Fixed Point Theory, Volume 6, No. 2, 2005, 279-284
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.htm

ON EXISTENCE OF INTEGRAL CURVES OF CONTINUOUS RIGHT-INVARIANT VECTOR FIELDS ON GROUPS OF DIFFEOMORPHISMS

A. Y. GLIKLIKH

Mathematics Faculty
Voronezh State University
Universitetskaja pl. 1
394006, Voronezh Russia
E-mail: agliklikh@gmail.com

Abstract. We prove existence of integral curves for continuous right-invariant vector fields. The main technical fact is the construction of extension of the field onto the tubular neighborhood in enveloping space such that it is locally Lipschitz continuous.
Key Words and Phrases: Groups of Sobolev diffeomorphisms, continuous right-invariant vector fields, integral curves, local Lipschitz continuity
2000 Mathematics Subject Classification: 58D05, 58D25, 34A35.

We investigate the differential equations on groups of diffeomorphisms of compact n-dimensional manifolds. The interest to this themes is connected with the fact that those groups are natural configuration spaces for systems of hydrodynamics in the framework of the approach by Arnold-Ebin-Marsden and the differential equations there describe the flows of fluids (see [1, 2]). The groups of diffeomorphisms have a property that has no analogy in finite-dimensional case. A vector X in the tangent space at the unit (i.e., the identity map) to the group is a vector field on the finite dimensional manifold. For $s > \frac{1}{2}n + 1$ the right-invariant vector field, generated by X on the group of Sobolev H^s diffeomorphisms is C^k smooth if and only if the field X belongs to Sobolev class H^{s+k}. In particular, if X is H^s, the corresponding right-invariant vector field is only continuous. It is well known that in infinite-dimensional

The research is partially supported by Grant 05-01-00100 from RFBF.

case a differential equation with continuous right-hand side may not have solutions (by this reason in [2] for getting the existence of integral curves for right-invariant vector fields, the field X is assumed belonging at least to H^{s+1} so that the right-invariant field is at least C^1). The main result of this note is that nevertheless the integral curves of continuous right invariant vector field do exist.

For preliminary information and notations we refer the reader to [2].

Let M be a compact n-dimensional Riemannian manifold without boundary. Let M be isometrically imbedded into R^k with k large enough, and $D^s(M)$ be the group of Sobolev H^s diffeomorphisms of M, $s > \frac{n}{2} + 1$. Let μ be the Riemannian volume form on M. Denote by V^{s-1} the set of volume forms on M that are H^{s-1} smooth, define the same orientation as μ, and such that $\int\limits_M \mu(dm) = \int\limits_M \nu(dm)$ for any $\nu \in V^{s-1}$. Define the map $\Psi_\mu : D^s(M) \to V^{s-1}$ as $\Psi_\mu(g) = (g^{-1})^*\mu$ where $(g^{-1})^* = T^*(g^{-1})\mu$ is the pull back of μ with respect to the map g^{-1}. It is well known, that two volume forms differ one from another by a multiplier that is a scalar function on M. So $\Psi_\mu(g) = \rho(g) \circ \mu$, where $\rho(g) : M \to (0, \infty)$ is a function of class H^{s-1}.

Define the standard scalar product on $H^s(M, R^k)$ as

$$(u, v)^s = \int\limits_M \sum_{|\alpha| \le s} (D^\alpha u(x), D^\alpha v(x))dx,$$

where α is the multi index.

Since the manifold M is isometrically imbedded into R^k, the diffeomorphisms group $D^s(M) \subset H^s(M, M)$ is isometrically imbedded into $H^s(M, R^k)$ and so we have a contraction of the scalar product of the space $H^s(M, R^k)$ as a Riemannian metric on the manifold $D^s(M)$.

From the definition of ρ it follows that for $g \in D^s(M)$ and $\bar{X}_g = X \circ g, X \in T_eD^s$, $\bar{Y}_g = Y \circ g, Y \in T_eD^s$ the relation $\int\limits_M \langle \bar{X}_g, \bar{Y}_g \rangle_{g(m)} \mu(dm) = \int\limits_M \frac{1}{\rho^2(g(m))} \langle X, Y \rangle_e \mu(dm)$ holds.

Define the distance between to points on $D^s(M)$ in a standard way as infinum of the lengths of curves connecting the points on the manifold $D^s(M)$. Denote this distance function by $dist$.

For any curve $X(t) \in TD^s(M), t \in [a, b]$ define its length as

$$\int_a^b \sqrt{(\frac{d}{dt}\pi X(t), \frac{d}{dt}\pi X(t))^s_{\pi X(t)}}dt+ \tag{1}$$

$$+\|TR_{X(b)^{-1}}X(b) - TR_{X(a)^{-1}}X(a)\|,$$

where R_g is the right shift.

Then for any $X^1, X^2 \in TD^s(M)$ we define the distance $d(X^1, X^2)$ as infinum of the lengths of curves, connecting the points, where the length is defined by the previous formula.

