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A FUNCTIONAL INTEGRO-DIFFERENTIAL INCLUSION IN BANACH ALGEBRAS

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Abstract. In this paper an existence theorem for a functional differential inclusion in Banach algebras is proved via a fixed point principle of Leray-Schauder type under generalized Lipschitz and Carathéodory conditions. The existence of extremal solutions is also obtained under certain monotonicity conditions.
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1. Statement of Problem

Given a closed and bounded interval $J = [0, 1]$ in $\mathbb{R}$, the set of real numbers, consider the functional differential inclusion (in short FDI)

$$\left.\begin{aligned} \frac{d}{dt}\left[\frac{x(t)}{f(t, x(t), x(\theta(t))}\right] &\in G\Big(t, x(\mu(t)), \int_0^{\sigma(t)} k(t, s, x(\eta(s))\, ds\Big) \text{ a.e. } t \in J \\ x(0) &= x_0 \in \mathbb{R} \end{aligned}\right\} \tag{1.1}$$

where $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} - \{0\}$ is continuous, $G : J \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}_{cp,cv}(\mathbb{R})$ and $\theta, \mu, \sigma, \eta J \to J$ are continuous with $\theta(0) = 0$.

By a solution to FDI (1.1) we mean a function $x \in C(J, \mathbb{R})$ that satisfies

$$\frac{d}{dt}\left[\frac{x(t)}{f(t, x(t), x(\theta(t)))}\right] = v(t),\ t \in J$$

for some $v \in L^1(J, \mathbb{R})$ satisfying $v(t) \in G\Big(t, x(\mu(t)), \int_0^{\sigma(t)} k(t,s,x(\eta(s))\,ds\Big)$ a.e. $t \in J$ and $x(0) = x_0$.

The FDI (1.1) is new to the theory of differential inclusions and the special cases of it have been discussed in the literature extensively. For example, the special case of FDI (1.1) in the form

$$\left.\begin{array}{c} \dfrac{d}{dt}\left[\dfrac{x(t)}{f(t,x(t))}\right] \in G(t,x(t)) \quad \text{a.e.} \quad t \in J \\ x(0) = x_0 \in \mathbb{R} \end{array}\right\} \tag{1.2}$$

has been studied in Dhage [9] for the existence of solutions. Again if $f(t,x,y) = 1$, then the FDI (1.1) reduces to FDI

$$\left.\begin{array}{c} x'(t) \in G\Big(t, x(\mu(t)), \int_0^{\sigma(t)} k(t,s,x(\eta(s))\,ds\Big) \quad \text{a.e.} \quad t \in J \\ x(0) = x_0 \in \mathbb{R}. \end{array}\right\} \tag{1.3}$$

There is a considerable work available in the literature for some special cases of FDI (1.3). See Aubin and Cellina [3], Deimling [6] and Hu and Papageorgiou [15] etc. Similarly in the special case when $G(t,x,y) = \{g(t,x,y)\}$ we obtain the differential equation

$$\left.\begin{array}{c} \dfrac{d}{dt}\left[\dfrac{x(t)}{f(t,x(t),x(\theta(t)))}\right] = g\Big(t, x(\mu(t)), \int_0^{\sigma(t)} k(t,s,x(\eta(s))\,ds\Big) \text{ a.e.} t \in J \\ x(0) = x_0 \in \mathbb{R}. \end{array}\right\} \tag{1.4}$$

The functional differential equation (1.4) is again new to the literature and a special case of differential equation (1.4) with $f(t,x,y) = f(t,x)$ has been studied recently in Dhage [7] and Dhage and O'Regan [11] for the existence of solutions. Therefore, it of interest to discuss the the FDI (1.4) for various aspects of its solution under suitable conditions. In this section we shall prove the existence of the solutions of FDI (1.4) in the space $C(J, \mathbb{R}$ of continuous real-valued functions on J, under the mixed generalized Lipschitz and Carathéodory conditions.

