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FIXED POINT CURVES GENERATED BY NONEXPANSIVE MAPPINGS

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Abstract. In this paper we consider a nonexpansive map T from a nonempty closed bounded and convex set K into K and investigate the properties of the fixed point curves generated from it.

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1. Introduction

Let X be a Banach space and T_1, T_2 two contraction maps from X into X. In 1988 S. Nadler and K. Ushijima [8] considered the contractions $T_t = (1-t)T_1 + tT_2$ for each t in $[0,1]$ and investigated the properties of the curve generated by their fixed points. They proved a necessary and sufficient condition for a curve to be a fixed point curve for two contractions $T_1, T_2 : \mathbb{R}^1 \to \mathbb{R}^1$ and asked if this condition applies for every strictly convex Banach space.

In 1990 in [9] it was given an example which shows that this is not true, even in the case of contraction maps from $\mathbb{R}^2$ into $\mathbb{R}^2$.

In this paper we first give a necessary and sufficient condition for a curve in a Banach space X to be a fixed point curve for two contractions maps from X into X. Then using the above technique, we consider a nonexpansive map T from a nonempty closed, convex and bounded set K of a Banach space X into K and investigate the properties of the fixed point curves of the contraction maps $T_t = (1-t)y + tT$ for $y \in K$ and $t \in [0,1]$.

2. Preliminaries and notations

With a contraction map we mean a map T from a Banach space X into X such that

$$\|Tx - Ty\| \leq L\|x - y\|, \quad 0 < L < 1,$$

for every $x, y \in X$, while a nonexpansive map is a map $T : X \to X$ which satisfies the condition

$$\|Tx - Ty\| \leq \|x - y\|,$$

for every $x, y \in X$.

Let $T_1, T_2 : X \to X$ be contraction maps. In [8] the authors constructed a curve

$$G : [0, 1] \to X$$

which maps every $t \in [0, 1]$ to the fixed point of the contraction map

$$T_t = (1 - t)T_1 + tT_2. \tag{1}$$

They proved that the condition

$$m|t_1 - t_2| \leq \|G(t_1) - G(t_2)\| \leq M|t_1 - t_2| \tag{2}$$

for $t_1, t_2 \in [0, 1]$ and $m > 0, M > 0$ is necessary for a curve to be the fixed point curve of the maps T_t in (1) if X is an arbitrary Banach space and necessary and sufficient if $X = \mathbb{R}^1$. They asked if condition (2) generalize to $\mathbb{R}^n$.

In [9] the author used the Archimedean spiral $G(t) = 2\pi t, t \in [0, 1]$, to show that this in tot true even if $X = \mathbb{R}^2$.

In Section 3 we give a necessary and sufficient condition on a curve to be a fixed point curve for the maps in (1) valid for every Banach space X and with the aid of this condition we give another proof to Theorem 1.1 in [8].

In Section 4 we consider a nonexpansive map $T : K \to K$ where K is a nonempty, bounded, closed and convex subset of a Banach space X. We consider the curves which define the fixed points of the contractions

$$T_t = (1 - t)y + tT, \quad y \in K \text{ fixed}, \ t \in [0, 1]$$

and give some results on the form of these curves and on the relations between such curves, for the various $y \in K$.

3. The fixed point curve for two contractions

Let X be a Banach space and $T_1, T_2 : X \to X$ two contraction maps,

$$\|T_1x - T_1y\| \leq L_1\|x - y\|, \quad \|T_2x - T_2y\| \leq L_2\|x - y\| \tag{3}$$

with $0 < L_1, L_2 < 1$. Then the maps

$$T_t = (1-t)T_1 + tT_2, \quad t \in [0,1] \tag{4}$$

are also contractions with $L = \max\{L_1, L_2\}$. So by the Banach contraction principle T_t has a unique fixed point, say x_t. These fixed points $x_t, t \in [0,1]$, form a curve which we name "*the fixed point curve for* T_1, T_2" and denote by $F(T_1, T_2)$.

Thus for every $x_t \in F(T_1, T_2)$ we will have

$$T_t x_t = (1-t)T_1x_t + tT_2x_t = x_t. \tag{5}$$

Relation (5) gives a necessary condition for the fixed point curve $F(T_1, T_2)$. According to it the vectors $T_1x_t - x_t$ and $x_t - T_2x_t$ are collinear and for every $t \in (0,1)$

$$\frac{\|T_1x_t - x_t\|}{\|x_t - T_2x_t\|} = \frac{t}{1-t}. \tag{6}$$

With the aid of (6) we prove the following:

Theorem 3.1. *(*[8]*, Theorem 1.1). Either* $F(T_1, T_2)$ *is constant or it is* $1-1$.

