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WEAKLY PICARD OPERATORS ON A SET WITH TWO METRICS

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Abstract. In this paper we present some results for weakly Picard operators on a set with two metrics. Applications to Fredholm integral equations are given.

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1. Introduction

Let (X, d) be a metric space and $A : X \to X$ an operator. The operator A is weakly Picard operator (WPO) if the sequence of successive approximation, $(A^n(x))_{n\in\mathbb{N}}$, converges for all $x \in X$ and the limit (which may depend on x) is a fixed point of A. If A is WPO and $F_A = \{x^*\}$, then by definition A is a Picard operator (PO).

For an WPO A we consider the operator A^∞ defined by

$$A^\infty : X \to X, \quad A^\infty(x) := \lim_{n\to\infty} A^n(x).$$

Let $c > 0$ be. Then an WPO A is c-WPO if

$$d(x, A^\infty(x)) \le cd(x, A(x)),\ \forall\ x \in X.$$

For the basic results of the WPO theory see I. A. Rus [23], [24].

The aim of this paper is to study the WPOs on a set with two metrics. Some applications to Fredholm integral equations are given.

Throughout this paper we follow the terminologies and notations in [24].

2. Weakly Picard operators

Let X be a nonempty set, d and ρ two metrics on X and $A : X \to X$ an operator. We have

Theorem 2.1. *We suppose that*

(i) there exists $c_1 > 0$ *such that*

$$d(A(x), A(y)) \le c_1\rho(x, y),\ \forall\ x, y \in X;$$

(ii) (X, d) *is a complete metric space;*

(iii) $A : (X, d) \to (X, d)$ *is closed;*

(iv) there exists $\alpha \in]0, 1[$ *such that*

$$\rho(A^2(x), A(x)) \le \alpha\rho(x, A(x)),\ \forall\ x \in X.$$

Then $A : (X, d) \to (X, d)$ *is WPO.*

If in addition we suppose that

(v) there exists $c_2 > 0$ *such that*

$$\rho(x, y) \le c_2 d(x, y),\ \forall\ x, y \in X,$$

then $A : (X, d) \to (X, d)$ *is c-WPO with* $c = 1 + \dfrac{c_1c_2}{1-\alpha}$.

Proof. Let $x \in X$. From (iv) we have that $(A^n(x))_{n\in\mathbb{N}}$ is a Cauchy sequence in (X, ρ). From (i) we have that $(A^n(x))_{n\in\mathbb{N}}$ is Cauchy sequence in (X, d). From (ii) it converges in (X, d) and from (iii) the limit $x^*(x)$ is a fixed point of A. So, A is WPO in (X, d).

From (iv), (i) and (v) we have

$$\begin{aligned} d(x, A^p(x)) &\leq d(x, A(x)) + d(A(x), A^p(x)) \\ &\leq d(x, A(x)) + d(A(x), A^2(x)) + \cdots + d(A^{p-1}(x), A^p(x)) \\ &\leq d(x, A(x)) + \frac{c_1}{1-\alpha}\rho(x, A(x)) \\ &\leq d(x, A(x)) + \frac{c_1 c_2}{1-\alpha} d(x, A(x)) \\ &= \left(1 + \frac{c_1 c_2}{1-\alpha}\right) d(x, A(x)),\ \forall\, x \in X,\ \forall\, p \geq 1,\ p \in \mathbb{N}. \end{aligned}$$

Hence

$$d(x, A^\infty(x)) \leq \left(1 + \frac{c_1 c_2}{1-\alpha}\right) d(x, A(x)),\ \forall\, x \in X.$$

Theorem 2.2. *We suppose that the conditions (i), (ii) and (iii) in Theorem 2.1 are satisfied. If*

(iv') there exists $\varphi : X \to \mathbb{R}_+$ such that

$$\rho(x, A(x)) \leq \varphi(x) - \varphi(A(x)),\ \forall\, x \in X,$$

then $A : (X, d) \to (X, d)$ is WPO.

If in addition we have (v) in Theorem 2.1 and

(vi) $\varphi(x) \leq c_3 \rho(x, A(x)),\ \forall\, x \in X$,

then A is c-WPO in (X, d) with $c = 1 + c_1 c_2 c_3$.

