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NOTE ON GENERALIZED C-CONTRACTIONS IN PROBABILISTIC METRIC SPACES

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Abstract. Two results from a recent paper of this journal are improved.
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1. Introduction

The following class of probabilistic contractions, intensively studied in the fixed point theory in probabilistic metric spaces (see e.g. [1], [2]), has been introduced by T. L. Hicks.

Definition 1.1. ([4]) If (S, F) is a probabilistic metric space, a mapping $f : S \to S$ is called *C-contraction* (or *Hicks C-contraction*) if there exists $k \in (0, 1)$ such that the following implication holds for every $p, q \in S$ and $t > 0$:

$$(H) \quad F_{pq}(t) > 1 - t \Rightarrow F_{fpfq}(kt) > 1 - kt.$$

In [6] Radu showed that every C-contraction in a Menger space (S, F, T) with $T \geq T_L$ is actually a Banach contraction in the metric space (S, K), where $K(p, q) = \sup\{t \geq 0 : t \leq 1 - F_{pq}(t)\}$ (Hicks had proved this only for Menger spaces under the t-norm T_M) and he also proved the following more general result:

Theorem 1.2. ([6]) *Let* (S, F, T) *be a complete Menger space such that* $\sup_{a<1} T(a, a) = 1$*. Then every* C*-contraction* f *on* S *has a unique fixed point which is the limit of the sequence* $(f^n p)_{n\in N}$ *for every* $p \in S$*.*

There are many fixed point theorems concerning probabilistic contractions of Hicks type in Menger spaces, generalizing Radu's result. The aim of this note is to improve two recently appeared in this journal such theorems, belonging to Radu.

2. Preliminaries

In this section we briefly recall some facts from probabilistic metric spaces theory that we will use in the sequel. For more details we refer the reader to the books [2] and [10].

A *triangular norm* T (shorter *t-norm*) is a binary operation on the unit interval $[0, 1]$, which is associative, commutative, nondecreasing at both places and $T(x, 1) = x$ for every $x \in [0, 1]$. Basic examples are the t-norms T_L (*Łukasiewicz* t-norm), T_P and T_M,defined by $T_L(a, b) = \max\{a + b - 1, 0\}$, $T_P(a, b) = ab$ and $T_M(a, b) = \min\{a, b\}$.

Any continuous, strictly decreasing mapping $f : [0, 1] \to [0, \infty)$ with $f(1) = 0$ determines a continuous t-norm T_f by

$$T_f(a, b) := f^{(-1)}(f(a) + f(b)),\ \forall a, b \in [0, 1]$$

where $f^{(-1)} : [0, \infty) \to [0, 1]$, $f^{(-1)}(x) = f^{-1}(x, f(0))$ is the pseudo-inverse of f.

A *distance distribution function* is any mapping $F : [0, \infty) \to [0, 1]$ which is nondecreasing, left continuous on $(0, \infty)$ and $F(0) = 0$. The class of all distribution functions is denoted by Δ_+. D_+ is the subset of Δ_+ containing all functions F which also satisfy the condition $\lim_{t\to\infty} F(t) = 1$.

If X is a nonempty set, a mapping F from $X \times X$ to Δ_+ is called *a probabilistic distance on* X and $F(x, y)$ is denoted by F_{xy}.

Definition 2.1. If S is a nonempty set and F is a probabilistic distance on S with the properties:

$(PM0):\quad F_{pq} = \varepsilon_0 \Leftrightarrow p = q$

$(PM1):\quad F_{pq}=F_{qp}\ \ \forall p,q\in S$

then the pair (S,F) is called *a probabilistic semimetric space (*shortly PSM-*space).* A PSM-space that satisfies a certain kind of "triangle inequality" is called *a probabilistic metric space.*

A *Menger space* is a triple (S,F,T) where (S,F) is a PSM space, T is a t-norm and the following triangle inequality holds:

$$(PM2_M)\qquad F_{pr}(x+y)\geq T(F_{pq}(x),F_{qr}(y))\ \ \forall p,q,r\in S,\ \ \forall x,y\geq 0..$$

Definition 2.2. *([5]) A PSM*-space (S,F) satisfying the triangle inequality

$$(PM2_H)\ \ \forall\varepsilon>0\ \exists\delta>0: F_{pq}(\delta)>1-\delta,\ F_{qr}(\delta)>1-\delta\Rightarrow F_{pr}(\varepsilon)>1-\varepsilon$$

is called an *H-space.*

Note that every Menger space (S,F,T) with T satisfying $\sup\limits_{a<1}T(a,a)=1$ is an H-space.

