathbb{R}^n)$ a solution of Darboux-Ionescu Problem (3.17) + (1.2_n)

$$\frac{\partial^3 u(x,y,z)}{\partial x\,\partial y\,\partial z} \in F_n(x,y,z,u(f(x,y,z),g(x,y,z),h(x,y,z))), \qquad (3.17)$$

$$(x,y,z) \in D,\ u \in \mathbb{R}^n$$

where $F_n : D \times \mathbb{R}^n \to 2^{\mathbb{R}^n}$ is a sequence of multifunctions convergent to F [25];

Then $u_n^* \to u^*$ in $C^*(D;\mathbb{R}^n)$.

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