Proof. Let $x \in X$ be. From (iv') we have that

$$\sum_{n \in \mathbb{N}} \rho(A^n(x), A^{n+1}(x)) \leq \varphi(x),\ \forall\, x \in X. \qquad (*)$$

Hence $(A^n(x))$ is Cauchy sequence in (X, ρ). Now the proof that A is WPO is as in the proof of Theorem 2.1.

From $(*)$, (v) and (vi) we have

$$\begin{aligned} d(x, A^{n+1}(x)) &\leq d(x, A(x)) + c_1 \rho(x, A^n(x)) \\ &\leq d(x, A(x)) + c_1 \varphi(x) \leq d(x, A(x)) + c_1 c_3 \rho(x, A(x)) \\ &\leq d(x, A(x)) + c_1 c_2 c_3 d(x, A(x)) = (1 + c_1 c_2 c_3) d(x, A(x)). \end{aligned}$$

So,

$$d(x, A^\infty(x)) \leq (1 + c_1 c_2 c_3) d(x, A(x)),\ \forall\, x \in X.$$

Remark 2.1. For other fixed point theorems in a set with two metrics see [1], [2], [5], [8]-[12], [17]-[20], [22], [28].

Remark 2.2. We can generalize the above theorems in the case of the generalized metrics. For example in the case when $d(x,y), \rho(x,y) \in \mathbb{R}^m_+$, we have:

Theorem 2.1'. *We suppose that:*

(i) there exists a matrix $C_1 \in \mathcal{M}_m(\mathbb{R}_+)$ such that

$$d(A(x),A(y)) \leq C_1\rho(x,y),\ \forall\ x,y \in X;$$

(ii) (X,d) is a complete metric space;

(iii) $A:(X,d) \to (X,d)$ is closed;

(iv) there exists a matrix $S \in \mathcal{M}_m(\mathbb{R}_+)$ that converges to zero, such that

$$\rho(A^2(x),A(x)) \leq S\rho(x,A(x)),\ \forall\ x \in X.$$

Then $A:(X,d) \to (X,d)$ is WPO.

If in addition we suppose that

(v) there exists $C_2 \in \mathcal{M}_m(\mathbb{R}_+)$ such that

$$\rho(x,y) \leq C_2 d(x,y),\ \forall\ x,y \in X,$$

then $A:(X,d) \to (X,d)$ is c-WPO with $c = I_m + C_1C_2(I_m - S)^{-1}$.

Theorem 2.2'. *We suppose that the conditions (i), (ii) and (iii) in Theorem 2.1' are satisfied. If*

(iv') there exists $\varphi : X \to \mathbb{R}^m_+$ such that

$$\rho(x,A(x)) \leq \varphi(x) - \varphi(A(x)),\ \forall\ x \in X,$$

then $A:(X,d) \to (X,d)$ is WPO.

If in addition we have (v) in Theorem 2.1' and

(vi) there exists $C_3 \in \mathcal{M}_m(\mathbb{R}_+)$ such that

$$\varphi(x) \leq C_3\rho(x,A(x)),\ \forall\ x \in X,$$

then A is c-WPO in (X,d) with $c = I_m + C_1C_2C_3$.

Remark 2.3. Let $(X,d,\leq)$ be an ordered metric space, (X,ρ) a metric space and $A : X \to X$ an operator. We suppose that we are in the conditions (i)-(iv) of the Theorem 2.1, or in the conditions (i)-(iii), (iv') of the Theorem 2.2 and in addition the operator $A:(X,\leq) \to (X,\leq)$ is increasing. Then

(a) $x \leq A(x) \Rightarrow x \leq A^\infty(x),\ \forall\ x \in X$;

(b) $x \geq A(x) \Rightarrow x \geq A^\infty(x),\ \forall\ x \in X$.

These implications follow from Lemma 3.5 in I. A. Rus [24].