2. Preliminaries

Before stating the main fixed point theorems, we give some useful definitions and preliminaries that will be used in the sequel. Let X be a Banach space and let $\mathcal{P}(X)$ denote the class of all subsets of X. Denote

$$\mathcal{P}_p(X) = \{A \subset X \mid A \text{ is non-empty and has a property } p\}.$$

Thus $\mathcal{P}_{bd}(X), \mathcal{P}_{cl}(X), \mathcal{P}_{cv}(X)$, $\mathcal{P}_{cp}(X)$, $\mathcal{P}_{cl,bd}(X)$, $\mathcal{P}_{cp,cv}(X)$ denote the classes of all bounded, closed, convex , compact , closed-bounded and compact-convex subsets of X respectively. Similarly $\mathcal{P}_{cl,cv,bd}(X)$ and $\mathcal{P}_{cp,cv}(X)$ denote the classes of closed, convex and bounded and compact, convex subsets of X respectively. A correspondence $T : X \to \mathcal{P}_p(X)$ is called a multi-valued operator or multi-valued mapping on X. A point $u \in X$ is called a fixed point of T if $u \in Tu$. The multi-valued operator T is called lower semi-continuous (in short l.s.c.)if G is any open subset of X, then

$$T^{-1(w)}(G) = \{x \in X \mid Tx \cap G \neq \emptyset\}$$

is an open subset of X. Similarly the multi-valued operator T is called upper semi-continuous (in short u.s.c.) if the set

$$T^{-1}(G) = \{x \in X \mid Tx \subset G\}$$

is open in X for every open set G in X. Finally T is called continuous if it is lower as well as upper semi-continuous on X. A multi-valued map $T : X \to \mathcal{P}_{cp}(X)$ is called **compact** if $\overline{T(S)}$ is a compact subset of X for any $S \subset X$. T is called **totally bounded** if for any bounded subset S of X, $T(S) = \bigcup_{x \in S} Tx$ is a totally bounded subset of X. It is clear that every compact multi-valued operator is totally bounded, but the converse may not be true. However the two notions are equivalent on a bounded subset of X. Finally T is called **completely continuous** if it is upper semi-continuous and totally bounded on X.

For any $A, B \in \mathcal{P}_p(X)$, let us denote

$$\begin{aligned} A \pm B &= \{a \pm b \mid a \in A, b \in B\} \\ A \cdot B &= \{ab \mid a \in A, b \in B\} \\ \lambda A &= \{\lambda a \mid a \in A\}. \end{aligned}$$

for $\lambda \in \mathbb{R}$. Similarly denote

$$|A| = \{|a| \mid a \in A\}$$

and

$$\|A\|_{\mathcal{P}} = \sup\{|a| \mid a \in A\}.$$

Let $A, B \in \mathcal{P}_{cl,bd}(X)$ and let $a \in A$. Then by

$$D(a, B) = \inf\{\|a - b\| \mid b \in B\}$$

and

$$\rho(A, B) = \sup\{D(a, B) \mid a \in A\}.$$

The function $H : \mathcal{P}_{cl,bd}(X) \times \mathcal{P}_{cl,bd}(X) \to \mathbb{R}^+$ defined by

$$H(A, B) = \max\{\rho(A, B),\ \rho(B, A)\}$$

is metric and is called the Hausdorff metric on X. It is clear that

$$H(0, C) = \|C\|_{\mathcal{P}} = \sup\{\|c\| \mid c \in C\}$$

for any $C \in \mathcal{P}_{cl,bd}(X)$.

Definition 2.1. *Let $T : X \to \mathcal{P}_{cl,bd}(X)$ be a multi-valued operator. Then T is called a multi-valued Lipschitz if there exists a constant $k > 0$ such that for each $x, y \in X$ we have*

$$H(Tx, Ty) \le k\|x - y\|.$$

The constant k is called a Lipschitz constant of T. Similarly a single-valued mapping $T : X \to X$ is called Lipschitz if there exists a constant $\alpha > 0$ such that

$$\|Tx - Ty\| \le \alpha\|x - y\|$$

for all $x, y \in X$. The constant α is called a Lipschitz constant of T.

