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WEAK CONVERGENCE THEOREM BY CESÁRO MEANS FOR NONEXPANSIVE MAPPINGS AND MONOTONE MAPPINGS

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Abstract. In this paper we introduce an iterative process by Cesáro means for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of the variational inequality problem for a monotone, Lipschitz continuous mapping in a Hilbert space. The iterative process is based on two ideas: ergodic iterates (Cesáro means) and so-called extragradient method. We obtain a weak convergence theorem for the sequence generated by this process. As a direct corollary of our theorem we obtain the well-known ergodic theorem proved by Baillon[1].
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1. Introduction

Let C be a closed convex subset of a real Hilbert space H and let P_C be the metric projection of H onto C. A mapping S of C into itself is called *nonexpansive* if

$$\|Su - Sv\| \leq \|u - v\|$$

for all $u, v \in C$; see [11]. We denote by $F(S)$ the set of fixed points of S. In 1975, Baillon [1] proved the first nonlinear ergodic theorem. Baillon's result can be stated as the following. Define

$$z_n = \frac{1}{n}\sum_{k=1}^{n} S^{k-1}x \tag{1.1}$$

for every $n = 1, 2, ...$ and $x \in C$ and suppose $F(S) \neq \emptyset$. Then the sequence $\{z_n\}$, generated by (1.1), converges weakly to some element of $F(S)$.

A mapping A of C into H is called *monotone* if

$$\langle Ax - Ay, x - y\rangle \geq 0$$

for all $x, y \in C$. We denote by $VI(C, A)'$ the set of $u \in C$ such that $\langle v - u, Av\rangle \geq 0$ for all $v \in C$. For finding an element of $VI(C, A)'$, Bruck [3] introduced the following iterative scheme: $x_{n+1} = P_C(x_n - \lambda_n Ax_n)$,

$$z_n = \frac{\sum_{k=1}^{n} \lambda_k x_k}{\sum_{k=1}^{n} \lambda_k} \tag{1.2}$$

for every $n = 1, 2, ...$, where $x_1 = x \in C$ and $\{\lambda_n\}$ is a sequence of positive real numbers such that $\sum_{n=1}^{\infty} \lambda_n = \infty$ and $\sum_{n=1}^{\infty} \|\lambda_n Ax_n\|^2 < \infty$. He showed that the sequence $\{z_n\}$, generated by (1.2), converges weakly to some element of $VI(C, A)'$. However, Bruck notices in his paper that Baillon's result does not follow from his theorem, since the condition $\sum_{n=1}^{\infty} \|\lambda_n Ax_n\|^2 < \infty$ may not be satisfied.

The *variational inequality problem* is to find a $u \in C$ such that

$$\langle Au, v - u\rangle \geq 0$$

for all $v \in C$. The set of solutions of the variational inequality problem is denoted by $VI(C, A)$. For solving the variational inequality problem in a finite-dimensional Euclidean space $\mathbb{R}^n$ under the assumption that a set $C \subset \mathbb{R}^n$ is closed and convex, a mapping A of C into $\mathbb{R}^n$ is monotone and k-Lipschitz-continuous and $VI(C, A)$ is nonempty, Korpelevich [6] introduced the following so-called extragradient method:

$$\begin{cases} x_0 = x \in C \\ \overline{x}_n = P_C(x_n - \lambda Ax_n) \\ x_{n+1} = P_C(x_n - \lambda A\overline{x}_n) \end{cases} \tag{1.3}$$

for every $n = 0, 1, 2, ...$, where $\lambda \in (0, 1/k)$. He showed that the sequences $\{x_n\}$ and $\{\overline{x}_n\}$, generated by (1.3), converge to the same point $z \in VI(C, A)$.

In this paper, motivated by the ideas of results of Baillon, Bruck and Korpelevich, we introduce an iterative process by Cesáro means for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of the variational inequality problem for a monotone, Lipschitz continuous mapping in a real Hilbert space. We obtain a weak convergence theorem for the sequence generated by this process. As a direct corollary of our theorem we obtain the well-known ergodic theorem proved by Baillon [1].