Remark 2.4. Let $(X, d, \leq)$ be an ordered metric space, (X, ρ) a metric space and $A, B, C : X \to X$ three operators. We suppose that the operators A, B, C satisfy the conditions (i)-(iv) in the Theorem 2.1, or the conditions (i)-(iii) and (iv') in Theorem 2.2. In addition we suppose that $A \leq B \leq C$ and $B : (X, \leq) \to (X, \leq)$ is increasing. Then

$$x \leq y \leq z \Rightarrow A^{\infty}(x) \leq A^{\infty}(y) \leq A^{\infty}(z).$$

This implication follows from Lemma 3.2 in [24].

Remark 2.5. Let $A, B : X \to X$ as in the Theorem 2.1 or in the Theorem 2.2. In addition we suppose that there exists $\eta > 0$ such that $d(A(x), B(x)) \leq \eta,\ \forall\ x \in X$. Then

$$H_d(F_A, F_B) \leq c\eta,$$

where H_d is the functional of Pompeiu-Hausdorff.

The above estimation follows from Theorem 2.2 in [24].

Remark 2.6. For the WPOs theory in L-spaces see I. A. Rus [26].

3. Applications to Fredholm functional-integral equations

Let $\Omega \subset \mathbb{R}^m$ be a bounded domain. We consider the following functional-integral equation

$$x(t) = h(t, x|_{\partial\Omega}) + \int\limits_{\Omega} K(t, s, x(s))ds, \quad t \in \overline{\Omega}. \tag{3.1}$$

We suppose that

(1) $K(t, s, u) = 0,\ \forall\ t \in \partial\Omega,\ s \in \overline{\Omega},\ u \in \mathbb{R}$;

(2) $h(\cdot, x|_{\partial\Omega}) \in C(\overline{\Omega}),\ \forall\ x \in C(\overline{\Omega})$;

(3) $h(t, x|_{\partial\Omega}) = x(t),\ \forall\ t \in \partial\Omega$;

(4) $K \in C(\overline{\Omega} \times \overline{\Omega} \times \mathbb{R})$;

(5) there exists $L \in C(\overline{\Omega} \times \overline{\Omega})$ such that

$$|K(t, s, u) - K(t, s, v)| \leq L(t, s)|u - v|,$$

for all $t, s \in \overline{\Omega}$ and $u, v \in \mathbb{R}$;

(6) $\int\limits_{\Omega\times\Omega} (L(t, s))^2 dtds < 1.$

Let $A : C(\overline{\Omega}) \to C(\overline{\Omega})$ be the operator defined by $A(x)(t) :=$ second part of (3.1). Consider on $C(\overline{\Omega})$ the following metrics

$$d(x,y) := \|x-y\|_{C(\overline{\Omega})} \text{ and } \rho(x,y) = \|x-y\|_{L^2(\Omega)}.$$

It is clear that (see Theorem 2.1),

$$c_1 = \sup_{t\in\overline{\Omega}} \left(\int_\Omega (L(t,s))^2 ds \right)^{\frac{1}{2}}, \quad c_2 = m(\Omega)$$

and

$$\alpha = \left(\int_{\Omega\times\Omega} (L(t,s))^2 dtds \right)^{\frac{1}{2}}.$$

Let us prove the result for α. For this we consider the following partition of $C(\overline{\Omega})$, $C(\overline{\Omega}) = \bigcup_{\varphi\in C(\partial\Omega)} X_\varphi$, where $X_\varphi = \{x \in C(\overline{\Omega}) |\ x|_{\partial\Omega} = \varphi\}$. We remark that X_φ is a closed subset of $(C(\overline{\Omega}), \|\cdot\|_C)$ and from (3) we have that $A(X_\varphi) \subset X_\varphi$. Moreover from (5) and (6) it follows that $A|_{X_\varphi} : (X_\varphi, \|\cdot\|_{L^2}) \to (X_\varphi, \|\cdot\|_{L^2})$ is α-contraction, for each $\varphi \in C(\partial\Omega)$. So, we are in the conditions of the Theorem 2.1 and we have

Theorem 3.1. *We suppose that we are in the conditions (1)-(6). Let $S_A \subset C(\overline{\Omega})$ be the solution set of the equation (3.1). Then $Card\, S_A = Card\, C(\partial\Omega)$.*

Theorem 3.2. *We suppose that the conditions (1)-(6) are satisfied. In addition we suppose that*

(7) $h(t,\cdot) : C(\partial\Omega) \to \mathbb{R}$ and $K(t,s,\cdot) : \mathbb{R} \to \mathbb{R}$ are increasing.