It is obvious that the distance d between the vectors in $TD^s(M)$ is the sum of the distance between their projections onto $D^s(M)$ and of the norm of the difference between the right shifts of those vectors into the unit point (identical map). Thus, for any $X^1, X^2 \in TD^s(M)$ we have:

$$d(X^1, X^2) = dist(\pi X^1, \pi X^2) + \|TR_{\pi X^{2-1}}X^1 - TR_{\pi X^{1-1}}\pi X^1\|. \tag{2}$$

Since the manifold M is isometrically imbedded into R^k, it has a tubular neighborhood $V \subset R^k$. Let $R : V \to M$ be the retraction of this neighborhood onto M. Then R generates the retraction of the tubular neighborhood $U = R^{-1}(D^s(M)) \subset H^s(M, V)$ of $D^s(M)$ onto $D^s(M)$. Thus the manifold $D^s(M)$ is imbedded into $H^s(M, R^k)$ as a neighborhood retract.

Introduce the map $\mathbf{R} : D^s \times T_eD^s \to TD^s$ by the formula $\mathbf{R}(\eta, X) = TR_\eta X$, $\eta \in D^s$, $X \in T_eD^s$. It is obvious that this map is a homeomorphism.

Recall that the manifold $D^s(M)$ is imbedded into $H^s(M, R^k)$. Denote the imbedding by i.

Theorem 1. *For any vector $X \in TD^s(M)$ and for any constant $C > 0$ there exist constants $A > 0$ $K > 0$ and a neighborhood V of X in $TD^s_\mu(M)$ such that for any Y, Z from V the following inequality $\|TiY - TiZ\| \leq (1 + A + (C + \|X\|)K)d(Y, Z)$ holds.*

Proof. Let $X \in TD^s$ and let $\pi X = \xi \in D^s$. Consider the derivative of i at this point. This derivative is a linear operator $d_\xi i : T_\xi D^s(M) \to H^s(M, R^k)$ and since i is a C^∞ map, this derivative is continuous in ξ. It is obvious that $TiX = i(\xi) + d_\xi i(X)$.

Take a constant $C > 0$ and choose some neighborhood $V_1(X) \subset T_\xi D^s(M)$ of X such that for any $Y \in V_1(X)$ we have $\|X\| \leq \|Y\| + C$. Since the operator $d_\eta iTR_{\eta\circ\xi^{-1}} : T_\xi D^s(M) \to H^s(M, R^k)$ is continuous in η there is a constant $K > 0$ and a neighborhood $V_2(\xi) \subset D^s(M)$ of the point ξ such that for any $\eta, \theta \in V_2(\xi)$ the norm of the operator $d_\eta iTR_{\eta\circ\theta^{-1}} - d_\theta i = d_\eta iTR_{\eta\circ\theta^{-1}} - d_\theta iTR_{\theta\circ\theta^{-1}}$ is less or equal to $Kdist(\eta, \theta)$. Indeed, if it is not the case, there exist sequences η_i, θ_i converging to ξ for which the stated norm is converging to the infinity while by its continuity it should converge to zero.

Since the operator $d_\eta i$ is continuous we can choose a constant $A > \|d_\xi i\|$ and then find a neighborhood $V_3(\xi)$ of ξ such that $\|d_\eta i\| \le A$ when $\eta \in V_3(\xi)$.

Consider the set

$$V = \mathbf{R}(V_2(\xi) \cap V_3(\xi), V_1(X)) \subset TD^s(M).$$

Since the imbedding is isometric, from the definitions of distances we get $\|i(\eta) - i(\theta)\| \le dist(\eta, \theta)$. Consider $\eta, \theta \in V$ such that $\pi^{-1}\eta = Y$ and $\pi^{-1}\theta = Z$. Then

$$\|TiY - TiZ\| \le \|i(\eta) - i(\theta)\| + \|d_\eta iY - d_\theta iZ\| =$$

$$\|i(\eta) - i(\theta)\| + \|d_\eta iY - d_\eta iTR_{\eta\circ\theta^{-1}}Z + d_\eta iTR_{\eta\circ\theta^{-1}}Z - d_\theta iZ\| \le$$

$$dist(\eta, \theta) + \|d_\eta i(Y - TR_{\eta\circ\theta^{-1}}Z(\theta))\| + \|(d_\eta iTR_{\eta\circ\theta^{-1}} - d_\theta i)Z(\theta))\| \le$$

$$d(Y, Z) + \|d_\eta i\|\|Y - TR_{\eta\circ\theta^{-1}}Z(\theta)\| + (C + \|X\|)Kdist(\eta, \theta) \le$$

$$d(Y, Z) + Ad(Y, Z) + (C + \|X\|)Kd(Y, Z) =$$

$$(1 + A + (C + \|X\|)K)d(Y, Z).$$

□

Let $\tilde{X}$ be a right invariant vector field on $D^s(M)$ generated by the vector $X \in T_eD^s(M)$ that belongs to Sobelev class H^s (i.e., H^s-vector field on M, see [2]). Under these conditions $\tilde{X}$ is a continuous vector field on $D^s(M)$. Denote by $\hat{X}$ the vector field on the tubular neighborhood U that is the extension of $Ti\tilde{X}$ by the formula $\hat{X}(x) = Ti\tilde{X}(R(x))$, $x \in U$, where R is retraction (see above).