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$ and let C be a closed convex subset of H. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x and $x_n \to x$ to indicate that $\{x_n\}$ converges strongly to x. For every point $x \in H$ there exists a unique nearest point in C, denoted by $P_C x$, such that $\|x - P_C x\| \le \|x - y\|$ for all $y \in C$. P_C is called *the metric projection* of H onto C. We know that P_C is a nonexpansive mapping of H onto C. It is also known that P_C is characterized by the following properties: $P_C x \in C$ and

$$\langle x - P_C x, P_C x - y \rangle \ge 0; \tag{2.1}$$

$$\|x - y\|^2 \ge \|x - P_C x\|^2 + \|y - P_C x\|^2 \tag{2.2}$$

for all $x \in H$, $y \in C$; see [11] for more details. Let A be a monotone mapping of C into H. In the context of variational inequality problem this implies

$$u \in VI\,(C, A) \Leftrightarrow u = P_C\,(u - \lambda A u) \quad \forall \lambda > 0. \tag{2.3}$$

It is also known that H satisfies Opial's condition [8], i.e., for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$ the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

holds for every $y \in H$ with $y \neq x$.

A set-valued mapping $T : H \to 2^H$ is called *monotone* if for all $x, y \in H$, $f \in Tx$ and $g \in Ty$ imply $\langle x - y, f - g \rangle \ge 0$. A monotone mapping $T : H \to 2^H$ is *maximal* if its graph $G\,(T)$ is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal if and only if for $(x, f) \in H \times H$, $\langle x - y, f - g \rangle \ge 0$ for every $(y, g) \in G\,(T)$ implies $f \in Tx$. Let A be a monotone, $k-$Lipschitz-continuous mapping of C into H and $N_C v$ be the normal cone to C at $v \in C$, i.e. $N_C v = \{w \in H : \langle v - u, w \rangle \geq 0, \forall u \in C\}$. Define

$$Tv = \begin{cases} Av + N_C v, & \text{if } v \in C \\ \emptyset, & \text{if } v \notin C \end{cases}$$

Then T is maximal monotone and $0 \in Tv$ if and only if $v \in VI(C, A)$; see [9].

3. Weak Convergence Theorem

In this section we prove a weak convergence theorem for nonexpansive mappings and monotone mappings. For our proof we need the foolowing lemma. Originally it was proved by Takahashi and Toyoda [13].

Lemma 3.1. *Let H be a real Hilbert space and D be a nonempty closed convex subset of H. Let $\{x_n\}$ be a sequence in H. Suppose that for all $u \in D$*

$$\|x_{n+1} - u\| \leq \|x_n - u\|$$

for every $n = 0, 1, 2, ...$ Then the sequence $\{P_D x_n\}$ converges strongly to some $z \in D$.

Now we can state a weak convergence theorem.

Theorem 3.1. *Let C be a closed convex subset of a real Hilbert space H. Let A be a monotone, k-Lipschitz-continuous mapping of C into H and S be a nonexpansive mapping of C into itself such that $F(S) \cap VI(C, A) \neq \emptyset$. Let $\{z_n\}$ be a sequence generated by*

$$\begin{cases} x_1 = x \in C \\ y_n = P_C(x_n - \lambda_n A x_n) \\ x_{n+1} = S P_C(x_n - \lambda_n A y_n) \\ z_n = \frac{1}{n} \sum\limits_{k=1}^{n} x_k \end{cases}$$

for every $n = 1, 2, ...,$ where $\{\lambda_n\} \subset [a, b]$ for some a, b with $0 < a < b < 1/k$. Then the sequence $\{z_n\}$ converges weakly to $z \in F(S) \cap VI(C, A)$, where $z = \lim\limits_{n\to\infty} P_{F(S)\cap VI(C,A)} x_n$.