Let x and y be two solutions of the equation (3.1). If $x|_{\partial\Omega} \le y|_{\partial\Omega}$ then $x \le y$.

Proof. From (1)-(6) we have that A is WPO. Because of (7) A is increasing. Let $\widetilde{x} \in X_{x|_{\partial\Omega}}$ and $\widetilde{y} \in X_{y|_{\partial\Omega}}$ be such that $\widetilde{x} \le \widetilde{y}$, $\widetilde{x}|_{\partial\Omega} = x|_{\partial\Omega}$, $\widetilde{y}|_{\partial\Omega} = y|_{\partial\Omega}$. Now the proof follows because the operator A^∞ is increasing (see Lemma 3.1, [24]).

Let us consider the following functional-integral equations

$$x(t) = h_i(t, x|_{\partial\Omega}) + \int_\Omega K_i(t,s,x(s))ds, \quad t \in \overline{\Omega},\ i = \overline{1,3}. \tag{3.2}$$

We suppose that

(8) the corresponding conditions (1)-(6) are satisfied for h_i, K_i, L_i, $i = \overline{1,3}$;
(9) $h_1 \leq h_2 \leq h_3$ and $K_1 \leq K_2 \leq K_3$;
(10) $h_2(t,\cdot) : C(\partial\Omega) \to \mathbb{R}$ and $K(t,s,\cdot) : \mathbb{R} \to \mathbb{R}$ are increasing.

Let x_i be a solution of the corresponding equation (3.2) and A_i the corresponding operator, $\forall\ i = \overline{1,3}$.

If $x_1|_{\partial\Omega} \leq x_2|_{\partial\Omega} \leq x_3|_{\partial\Omega}$, then $x_1 \leq x_2 \leq x_3$.

Proof. From (8) we have that A_i are WPO, $i = \overline{1,3}$. Because of (10), the operator A_2 is increasing. From (9) we have that $A_1 \leq A_2 \leq A_3$. So, we are in the conditions of Lemma 3.2 in [24].

Remark 3.1. More general we can consider the following equation

$$x(t) = h(t, x|_{\partial\Omega}) + g(t, x) \tag{3.2}$$

We have

Theorem 3.1'. *We suppose that*
(1) $g(t,x) = 0,\ \forall\ t \in \partial\Omega$ *and* $x \in C(\overline{\Omega})$*;*
(2) $g(\cdot, x) \in C(\overline{\Omega}),\ \forall\ x \in C(\overline{\Omega})$*;*
(3) $h(\cdot, x) \in C(\overline{\Omega}),\ \forall\ x \in C(\overline{\Omega})$*;*
(4) $h(t, x|_{\partial\Omega}) = x(t),\ \forall\ t \in \partial\Omega$
(5) there exists $\alpha \in]0,1[$ *such that*

$$|g(t,x) - g(t,y)| \leq \alpha\rho(x,y),\ \forall\ x,y \in C(\overline{\Omega}).$$

Let S *be the solution set of the equation (3.2). Then* $CardS = CardC(\partial\Omega)$.

In the case of Dirichlet problem

$$-\Delta x = f(t,x), \quad x \in C^2(\Omega) \cap C(\overline{\Omega})$$

$$x|_{\partial\Omega} = \varphi,$$

where $f \in C(\overline{\Omega} \times \mathbb{R})$ and $\varphi \in C(\partial\Omega)$, the corresponding operator is the following

$$A_f(x)(t) := \int_{\partial\Omega} \frac{\partial G(t,s)}{\partial n_s} x(s) d\sigma_s + \int_{\Omega} G(t,s) f(s, x(s)) ds.$$

Here G is the Green function. The WPOs' technique to this problem has been used by Dincuţă ([6]), A. Buică ([4]) and I. A. Rus ([26]).

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