The Kuratowskii measure α and the Hausdorff measure β of noncompactness of a bounded set S in a Banach space are the nonnegative real numbers defined by

$$\alpha(S) = \inf\Big\{r > 0 : S \subset \bigcup_{i=1}^{n} S_i,\ \text{and } \operatorname{diam}(S_i) \le r,\ \forall\, i\Big\} \tag{2.1}$$

and

$$\beta(S) = \inf\Big\{r > 0 \mid S \subset \bigcup_{i=1}^{n} \mathcal{B}_i(x_i, r),\ \text{for some } x_i \in X\Big\}, \tag{2.2}$$

where $\mathcal{B}_i(x_i, r) = \{x \in X \mid d(x, x_i) < r\}$.

It is known that the Kuratowskii measure α of noncompactness has all the properties of Hausdorff measure β of noncompactness (β_1) through (β_6) mentioned below.

(β_1) $\beta(A) = 0 \Longleftrightarrow \overline{A}$ is compact.

(β_2) $\beta(A) = \beta(\overline{A}) = \beta(\overline{co}A)$, where $\overline{A}$ and $\overline{co}$ denote respectively the closure and the closed convex hull of A.

(β_3) $A \subset B \Rightarrow \beta(A) \leq \beta(B)$.

(β_4) $\beta(A \cup B) = \max\{\beta(A), \beta(B)\}$.

(β_5) $\beta(\lambda A) = |\lambda|\beta(A), \forall \lambda \in \mathbb{R}$.

(β_6) $\beta(A + B) \leq \beta(A) + \beta(B)$.

The details of Hausdorff measure of noncompactness and its properties appear in Akhmerov et. al. [1] and the references therein. The following results appear in Akhmerov et. al. [1].

Lemma 2.1. *Let α and β be respectively the Kuratowskii and Hausdorff measure of noncompactness in a Banach space, then for any bounded set S in a Banach space X we have*

$$\alpha(S) \leq 2\,\beta(S).$$

Lemma 2.2. *If $A : X \to X$ is a single-valued Lipschitz map with a Lipschitz constant k, then we have $\alpha(A(S)) \leq k\,\alpha(S)$ for any bounded subset S of X.*

Lemma 2.3. (Banas and Lecko [4]) *If $A, B \in \mathcal{P}_{bd}(X)$, then*

$$\beta(AB) \leq \|A\|_{\mathcal{P}}\,\beta(B) + \|B\|_{\mathcal{P}}\,\beta(A).$$

Definition 2.2. *A multi-valued mapping $T : X \to \mathcal{P}_{bd}(X)$ is called β-***condensing** *if for any $S \in \mathcal{P}_{bd}(X)$, we have that $\beta(T(S) < \beta(S)$ for $\beta(S) > 0$.*

The following extension of Leray-Schauder principle due to Martelli [17] is well-known in the literature.

Theorem 2.1. (Martelli [17]) *Let X be a Banach space and let $T : X \to P_{cp,cv}(X)$ be a upper semi-continuous and β-condensing multi-valued operator. Then either*

(i) *the operator inclusion $x \in Tx$ has a solution, or*

(ii) *the set* $\mathcal{E} = \{u \in X \mid \lambda u \in Tu,\ \lambda > 1\}$ *is unbounded.*

Theorem 2.2. *Let* X *be a Banach algebra and let* $A : X \to X$*, and* $B : X \to \mathcal{P}_{cl,cv}(X)$ *be two multi-valued operators satisfying*

(a) A *is single-valued Lipschitz with a Lipschitz constant* k*,*
(b) B *is compact and upper semi-continuous,*
(c) $AxBx$ *is a convex subset of* X *for each* $x \in X$*, and*
(d) $2Mk < 1$*, where* $M = \|B(X)\|_{\mathcal{P}}$*.*