Proof. Put $t_n = P_C(x_n - \lambda_n A y_n)$ for every $n = 1, 2, ...$ Let $u \in F(S) \cap VI(C, A)$. From (2.2), we have

$$\|t_n - u\|^2 \leq \|x_n - \lambda_n A y_n - u\|^2 - \|x_n - \lambda_n A y_n - t_n\|^2$$

$$= \|x_n - u\|^2 - \|x_n - t_n\|^2 + 2\lambda_n \langle Ay_n, u - t_n\rangle$$

$$= \|x_n - u\|^2 - \|x_n - t_n\|^2$$

$$+2\lambda_n (\langle Ay_n - Au, u - y_n\rangle + \langle Au, u - y_n\rangle + \langle Ay_n, y_n - t_n\rangle)$$

$$\leq \|x_n - u\|^2 - \|x_n - t_n\|^2 + 2\lambda_n \langle Ay_n, y_n - t_n\rangle$$

$$= \|x_n - u\|^2 - \|x_n - y_n\|^2 - 2\langle x_n - y_n, y_n - t_n\rangle - \|y_n - t_n\|^2$$

$$+2\lambda_n \langle Ay_n, y_n - t_n\rangle$$

$$= \|x_n - u\|^2 - \|x_n - y_n\|^2 - \|y_n - t_n\|^2 + 2\langle x_n - \lambda_n Ay_n - y_n, t_n - y_n\rangle .$$

Further, from (2.1), we have

$$\begin{aligned}
&\langle x_n - \lambda_n Ay_n - y_n, t_n - y_n\rangle \\
&= \langle x_n - \lambda_n Ax_n - y_n, t_n - y_n\rangle + \langle \lambda_n Ax_n - \lambda_n Ay_n, t_n - y_n\rangle \\
&\leq \langle \lambda_n Ax_n - \lambda_n Ay_n, t_n - y_n\rangle \\
&\leq \lambda_n k \|x_n - y_n\| \|t_n - y_n\| .
\end{aligned}$$

So, we have

$$\|t_n - u\|^2 \leq \|x_n - u\|^2 - \|x_n - y_n\|^2 - \|y_n - t_n\|^2 + 2\lambda_n k \|x_n - y_n\| \|t_n - y_n\|$$

$$\leq \|x_n - u\|^2 - \|x_n - y_n\|^2 - \|y_n - t_n\|^2 + \lambda_n^2 k^2 \|x_n - y_n\|^2 + \|y_n - t_n\|^2$$

$$\leq \|x_n - u\|^2 + \left(\lambda_n^2 k^2 - 1\right) \|x_n - y_n\|^2$$

$$\leq \|x_n - u\|^2 .$$

We also have

$$\|x_{n+1} - u\| = \|St_n - u\| \leq \|t_n - u\| \leq \|x_n - u\| .$$

Therefore, there exists $c = \lim_{n\to\infty} \|x_n - u\|$ and the sequences $\{x_n\}$, $\{z_n\}$ are bounded. From

$$\|x_{n+1} - u\|^2 \leq \|t_n - u\|^2 \leq \|x_n - u\|^2 + \left(\lambda_n^2 k^2 - 1\right) \|x_n - y_n\|^2$$

we also obtain

$$\|x_n - y_n\|^2 \leq \frac{1}{(1 - \lambda_n^2 k^2)} \left(\|x_n - u\|^2 - \|x_{n+1} - u\|^2\right)$$

Hence, $x_n - y_n \to 0$, $n \to \infty$.

As $\{z_n\}$ is bounded, there is a subsequence $\{z_{n_i}\}$ of $\{z_n\}$ that converges weakly to some z. We can obtain that $z \in F(S) \cap VI(C, A)$. First, we show $z \in VI(C, A)$. Let

$$Tv = \begin{cases} Av + N_C v, & \text{if } v \in C, \\ \emptyset, & \text{if } v \notin C. \end{cases}$$