Proof. Suppose $F(T_1, T_2)$ is not constant and for $t_1, t_2 \in [0,1]$ $x_{t_1} = x_{t_2}$. From equation (6) it follows that

$$\frac{t_1}{1-t_1} = \frac{t_2}{1-t_2}$$

which implies that $t_1 = t_2$. □

The following theorem shows that condition (5) is also sufficient for the fixed point curve $F(T_1, T_2)$.

Theorem 3.2. *Let* $F : [0,1] \to X$ *be a map. The necessary and sufficient condition that* F *is the fixed point curve for the contractions* $T_1, T_2 : X \to X$, *is that for every* $t \in [0,1]$ *the point* $F(t) := x_t$ *satisfies the condition*

$$x_t = (1-t)T_1x_t + tT_2x_t.$$

Proof. The necessity follows immediately from (5). For the sufficiency suppose for a contradiction that for a $t_0 \in [0,1]$ there exists a z_{t_0} which satisfies (5) i.e.

$$z_{t_0} = (1-t_0)T_1 z_{t_0} + t_0 T_2 z_{t_0} \tag{7}$$

and such that $z_{t_0} \notin F(T_1, T_2)$. Now for this t_0, the point $x_{t_0} \in F(T_1, T_2)$, satisfies

$$x_{t_0} = (1-t_0)T_1 x_{t_0} + t_0 T_2 x_{t_0}. \tag{8}$$

¿From (7) and (8) it follows that

$$\begin{aligned} \|z_{t_0} - x_{t_0}\| &= \|(1-t_0)T_1 z_{t_0} + t_0 T_2 z_{t_0} - (1-t_0)T_1 x_{t_0} - t_0 T_2 x_{t_0}\| \\ &\leq (1-t_0)\|T_1 z_{t_0} - T_1 x_{t_0}\| + t_0\|T_2 z_{t_0} - T_2 x_{t_0}\| \\ &\leq (1-t_0)L_1\|z_{t_0} - x_{t_0}\| + t_0 L_2\|z_{t_0} - x_{t_0}\| \\ &\leq L\|z_{t_0} - x_{t_0}\|, \end{aligned}$$

where L_1, L_2 are defined by (3) and $L = \max\{L_1, L_2\} < 1$, which is impossible. The proof of the theorem is complete. ☐

4. THE FIXED POINT CURVES FOR A NONEXPANSIVE MAP

Let K be a nonempty, bounded, closed and convex subset in a Banach space X and $T : K \to K$ a nonexpansive map. Fix $y \in K$. Then the maps

$$T_t := (1-t)y + tT, \quad t \in [0,1] \tag{9}$$

are contraction maps and thus every T_t has a fixed point, say x_t. The totallity of these fixed points form a curve, which we name "*the fixed point curve of T with respect to y*" and denote by $F(y, T)$. Thus for every $y \in K$ we have a curve $F : [0,1] \to K$ with

$$F(t) = x_t \in F(y, T), \quad t \in [0,1]. \tag{10}$$

For $t = \frac{1}{n}, n \in \mathbb{N}$, every $F(y, T)$ define a sequence $\{x_n\}$. Such sequences are very important in the fixed point theory and widely investigated, see [1, 2, 3, 4, 5, 6, 7].

In this section we characterize and give some properties of the curves $F(y, T)$.

Lemma 4.1. *For every $y \in K$ the fixed point curve $F(y,T)$ is continuous in* $(0,1)$.