Then either

(i) *the operator inclusion* $x \in AxBx$ *has a solution, or*
(ii) *the set* $\mathcal{E} = \{u \in X \mid \lambda u \in AuBu,\ \lambda > 1\}$ *is unbounded.*

Proof. Define a multi-valued mapping $T : X \to \mathcal{P}_p(X)$ by

$$Tx = Ax\,Bx,\ x \in X.$$

We show that T satisfies all the conditions of Theorem 2.1. First of all we show that T has convex and compact values on X. Let $z_1, z_2 \in Ax\,Bx$ be any two elements. Then there are points $u_1, u_2 \in Bx$ such that $z_1 = (Ax)\,u_1$ and $z_2 = (Ax)\,u_2$. Now for any $\lambda \in [0, 1]$, one has

$$\begin{aligned}\lambda z_1 + (1-\lambda)z_2 &= \lambda(Ax\,u_1) + (1-\lambda)(Ax\,u_2)\\ &= (Ax)\,(\lambda u_1) + Ax\big[(1-\lambda)\ u_2\big]\\ &= (Ax)\big[(\lambda u_1) + (1-\lambda)u_2\big]\\ &= (Ax)\,z.\end{aligned}$$

Since Bx is a convex set, one has $z = \lambda u_1 + (1-\lambda)u_2 \in Bx$, and hence T has convex values on X. Again in view of Lemma 2.3 we obtain

$$\beta(Tx) = \beta(Ax\,Bx) \le \|Ax\|\beta(Bx) + \|Bx\|_{\mathcal{P}}\beta(Ax) = 0,$$

and therefore, T has compact values on X. As a result, T defines a multi-valued mapping $T : X \to \mathcal{P}_{cp,cv}(X)$.

Next we show that T is β-condensing on X. Let S be a bounded set in X. Then $T(S) \subset A(S)B(S)$. Now by Lemma 2.3,

$$\begin{aligned}\beta(T(S)) &\leq \|A(S)\|_{\mathcal{P}}\,\beta(B(S)) + \|B(S)\|_{\mathcal{P}}\,\beta(A(S)) \\ &\leq M\alpha(A(S)) \\ &\leq Mk\,\alpha(S) \\ &\leq 2Mk\beta(S) \\ &< \beta(S),\end{aligned}$$

provided that $\beta(S) > 0$. This shows that T is β-condensing on X. Now an application of Theorem 2.1 yields that either

(i) the operator inclusion $x \in AxBx$ has a solution, or
(ii) the set $\mathcal{E} = \{u \in X \mid \lambda u \in AuBu,\ \lambda > 1\}$ is unbounded.

This completes the proof. □

3. Existence Results

In this section we prove the existence theorems for the differential inclusions in Banach algebras by the applications of the abstract results of the previous section under generalized Lipschitz and Carathéodory conditions.

Define a norm $\|\cdot\|$ in $C(J, \mathbb{R})$ by

$$\|x\| = \sup_{t\in J} |x(t)|.$$

Again define a multiplication "$\cdot$" by

$$(x.y)(t) = x(t)y(t) \quad \forall \quad t \in J.$$

Then $C(J, \mathbb{R})$ is a Banach algebra with the above norm and multiplication in it.

We need the following definitions in the sequel.

Definition 3.1. *A multi-valued map $F : J \to \mathcal{P}_p(\mathbb{R})$ is said to be measurable if for any $y \in X$, the function $t \to d(y, F(t)) = \inf\{|y - x| : x \in F(t)\}$ is measurable.*

Definition 3.2. *A measurable multi-valued function $F : J \to \mathcal{P}_{cp}(\mathbb{R})$ is said to be integrably bounded if there exists a function $h \in L^1(J, \mathbb{R})$ such that $\|v\| \leq h(t)$ a.e. $t \in J$ for all $v \in F(t)$.*

Remark 3.1. *It is known that if $F : J \to \mathcal{P}_{cl,bd}(\mathbb{R})$ is an integrably bounded multi-valued function , then the set S^1_F of all Lebesgue integrable selections of F is closed and non-empty. See Deimling* [6].