Take an arbitrary point $\xi \in D^s(M)$. By Theorem 1.4 from [3], we know, that there exists a ball $B \subset U$, centred at ξ, with a small enough radius such that the retraction R is Lipshitz continuous with the constant $q = 2$ with respect to the distance $dist$ on $D^s(M)$ and the norm of $H^s(M, R^k)$ on the ball. Without loss of generality we can suppose that $B \bigcap U \subset V$, where V is the neighborhood from Theorem 1.

Theorem 2. *The vector field $\hat{X}$ is Lipshitz continuous on the ball B with the constant* $2[1 + A + (C + \|X(\xi)\|)K)]$.

Proof. Immediately from the construction of the distances d and $dist$ and from the definition of the right invariant field it follows that $d(\tilde{X}(\xi), \tilde{X}(\eta)) = dist(\xi, \eta)$ for any $\xi, \eta \in D^s(M)$. Let $x, y \in B$. Then from Theorem 1, Theorem 1.4 of [3] and from the construction of the field $\hat{X}$ we see that

$$\|\hat{X}(x) - \hat{X}(y)\| = \|Ti\tilde{X}(Rx) - Ti\tilde{X}(Ry)\| \leq$$
$$(1 + A + (C + \|X\|)K)d(\tilde{X}(Rx), \tilde{X}(Ry) =$$
$$(1 + A + (C + \|X\|)K)dist(Rx, Ry) <$$
$$2(1 + A + (C + \|X\|)K)\|x - y\|.$$

□

Theorem 3. *Let a vector $X \in T_e D^s_\mu(M)$ belong to the class H^s (this means that it is an H^s vector field $X(m)$ on M). Let $\tilde{X}(t)$ be the corresponding right invariant vector field on $D^s(M)$. Then the Cauchy problem*

$$\dot{\xi}(t) = \tilde{X}, \qquad \xi(0) = \xi_0 \in D^s(M)$$

has a solution for any $t \in [0, \infty)$ and this solution is unique.

Proof. Let $\hat{X}(t)$ be the extension of the vector field $Ti\tilde{X}(t)$ to the neighborhood U as it is described above. From Theorem 2 it follows that the vector field $\bar{X}$ is Lipshitz continuous on the ball B constructed for the point $\xi(0)$. Hence the solution of the Cauchy problem $\dot{\xi}(t) = \hat{X}$, $\xi(0) = \xi_0 \in D^s(M)$ exists and it is unique. This solution exists in B, in other words it exists when $t \in [0, \varepsilon)$ for some $\varepsilon > 0$. Since at the points of $D^s(M)$ the filed $\hat{X}$ is tangent to $D^s(M)$, this solution belongs to $D^s(M)$.

It is easy to see (see [2]), that the integral curve of the right invariant field $\tilde{X}(t)$ is the flow of the finite-dimensional vector field $X(m)$ on M. Since the field $X(m) \in H^s(M, TM)$ is C^1 smooth on M for $s \geq \frac{n}{2} + 1$ this flow is unique in the space C^1, and therefore in $D^s(M)$.

Since M is a compact manifold without boundary, this flow exists for any $t \in [0, \infty)$. □

By an obvious modification of the proof, analogous to that of Theorem 2.4 in [3], we obtain the following statement.

Corollary. *Let a vector $X(t)$ in $T_e D^s_\mu(M)$ be measurable in t and for each t belong to the class H^s (i.e. it is H^s-vector field on M). Let also $\|X(t)\|$ be* *integrable on each finite interval. Let* $\tilde{X}(t)$ *be the corresponding right invariant vector field on* $D^s(M)$*. Then the Cauchy problem*

$$\dot{\xi}(t) = \tilde{X}(t), \qquad \xi(0) = \xi_0 \in D^s(M)$$

has a solution for all $t \in [0, \infty)$ *and this solution is unique.*

References

[1] V. Arnol'd, *Sur la géométrie différentielle des groupes de Lie de dimension infinie et ses applications a l'hydrodynamique des fluides parfaits*, Ann. Inst. Fourier, **16**, **1**(1966), 319-361.

[2] D.G. Ebin, J. Marsden, *Groups of diffeomorphisms and the motion of an incompressible fluid*, Ann. Math., **92**(1970), 102-163.

[3] Yu. E. Gliklikh, V.V. Obukhovskiĭ, *Tubular Neighbouhoods of Hilbert manifolds and differential equations of Carathéodory type on groups of diffeomorphisms*, New Approaches in Nonlinear Analysis.- Palm Harbor, Fl. Hadronic Press, 1999, 109-123.

Received May 13, 2005.