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SEMI-LOCAL CONVERGENCE OF AN ALGORITHM FOR SOLVING NONLINEAR EQUATIONS

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Abstract. We propose a general coordinate relaxation algorithm $x^{k+1} = x^k - \omega_k t_{i_k} e^{i_k}$ for solving the nonlinear equation $F(x) = 0$, where $F : D \subset \mathbb{R}^n \to \mathbb{R}^n$ is a nonlinear mapping. Under appropriate conditions we prove a semi-local convergence theorem for this algorithm.

Key Words and Phrases: Local-semiconvergence, algorithm, nonlinear equation, relaxation method.

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1. Introduction

Consider the system of nonlinear equations

$$F(x) = 0, \quad x \in D \tag{1.1}$$

where $F : D \subset \mathbb{R}^n \to \mathbb{R}^n$ is a nonlinear mapping and D is a nonempty subset of $\mathbb{R}^n$. Write

$$F(x) = \begin{bmatrix} f_1(x_1, x_2, \cdots, x_n) \\ f_2(x_1, x_2, \cdots, x_n) \\ \vdots \\ f_n(x_1, x_2, \cdots, x_n) \end{bmatrix}.$$

Equation (1.1) is rewritten as

$$\begin{cases} f_1(x_1, x_2, \cdots, x_n) = 0, \\ f_2(x_1, x_2, \cdots, x_n) = 0, \\ \qquad \vdots \\ f_n(x_1, x_2, \cdots, x_n) = 0, \end{cases} \qquad x \in D.$$

In this paper we discuss the convergence for Eq. (1.1) of the following general coordinate relaxation method

$$x^{k+1} = x^k - \omega_k t_{i_k} e^{i_k}, \quad k = 0, 1, \cdots \tag{1.2}$$

where t_{i_k} is a step-size of the iteration, ω_k is a relaxation parameter, and $i_k \in \{1, 2, \cdots, n\}$.

If $F : D \subset \mathbb{R}^n \to \mathbb{R}^n$ is a gradient mapping (i.e., there is a function $g : D \subset \mathbb{R}^n \to \mathbb{R}$ such that $F(x)^T = g'(x)$ for $x \in D$) and if F is a continuously differentiable and uniformly monotone or strictly monotone mapping on D, Schechter [5] and respectively, Brewster-Kannan [2] proved the global convergence of the nonlinear SOR-Newton mehtod

$$x^{k+1} = x^k - \omega_k \frac{f_{i_k}(x^k)}{f_{i_k i_k}(x^k)} e^{i_k}, \quad k = 0, 1, \cdots$$

where $f_{ij}(x)$ denotes the $(i, j)-$entry of the Jacobi matrix of $F'(x)$.

Consider now the case where F is a nongradient mapping. In [1], Bers proved that if $F'(x)$ is uniformly diagonally dominant, then the Gauss-Seidel method

$$\begin{cases} x^{k+1} = x^k - \alpha_k e^{i_k} \\ \text{where } \alpha_k \text{ is such that } f_{ik}(x^k - \alpha_k e^{i_k}) = 0 \end{cases}$$

has global convergence.

While Gipser [3] proved the global convergence of the following method

$$\begin{cases} x^{k+1} = x^k + t_k e^{i_k} \\ \text{where } t_k \text{ is such that } f_{ik}(x^{k+1}) = c_k f_{ik}(x^k),\ c_k \in [0, 1 - \varepsilon],\ \varepsilon > 0, \end{cases}$$

where Gipser assumed that there exists a nonsingular diagonal matrix S such that either $F'(x)S$ is uniformly row diagonally dominant or $SF'(x)$ is uniformly column diagonally dominant.