Definition 3.3. *Let E be a Banach space. A multi-valued function let $\beta : J \times E \times E \to \mathcal{P}_{bd,cl}(E)$ is called Carathéodory if*

i) *$t \mapsto \beta(t, x, y)$ is measurable for each $x, y \in E$, and*
ii) *$(x, y) \mapsto \beta(t, x, y)$ is an upper semi-continuous almost everywhere for $t \in J$.*

Definition 3.4. *A Carathéodory multi-function $\beta(t, x, y)$ is called L^1-Carathéodory if for each real number $r > 0$ there exists a function $h_r \in L^1(J, \mathbb{R})$ such that*

$$\|\beta(t, x, y)\|_{\mathcal{P}} = \sup\{\|u\| : u \in \beta(t, x, y)\} \le h_r(t) \quad a.e.\ t \in J$$

for all $x, y \in E$ with $\|x| \le r$, $\|y\| \le r$.

Definition 3.5. *A Carathéodory multi-function $\beta(t, x, y)$ is called L^1_X-Carathéodory if there exists a function $h \in L^1(J, \mathbb{R})$ such that*

$$\|\beta(t, x, y)\|_{\mathcal{P}} \le h(t) \quad a.e. \quad t \in J$$

for all $x, y \in E$.

Denote

$$S^1_\beta(x) = \left\{ v \in L^1(J, E) \mid v(t) \in \beta\Big(t, x(\mu(t)), \int_0^{\sigma(t)} k(t, s, x(\eta(s))\, ds\Big) \text{ a.e. } t \in J \right\}$$

Then we have the following lemmas due to Lasota and Opial [16].

Lemma 3.1. *Let E be a Banach space. If $dim(E) < \infty$ and $\beta : J \times E \times E \to \mathcal{P}_{bd,cl}(E)$ is L^1-Carathéodory, then $S^1_\beta(x) \neq \emptyset$ for each $x \in E$.*

Lemma 3.2. *Let E be a Banach space, β a L^1-Carathéodory multi-map with $S^1_\beta \neq \emptyset$ and let $\mathcal{L} : L^1(J, E) \to C(J, E)$ be a continuous linear mapping. Then the operator*

$$\mathcal{L} \circ S^1_\beta : C(J, E) \to \mathcal{P}_{bd,cl}(C(J, E))$$

is a closed graph operator on $C(J, E) \times C(J, E)$.

We need the following definition in sequel. See Dhage [7].

Definition 3.6. *A function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is called a $\mathcal{D}$-**function** if it satisfies*

(i) ψ is continuous,
(ii) ψ is nondecreasing , and
(iii) ψ is scalarly submultiplicative, that is, $\psi(\lambda r) \leq \lambda\psi(r)$ for all $\lambda > 0$ and $r \in \mathbb{R}^+$.

The class of all $\mathcal{D}$-functions on $\mathbb{R}^+$ is denoted by Ψ. There do exist $\mathcal{D}$-functions on $\mathbb{R}$. Actually the function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ defined by $\psi(r) = \ell r,\ \ell > 0$ satisfies all the conditions $(i) - (iii)$ mentioned above and hence a $\mathcal{D}$-function on $\mathbb{R}^+$. We consider the following hypotheses in the sequel.

(H_1) The function $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \setminus \{0\}$ is continuous and there exists a bounded function $\ell : J \to \mathbb{R}$ with bound $\|\ell\|$ satisfying

$$|f(t, x_1, x_2) - f(t, y_1, y_2)| \leq \ell(t) \max\{|x_1 - y_1|, |x_2 - y_2|\} \quad \text{a.e.} \quad t \in J$$

for all $x, y \in \mathbb{R}$.

(H_2) The multi-function $G : J \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}_{cp,cv}(\mathbb{R})$ is L^1_X-Carathéodory.