If (X,F) is an H space, then the family

$$\{U_{\varepsilon,\lambda}\}_{\varepsilon>0,\lambda\in(0,1)},\text{ where } U_{\varepsilon,\lambda}=\{(x,y)\in X\times X:\ F_{xy}(\varepsilon)>1-\lambda\}$$

is a base for a metrizable uniformity on X, called *the F-uniformity* and denoted by $\mathcal{U}_F$. The F-uniformity is also generated by the family $\{V_\delta\}_{\delta>0}$, where $V_\delta:=U_{\delta,\delta}$. $\mathcal{U}_F$ naturally determines a topology $\mathcal{T}_F$ on X, named *the F-topology:* It has as neighborhoods base at x the family $\{U_x(\varepsilon,\lambda)\}_{\varepsilon>0,\lambda\in(0,1)}$, where $U_x(\varepsilon,\lambda)=\{y\in X: F_{xy}(\varepsilon)>1-\lambda\}$ or the family $\{U_x(\varepsilon)\}_{\varepsilon>0}$, where $U_x(\varepsilon)=\{y\in X: F_{xy}(\varepsilon)>1-\varepsilon\}$.

In what follows the probabilistic metric spaces are endowed with the F-uniformity.

3. Main results

First we recall some results from the quoted paper [9].

In the next definitions $\mathcal{M}$ is the family (firstly considered in [7]) of all functions $m:[0,\infty)\to[0,\infty)$ such that

$$\begin{aligned} i) &: \ m(t+s)\geq m(t)+m(s);\\ ii) &: \ m(t)=0\Leftrightarrow t=0;\\ iii) &: \ m \text{ is continuous}\end{aligned}$$

and Φ is the family of all functions $\phi : [0, \infty) \to [0, \infty)$ such that

$$i) \ : \ \phi \text{ is strictly increasing;}$$
$$ii) \ : \ \phi \text{ is right continuous;}$$
$$iii) \ : \ \lim_{n\to\infty} \phi^n(t) = 0 \ \forall t > 0.$$

Definition 3.1. ([9]) A PSM space (X, F) is called a *PM-space of type* $\mathcal{M}$ if the following triangle inequality holds:

$$(PM2^{\mu}) \ \ F_{pq}(t) > 1 - \mu(t), \ F_{qr}(s) > 1 - \mu(s) \Rightarrow F_{pr}(t+s) > 1 - \mu(t+s),$$

where $\mu \in \mathcal{M}$ is given.

Definition 3.2. ([9]) Let (S, F) be a probabilistic semimetric space. A mapping $A : S \to S$ is called a *φ-μ C-contraction* if, for some $\varphi \in \Phi$ and $\mu \in \mathcal{M}$,

$$p, q \in S, \ t > 0, \ F_{pq}(t)) > 1 - \mu(t) \Rightarrow F_{ApAq}(\mu(\varphi(t)) > 1 - \mu(\varphi(t)).$$

Theorem 3.3. ([9, Theorem 2.1.12]) *Let (S, F) be a complete Menger space of type $\mathcal{M}$, for which $(PM2^{\mu})$ holds. Then every φ-μ C-contraction has a unique fixed point which can be obtained by the successive approximations.*

Theorem 3.4. ([9, Theorem 2.1.15]) *Let (S, F, T_f) be a complete Menger space. Then every mapping $A : S \to S$ which satisfies, for some $\mu \in M$ and $\varphi \in \Phi$, the condition:*

$$p, q \in S, \ t > 0, \ f \circ F_{pq}(t) < \mu(t) \Rightarrow f \circ F_{ApAq}(\varphi(t)) < \mu(\varphi(t))$$

has a unique fixed point which is the limit of successive approximations.

In Theorem 3.6 below, $\mathcal{M}_1$ is the family of all functions $m : [0, \infty) \to [0, \infty)$ such that

$$\mu 1) \ : \ \mu(t) = 0 \ \Leftrightarrow t = 0;$$
$$\mu 2) \ : \ \lim_{t\to 0} \mu(t) = 0;$$
$$\mu 3) \ : \ \exists t > 0 : \mu(t) > 1$$

and Φ_1 is the family of all functions $\varphi : [0, \infty) \to [0, \infty)$ such that

$$\varphi 1) \ : \ \varphi(t) = 0 \Leftrightarrow \ t = 0;$$
$$\varphi 2) \ : \ \lim_{n\to\infty} \varphi^n(t_0) = 0 \ \text{ for some } t_0 \text{ with } \mu(t_0) > 1.$$

Notice that $\mathcal{M}_1$ includes $\mathcal{M}$, for each $m \in \mathcal{M}$ is an increasing, continuous bijection from $[0,\infty)$ to $[0,\infty)$.

Proposition 3.5. *Every PM-space of type $\mathcal{M}$ is an H-space.*

Proof. Suppose (X,F) is a PM-space of type $\mathcal{M}$. Let $\varepsilon > 0$ be given and $s > 0$ be such that $s < \varepsilon$ and $\mu(s) < \varepsilon$. Choosing $\delta = \min\{s/2, \mu(s/2)\}$ we have $\delta > 0$ and $F_{pq}(\delta) > 1-\delta,\ F_{qr}(\delta) > 1-\delta \Longrightarrow F_{pq}(s/2) > 1-\mu(s/2)$, $F_{qr}(s/2) > 1-\mu(s/2) \Longrightarrow F_{pr}(s) > 1-\mu(s) \Longrightarrow F_{pr}(\varepsilon) > 1-\varepsilon$.