In this paper we prove the semi-local convergence of the algorithm (1.2) in which we consider the following iteration order:

(I) i_k satisfies the equation

$$\max_{1\le j\le n} |F(x^k)^T F'(x^k)e^j| = |F(x^k)^T F'(x^k)e^{i_k}|. \tag{1.3}$$

Our step-size t_{i_k} satisfies the condition

$$\begin{aligned}\left(\frac{1-\beta}{1-\delta}\right)\frac{|F(x^k)^T F'(x^k)e^{i_k}|}{r_{i_k}} &\le t_{i_k}\mathrm{sgn}(F(x^k)^T F'(x^k)e^{i_k}) \\ &\le \left(\frac{1+\beta}{1+\delta}\right)\frac{|F(x^k)^T F'(x^k)e^{i_k}|}{r_{i_k}}\end{aligned}$$

and the relaxation parameters ω_k satisfies the condition

$$1-\delta < \omega_k < 1+\delta.$$

2. Semi-Local Convergence of the Algorithm

There has been little discussion on the semi-local convergence for the coordinate-relaxation algorithm 1.2. Below we give such a semi-local convergence theorem for the algorithm 1.2 with iteration order (I).

Theorem 2.1. *Let $F : D \subset \mathbb{R}^n \to \mathbb{R}^n$ be continuously differentiable on the set D. Assume that $F'(x)$ is nonsingular on D and there holds the following condition*

$$|F(x)^T F'(x)^T e^j - F(y)^T F'(y)e^j| \le r_j\|x-y\| \tag{2.1}$$

where $r_j > 0$, $j = 1, 2, \cdots, n$. Suppose

$$\nu = \max_{x\in D}\max_{1\le j\le n}\|F'(x)e^j\|_2 < \infty$$

and

$$r = \inf_{\|y\|_2=1}\inf_{x\in D}\max_{1\le j\le n}\|y^T F'(x)e^j\|_2 > 0. \tag{2.2}$$

If $x^0 \in D$ is such that the closed ball $S = S(x^0, t_0) := \{x : \|x-x^0\|_2 \le t_0\} \subset D$, where

$$t_0 = \frac{(1+\beta)\nu}{(1-q)r_{min}}\|F(x^0)\|_2,$$

with β being a real number such that $0 < \delta \le \beta \le 1$ and $(1-\beta^2)r^2 < r_{min}$, and $r_{min} = \min_{1\le j\le N} r_i$, and if the iteration order $\{i_k\}$ satisfies (I) with the *step-size* $\{t_{i_k}\}$ *satisfying the condition*

$$\begin{aligned}\left(\frac{1-\beta}{1-\delta}\right)\frac{|F(x^k)^TF'(x^k)e^{i_k}|}{r_{i_k}} &\leq t_{i_k}\mathrm{sgn}(F(x^k)^TF'(x^k)e^{i_k}) \qquad (2.3)\\ &\leq \left(\frac{1+\beta}{1+\delta}\right)\frac{|F(x^k)^TF'(x^k)e^{i_k}|}{r_{i_k}}\end{aligned}$$

where δ *and* β *are such that* $0<\delta\leq\beta<1$*, then for any* $1-\delta\leq\omega_k\leq 1+\delta$*, there hold the following conclusions for the algorithm 1.2:*

(i) $\{x^k\}\subset S$;
(ii) $x^k\to x^*\in S$ *as* $k\to\infty$ *and* $F(x*)=0$; *and*
(iii) $\|x^k-x^*\|_2\leq t_0q^k$.

Proof. First we prove the following conclusion by induction: for all $k\geq 1$,

(a) $x^k\in S$,
(b) $\|F(x^k)\|_2\leq q\|F(x^{k-1})\|_2$, and
(c) $\|x^k-x^{k-1}\|_2\leq \frac{(1+\beta)\nu}{r_{min}}\|F(x^0)\|_2q^{k-1}$.

Indeed, when $k=1$, we have

$$\begin{aligned}\|x^1-x^0\|_2 &= \omega_0|t_{i_0}|\\ &\leq (1+\beta)\frac{|F(x^0)^TF'(x^0)e^{i_0}|}{r_{i_0}}\\ &\leq \frac{1+\beta}{r_{min}}\|F(x^0)\|_2\|F'(x_0)e^{i_0}\|_2\\ &\leq \frac{(1+\beta)\nu}{r_{min}}\|F(x^0)\|_2<t_0.\end{aligned}$$