Then T is maximal monotone and $0 \in Tv$ if and only if $v \in VI(C, A)$; see [9]. Let $(v, w) \in G(T)$. Then, we have $w \in Tv = Av + N_C v$ and hence $w - Av \in N_C v$. So, we have $\langle v - u, w - Av \rangle \geq 0$ for all $u \in C$.
On the other hand, from $y_k = P_C(x_k - \lambda_k A x_k)$ and $v \in C$ we have

$$\langle x_k - \lambda_k A x_k - y_k, y_k - v \rangle \geq 0,$$

and hence

$$\left\langle v - y_k, \frac{y_k - x_k}{\lambda_k} + A x_k \right\rangle \geq 0.$$

Therefore, from $w - Av \in N_C v$ and $x_k \in C$, we have

$$\begin{aligned} \langle v - x_k, w \rangle &\geq \langle v - x_k, Av \rangle \\ &\geq \langle v - x_k, Av \rangle - \left\langle v - y_k, \frac{y_k - x_k}{\lambda_k} + A x_k \right\rangle \\ &= \langle v - x_k, Av - A x_k \rangle + \langle (v - x_k) - (v - y_k), A x_k \rangle \\ &\quad - \left\langle v - y_k, \frac{y_k - x_k}{\lambda_k} \right\rangle \\ &\geq \langle y_k - x_k, A x_k \rangle - \left\langle v - y_k, \frac{y_k - x_k}{\lambda_k} \right\rangle \\ &\geq \left(- \|A x_k\| - \frac{\|y_k - v\|}{\lambda_k} \right) \|y_k - x_k\| \\ &\geq \left(-K - \frac{L}{a} \right) \|y_k - x_k\| . \end{aligned}$$

for every $k = 1, 2, ...$, where $K = \sup \{\|A x_k\| : k \in \mathbf{N}\}$ and

$$L = \sup \{\|y_k - v\| : k \in \mathbf{N}\} .$$

Hence we have

$$\langle v - z_n, w \rangle \geq \left(-K - \frac{L}{a} \right) \frac{1}{n} \sum_{k=1}^{n} \|y_k - x_k\|.$$

Taking $n = n_i$, from $\|x_n - y_n\| \to 0$, we have $\langle v - z, w \rangle \geq 0$ as $i \to \infty$. Since T is maximal monotone, we have $z \in T^{-1}0$ and hence $z \in VI(C, A)$.

Let us show $z \in F(S)$. Let $u \in VI(C, A)$. We have

$$\|x_{k+1} - Su\| = \|St_k - Su\| \leq \|t_k - u\| \leq \|x_k - u\|$$

for every $k = 1, 2, ...$ For $u \in VI(C, A)$, we have

$$\begin{aligned} 0 &\leq \|x_k - u\|^2 - \|x_{k+1} - Su\|^2 \\ &= \|x_k - Su\|^2 + 2\langle x_k - Su, Su - u\rangle \\ &\quad + \|Su - u\|^2 - \|x_{k+1} - Su\|^2 \end{aligned}$$

for every $k = 1, 2, ...$ Then

$$0 \leq \frac{1}{n}\left(\|x - Su\|^2 - \|x_{n+1} - Su\|^2\right) + 2\langle z_n - Su, Su - u\rangle + \|Su - u\|^2.$$

Taking $n = n_i$, we have, as $i \to \infty$,

$$0 \leq 2\langle z - Su, Su - u\rangle + \|Su - u\|^2.$$

Putting $u = z$, we obtain $0 \leq -\|Sz - z\|^2$ and hence $z \in F(S)$. This implies $z \in F(S) \cap VI(C, A)$.

Put $u_n = P_{F(S)\cap VI(C,A)}x_n$. By $\|x_{n+1} - u\| \leq \|x_n - u\|$ for all $u \in F(S) \cap VI(C, A)$ and Lemma 3.1, we obtain that $\{u_n\}$ converges strongly to $w \in F(S) \cap VI(C, A)$. We show $z = w$. Since $u_k = P_{F(S)\cap VI(C,A)}x_k$ and $z \in F(S) \cap VI(C, A)$, we have

$$\langle z - u_k, u_k - x_k\rangle \geq 0$$

for every $k = 1, 2, ...$ So, we have

$$\begin{aligned} \langle z - w, x_k - u_k\rangle &= \langle z - u_k, x_k - u_k\rangle + \langle u_k - w, x_k - u_k\rangle \\ &\leq \|u_k - w\| \|x_k - u_k\| \leq M\|u_k - w\| \end{aligned}$$

for every $k = 1, 2, ...$, where $M = \sup\{\|x_k - u_k\| : k \in \mathbf{N}\}$. Hence we have

$$\left\langle z - w, z_n - \frac{1}{n}\sum_{k=1}^{n} u_k \right\rangle \leq \frac{M}{n}\sum_{k=1}^{n}\|u_k - w\|.$$