Proof. Fix $t_0 \in (0,1)$ and let x_t denote the fixed point of the contraction defined by (9). Then

$$\begin{aligned} \|x_t - x_{t_0}\| &= \|(1-t)y + tTx_t - (1-t_0)y - t_0Tx_{t_0}\| \\ &\leq |t-t_0|\|y\| + |t-t_0|\|Tx_t\| + t_0\|Tx_t - Tx_{t_0}\| \\ &\leq |t-t_0|\|y\| + |t-t_0|\|Tx_t\| + t_0\|x_t - x_{t_0}\| \end{aligned}$$

or

$$(1-t_0)\|x_t - x_{t_0}\| \leq |t-t_0|(\|y\| + \|Tx_t\|).$$

Since K is bounded we have

$$\lim_{t\to t_0} \|x_t - x_{t_0}\| = 0.$$

□

Lemma 4.2. *The function $G : [0,1] \to \mathbb{R}$ with*

$$G(t) = \|x_t - Tx_t\|, \quad x_t \in F(y,T),\ y \in K \tag{11}$$

is continuous and $\lim\limits_{t\to 1} G(t) = 0$.

Proof. For $t \in [0,1]$ we have

$$\begin{aligned} \|x_t - Tx_t\| &= \|(1-t)y + tTx_t - Tx_t\| \\ &= (1-t)\|y - Tx_t\| \end{aligned}$$

and since K is bounded $\lim_{t\to 1} G(t) = 0$. □

The next theorem gives a characterization of the fixed point curve $F(y,T)$.

Theorem 4.3. *Let K be a nonempty, bounded, closed and convex subset of a Banach space X, $y \in K$ and $T : K \to K$ a nonexpansive map. Then a point $x \in K$ lies in the fixed point curve $F(y,T)$ if and only if there exists a $t \in [0,1]$ such that*

$$(1-t)(y-x) = t(x - Tx). \tag{12}$$

Proof.($\Rightarrow$) It is clear since every point x of the fixed point curve $F(y,T)$ is a fixed point x_t of a contraction T_t defined by (9), i.e. if satisfies an equation of the form

$$x = x_t = T_t x_t = (1-t)y + tTx_t, \quad t \in [0,1]. \tag{13}$$

($\Leftarrow$) Suppose for a contradiction that there exists an $x_0 \in K - F(y,T)$ such that

$$(1-t_0)y + t_0 Tx_0 = x_0, \quad t_0 \in (0,1). \tag{14}$$

Then there exists also a point $x_{t_0} \in F(y,T)$ with

$$(1-t_0)y + t_0 Tx_{t_0} = x_{t_0}, \quad t_0 \in (0,1). \tag{15}$$

Equations (14) and (15) imply that

$$x_0 - t_0 Tx_0 = x_{t_0} - t_0 Tx_{t_0}$$

and further that

$$\|x_0 - x_{t_0}\| = t_0 \|Tx_0 - Tx_{t_0}\|$$

which is impossible, since T is nonexpansive and $t_0 \in (0,1)$. $\square$

Corollary 4.4. *With the assumptions of Theorem (4.3), the fixed point curve* $F(y,T)$ *is* $1-1$.

Proof. For $x_{t_1}, x_{t_2} \in F(y,T)$ equation (12) gives

$$(1-t_1)(y - x_{t_1}) = t_1(x_{t_1} - Tx_{t_1})$$

and

$$(1-t_2)(y - x_{t_2}) = t_2(x_{t_2} - Tx_{t_2})$$

which for $x_{t_1} = x_{t_2}$ implies that $t_1 = t_2$. $\square$

Corollary 4.5. *The function* $G : y \to F(y,T)$ *is* $1-1$.

Corollary 4.6. *If the fixed point curves* $F(y_1,T)$ *and* $F(y_2,T)$ *intersect at a point* x_0, *then the points* y_1, y_2, x_0 *and* Tx_0 *are collinear.*

Corollary 4.7. *The function* $G(t) = \|x_t - Tx_t\|$, $t \in [0,1]$ *is strictly decreasing.*

Proof. From Lemma 4.2 we have that $G(t)$ is continuous, $\lim_{t\to 1} G(t) = 0$ and $G(0) > 0$, and from Theorem 4.3 that $G(t)$ is $1-1$. $\square$

Our last corollary gives an information about maximal values of $\|x_t - Tx_t\|$.

Corollary 4.8. *Let* $x_0 \in K$. *Then*

$$\sup\{\|x_t - Tx_t\| : x_t = (1-\lambda)x_0 + \lambda Tx_0, \lambda \in \mathbb{R}\} = \|x_s - Tx_s\|,$$

with $x_s = (1-\lambda_0)x_0 + \lambda_0 Tx_0$, *where*

$$\lambda_0 = \inf\{\lambda \in \mathbb{R} : (1-\lambda)x_0 + \lambda Tx_0 \in K\}.$$

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