Hence $x^1\in S$. Put

$$\varphi(x)=\frac{1}{2}\|F(x)\|_2^2.$$

Then

$$\varphi(x^0)-\varphi(x^1)\geq\frac{1-\beta^2}{2r_{i_0}}\left[F(x^0)^TF'(x^0)e^{i_0}\right]^2. \qquad (2.4)$$

Notice that

$$|F(x^0)^TF'(x^0)e^{i_0}|=\max_{1\leq j\leq n}|F(x^0)^TF'(x^0)e^j|.$$

If $F(x^0)\neq 0$, then using $\gamma>0$ we infer that

$$\max_{1\leq j\leq n}|F(x^0)^TF'(x^0)e^j|\geq\gamma\|F(x^0)\|_2. \qquad (2.5)$$

From (2.4) and (2.5) it follows that

$$\|F(x^0)\|_2^2 - \|F(x^1)\|_2^2 \geq \frac{(1-\beta^2)r^2}{r_{max}}\|F(x^0)\|_2^2$$

and

$$\|F(x^1)\|_2 \leq q\|F(x^0)\|_2.$$

Consequently (a)-(c) hold for $k = 1$.

Assume that for some $k > 1$, (a)-(c) hold for all $1 \leq j \leq k$ and prove that (a)-(c) remain true for $k + 1$. Indeed we have

$$\begin{aligned}\|x^{k+1} - x^k\|_2 &= \omega_k |t_{i_k}| \\ &\leq (1+\beta)\frac{|F(x^k)F'(x^k)e^{i_k}|}{r_{i_k}} \\ &\leq \frac{1+\beta}{r_{min}}\|F(x^k)F'(x^k)e^{i_k}\|_2 \\ &\leq \frac{(1+\beta)\nu}{r_{min}}\|F(x^k)\|_2.\end{aligned}$$

By the induction assumption we get

$$\|x^{k+1} - x^k\|_2 \leq \frac{(1+\beta)\nu}{r_{min}}\|F(x^0)\|_2 q^k.$$

In the meanwhile we have

$$\begin{aligned}\|x^{k+1} - x^0\|_2 &\leq \sum_{i=0}^{k}\|x^{i+1} - x^i\|_2 \\ &\leq \frac{(1+\beta)\nu}{r_{min}}\|F(x^0)\|_2 \sum_{i=0}^{k} q^i < t_0.\end{aligned}$$

We therefore have that $x^{k+1} \in S$. Since $\gamma > 0$ and

$$\varphi(x^k) - \varphi(x^{k+1}) \geq \frac{1-\beta^2}{2r_{i_k}}|F(x^k)^T F'(x^k)e^{i_k}|^2, \tag{2.6}$$

we obtain

$$|F(x^k)^T F'(x^k)e^{i_k}| = \max_{1\leq j\leq n}|F(x^j)^T F'(x^j)e^j| \geq \gamma\|F(x^k)\|_2. \tag{2.7}$$

By (2.6) and (2.7) we get

$$\|F(x^k)\|_2^2 - \|F(x^{k+1}\|_2^2 \geq \frac{(1-\beta^2)\gamma^2}{r_{max}}\|F(x^k)\|^2.$$

Hence

$$\|F(x^k)\|_2^2 \leq q\|F(x^k)\|^2$$

and we have proved that (a)-(c) hold for all $k \geq 1$. Thus (i) holds.

For all integers k, $m \geq 1$, we have

$$\begin{aligned} \|x^{k+m} - x^k\|_2 &\leq \sum_{j=0}^{m-1} \|x^{k+j+1} - x^{k+j}\|_2 \\ &\leq \frac{(1+\beta)\nu}{r_{min}} \|F(x^0)\|_2 \sum_{j=0}^{m-1} q^{k+j} < t_0 q^k. \end{aligned} \tag{2.8}$$

It follows that $\{x^k\}$ is Cauchy and hence convergent to x^* (say) in S. By (a)-(c) we see that

$$\|F(x^k)\|_2 \leq q^k \|F(x^0)\|_2, \ 0 < q < 1.$$

Taking the limit as $k \to \infty$ to obtain that $F(x^*) = 0$. This verifies (ii). To see (iii) we let $m \to \infty$ in (2.8) to get $\|x^k - x^*\|_2 \leq t_0 q^k$. ☐

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