Taking $n = n_i$, from $\|u_n - w\| \to 0$, we obtain $\langle z - w, z - w\rangle \leq 0$ as $i \to \infty$ and hence $z = w$. Therefore, we obtain $z_n \rightharpoonup z$.

□

4. Applications.

Using Theorem 3.1, we prove some theorems in a real Hilbert space. The first one is a nonlinear ergodic theorem obtained by Baillon [1].

Theorem 4.1. *Let C be a closed convex subset of a real Hilbert space H and S be a nonexpansive mapping of C into itself such that $F(S) \neq \emptyset$. Let $\{z_n\}$ be a sequence generated by*

$$\begin{cases} x \in C \\ z_n = \frac{1}{n} \sum\limits_{k=1}^{n} S^{k-1} x \end{cases}$$

for every $n = 1, 2, ...,$ where $\{\lambda_n\} \subset [a, b]$ for some a, b with $0 < a < b < 1/k$. Then the sequence $\{z_n\}$ converges weakly to $z \in F(S)$.

Proof. In Theorem 3.1 put $Ax = 0$ for all $x \in C$. We have $C = VI(C, A)$ and $x_{n+1} = S^n x$. So, by Theorem 3.1, we obtain the desired result. □

Theorem 4.2. *Let H a real Hilbert space. Let A be a monotone, k-Lipschitz-continuous mapping of H into itself and S be a nonexpansive mapping of H into itself such that $F(S) \cap A^{-1}0 \neq \emptyset$. Let $\{z_n\}$ be a sequence generated by*

$$\begin{cases} x_1 = x \in H \\ x_{n+1} = S(x_n - \lambda_n A(x_n - \lambda_n A x_n)) \\ z_n = \frac{1}{n} \sum\limits_{k=1}^{n} x_k \end{cases}$$

for every $n = 1, 2, ...,$ where $\{\lambda_n\} \subset [a, b]$ for some a, b with $0 < a < b < 1/k$. Then the sequence $\{z_n\}$ converges weakly to $z \in F(S) \cap A^{-1}0$, where $z = \lim\limits_{n\to\infty} P_{F(S)\cap A^{-1}0} x_n$.

Proof. We have $A^{-1}0 = VI(H, A)$ and $P_H = I$. By Theorem 3.1, we obtain the desired result. □

Remark. Notice that $F(S) \cap A^{-1}0 \subset VI(F(S), A)$. See also Yamada [14] for the case when A is a strongly monotone and Lipschitz continuous mapping of a real Hilbert space H into itself and S is a nonexpansive mapping of H into itself.

Theorem 4.3. *Let H be a real Hilbert space. Let A be a monotone, k-Lipschitz-continuous mapping of H into itself and $B : H \to 2^H$ be a maximal* *monotone mapping such that* $A^{-1}0 \cap B^{-1}0 \neq \emptyset$*. Let* J_r^B *be the resolvent of* B *for each* $r > 0$*. Let* $\{z_n\}$ *be a sequence generated by*

$$\begin{cases} x_1 = x \in H \\ x_{n+1} = J_r^B\left(x_n - \lambda_n A\left(x_n - \lambda_n A x_n\right)\right) \\ z_n = \frac{1}{n}\sum\limits_{k=1}^{n} x_k \end{cases}$$

for every $n = 1, 2, ...,$ *where* $\{\lambda_n\} \subset [a, b]$ *for some* $a, b \in (0, 1/k)$*. Then the sequence* $\{z_n\}$ *converges weakly to some point* $z \in A^{-1}0 \cap B^{-1}0$*, where* $z = \lim\limits_{n\to\infty} P_{A^{-1}0\cap B^{-1}0} x_n$*.*