As we have already noted, any Menger space with $\sup_{a<1} T(a,a) = 1$ is also an H-space. Particularly, a Menger space (S,F,T_f) is an H space.

The following theorem improves Theorem 3.3. and Theorem 3.4.

Theorem 3.6. *Let (S,F) be a complete H space and $A : S \to S$ be a mapping with the property*

$$p,q \in S,\ t > 0,\ f \circ F_{pq}(t) < \mu(t) \Rightarrow f \circ F_{ApAq}(\varphi(t)) < \mu(\varphi(t)),$$

where $f : R \to R$ a continuous decreasing bijection with $f(1) = 0$, and $\mu \in \mathcal{M}_1, \varphi \in \Phi_1$. Then A has a unique fixed point, which is the limit of successive approximations.

Proof. Let $f^{-1} : R \to R$ be the inverse of f . We rewrite the contractivity condition as

$$p,q \in S,\ t > 0,\ F_{pq}(t) > f^{-1} \circ \mu(t) \Rightarrow F_{ApAq}(\varphi(t)) > f^{-1} \circ \mu(\varphi(t)).$$

As in the quoted paper of Radu [6], we will proceed in four steps:

Step 1: A is (uniformly) continuous

Let $\varepsilon > 0$ $\lambda \in (0,1)$ be given. We can choose $t > 0$ such that $\varphi(t) < \varepsilon$ and $\mu \circ \varphi(t) < f(1-\lambda)$ (such a t is $\varphi^m(t_0)$ for some m).

Let $\varepsilon_1 = t$ and $\lambda_1 = 1 - f^{-1}(\mu(t)$. Then $F_{pq}(\varepsilon_1) > 1-\lambda_1$ implies $F_{ApAq}(\varepsilon) > F_{ApAq}(\varphi(t)) > f^{-1} \circ \mu(\varphi(t)) > 1-\lambda$.

Step 2: $(x_n)_{n\in N}$ $x_n = A^n x$ $(A^0 x = x)$ is a Cauchy sequence for every $x \in X$.

Fix a t_0 such that $\mu(t_0) > f(0)$ and $\lim_{n\to\infty} \varphi^n(t_0) = 0$. We will show that, for every $n \in N$, the following relation (R) holds:

$$(R):\ F_{x_n x_{n+m}}(\varphi^n(t_0)) > f^{-1} \circ \mu(\varphi^n(t_0)) \quad \forall m \in N.$$

(R) is true for $n = 0$.

Next, $F_{x_n x_{n+m}}(\varphi^n(t_0)) > f^{-1} \circ \mu(\varphi^n(t_0))$ implies $F_{Ax_n Ax_{n+m}}(\varphi^{n+1}(t_0)) > f^{-1} \circ \mu(\varphi^{n+1}(t_0))$ that is,

$$\begin{aligned} F_{x_n x_{n+m}}(\varphi^n(t_0)) &> f^{-1} \circ \mu(\varphi^n(t_0)) \Longrightarrow \\ F_{x_{n+1} x_{n+1+m}}(\varphi^{n+1}(t_0)) &> f^{-1} \circ \mu(\varphi^{n+1}(t_0)). \end{aligned}$$

This completes the proof of (R).

Let now $\varepsilon > 0$ be given and $n_0 \in N$ be such that $f^{-1} \circ \mu(\varphi^n(t_0)) > 1 - \varepsilon$ and $\varphi^n(t_0)) < \varepsilon \; \forall n \geq n_0$. Then, for all $n \geq n_0$ and $m \in N$, $F_{x_n x_{n+m}}(\varepsilon) > 1 - \varepsilon$, which shows that (x_n) is a Cauchy sequence.

Step 3: A has at most one fixed point

If x, y in S are such that $Ax = x$, $Ay = y$ then, as above, $F_{xy}(\varphi^n(t_0))) > f^{-1} \circ \mu(\varphi^n(t_0)) \; \forall n \in N$. On taking $n \to \infty$ one obtains $F_{xy}(0+) = 1$, which implies $x = y$.

Step 4: A has a (unique) fixed point

Since S is complete, there exists $p \in S$ such that $\lim_{n\to\infty} x_n = p$. From the continuity of A it follows that $A(A^n x) \to Ap$, that is, $Ap = p$.

Corollary 3.7. *Let (S, F) be a complete H space and $A : S \to S$ be a mapping with the property that*

$$p, q \in S,\; t > 0,\; F_{pq}(t)) > 1 - \mu(t) \Rightarrow F_{ApAq}(\mu(\varphi(t)) > 1 - \mu(\varphi(t)),$$

where $\mu, \varphi : [0, \infty) \to [0, \infty)$ have the properties: $\mu1) - \mu3)$, $\varphi1) - \varphi2)$. Then A has a unique fixed point, which is the limit of successive approximations.

The proof follows from Theorem 3.6, with $f(t) = 1 - t$.

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