Proof. We have $A^{-1}0 = VI\left(H, A\right)$ and $F\left(J_r^B\right) = B^{-1}0$. Putting $P_H = I$, by Theorem 3.1 we obtain the desired result. $\square$

A mapping $T : C \to C$ is called *pseudocontractive* if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|(I - T)x - (I - T)y\|^2$$

for all $x, y \in C$, or, equivalently,

$$\langle Tx - Ty, x - y\rangle \leq \|x - y\|^2 \tag{4.1}$$

for all $x, y \in C$, see [2].

Theorem 4.4. *Let* C *be a closed convex subset of a real Hilbert space* H*. Let* T *be a pseudocontractive,* m*-Lipschitz-continuous mapping of* C *into itself and* S *be a nonexpansive mapping of* C *into itself such that* $F(T) \cap F(S) \neq \emptyset$*. Let* $\{z_n\}$ *be a sequence generated by*

$$\begin{cases} x_1 = x \in C \\ y_n = x_n - \alpha_n\left(x_n - Tx_n\right) \\ x_{n+1} = SP_C\left(x_n - \alpha_n\left(y_n - Ty_n\right)\right) \\ z_n = \frac{1}{n}\sum\limits_{k=1}^{n} x_k \end{cases}$$

for every $n = 1, 2, ...,$ *where* $\{\alpha_n\} \subset [a, b]$ *for some* a, b *with* $0 < a < b < \frac{1}{m+1}$*. Then the sequence* $\{z_n\}$ *converges weakly to some* $z \in F(T) \cap F(S)$*.*

Proof. Let $A = I - T$. Let us show the mapping A is monotone and $(m + 1)$-Lipschitz-continuous. From the definition of the mapping A and (4.1), we have

$$\begin{aligned}\langle Ax - Ay, x - y\rangle &= \langle x - y - Tx + Ty, x - y\rangle \\ &= \|x - y\|^2 - \langle Tx - Ty, x - y\rangle \geq \|x - y\|^2 - \|x - y\|^2 = 0.\end{aligned}$$

So, A is monotone. We also have

$$\begin{aligned}\|Ax - Ay\|^2 &= \|(I - T)\,x - (I - T)\,y\|^2 \\ &= \|x - y\|^2 + \|Tx - Ty\|^2 - 2\,\langle x - y, Tx - Ty\rangle \\ &\leq \|x - y\|^2 + m^2\,\|x - y\|^2 + 2\,\|x - y\|\,\|Tx - Ty\| \\ &\leq \|x - y\|^2 + m^2\,\|x - y\|^2 + 2m\,\|x - y\|^2 = (m + 1)^2\,\|x - y\|^2\,.\end{aligned}$$

So, we have $\|Ax - Ay\| \leq (m + 1)\,\|x - y\|$ and A is $(m + 1)$-Lipschitz-continuous.

Now let us show $F\,(T) = VI\,(C, A)$. In fact, we have, for $\lambda > 0$,

$$\begin{aligned}u \in VI\,(C, A) &\Leftrightarrow \langle y - u, Au\rangle \geq 0 \quad \forall y \in C \\ &\Leftrightarrow \langle u - y, u - \lambda Au - u\rangle \geq 0 \quad \forall y \in C \\ &\Leftrightarrow u = P_C\,(u - \lambda Au) \\ &\Leftrightarrow u = P_C\,(u - \lambda u + \lambda Tu) \\ &\Leftrightarrow \langle u - \lambda u + \lambda Tu - u, u - y\rangle \geq 0 \quad \forall y \in C \\ &\Leftrightarrow \langle u - Tu, u - y\rangle \leq 0 \quad \forall y \in C \\ &\Leftrightarrow u = Tu.\end{aligned}$$

By Theorem 3.1 we obtain the desired result. □

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