(H$_3$) The function $k : J \times J \times \mathbb{R} \to \mathbb{R}$ is continuous and there exists a function $\alpha \in L^1(J, \mathbb{R}^+)$, such that

$$\Big|\int_0^t k(t, s, y)\, ds\Big| \leq \alpha(t)|y| \text{ a.e. } t, s \in J \text{ and } y \in \mathbb{R}.$$

(H$_4$) There exists a function $p \in L^1(J, \mathbb{R}^+)$ and a $\mathcal{D}$-function $\psi \in \Psi$ such that

$$\|G(t, x, y)\|_{\mathcal{P}} \leq \gamma(t)\psi(|x| + |y|) \quad \text{a.e. } t \in J$$

for each $x, y \in \mathbb{R}$.

Theorem 3.1. *Assume that the hypotheses (H$_1$)-(H$_4$) hold. Further if $\theta(t) \leq t$, $\mu(t) \leq t$, $\sigma(t) \leq t$ and $\eta(t) \leq t$ for all $t \in J$ and*

$$\int_{C_1}^{\infty} \frac{ds}{\psi(s)} > C_2 \int_0^1 \gamma(s)\big[1 + \alpha(s)\big]\, ds \tag{3.1}$$

where

$$C_1 = \frac{F\Big|\dfrac{x_0}{f(0, x_0, x_0}\Big|}{1 - \|\ell\| \left(\Big|\dfrac{x_0}{f(0, x_0, x_0)}\Big| + \|h\|_{L^1}\right)},$$

$$C_2 = \frac{F}{1 - \|\ell\| \left(\left| \dfrac{x_0}{f(0, x_0, x_0)} \right| + \|h\|_{L^1} \right)}$$

$$F = \sup\{|f(t, 0, 0) \mid t \in J\}$$

and

$$2\,\|\ell\| \left(\left| \frac{x_0}{f(0, x_0, x_0)} \right| + \|h\|_{L^1} \right) < 1,$$

then the FDI (1.1) has a solution on J.

Proof. Let $X = C(J, \mathbb{R})$ and consider the two multi-valued mappings A and B on X defined by

$$Ax(t) = f(t, x(t), x(\theta(t))) \tag{3.2}$$

and

$$Bx(t) = \left\{ u \in X \mid u(t) = \frac{x_0}{f(0, x_0, x_0} + \int_0^t v(s)\, ds, \quad v \in S_G^1(x) \right\} \tag{3.3}$$

for all $t \in J$.

Then the FDI (1.1) is equivalent to the operator inclusion

$$x(t) \in Ax(t)\, Bx(t),\ t \in J. \tag{3.4}$$

We shall show that the multi-valued operators A and B satisfy all the conditions of Corollary 2.2. Clearly the operator B is well defined since $S_G^1(x) \neq \emptyset$ for each $x \in X$.

Step I : We first show that the operators A and B define the multi-valued operators $A, B : X \to \mathcal{P}_{cp,cv}(X)$. The case of A is obvious since it is a single-valued operator on S. We only prove the claim for the operator B. Let $\{u_n\}$ be a sequence in Bx converging to a point u. Then there is a sequence $\{v_n\} \subset S_G^1(x)$ such that

$$u_n(t) = \frac{x_0}{f(0, x_0, x_0)} + \int_0^t v_n(s)\, ds$$

and $v_n \to v$. Since $G(t, x)$ is closed for each $(t, x) \in J \times \mathbb{R}$, we have $v \in S_G^1(x)$. As a result

$$u(t) = \frac{x_0}{f(0, x_0, x_0)} + \int_0^t v(s)\, ds \in Bx(t),\ \forall\, t \in J.$$

Hence B has closed values on X. Again let $u_1, u_2 \in Ax$. Then the are $v_1, v_2 \in S^1_G(x)$ such that

$$u_1(t) = \frac{x_0}{f(0,x_0,x_0)} + \int_0^t v_1(s)\,ds,\ t \in J,$$

and

$$u_2(t) = \frac{x_0}{f(0,x_0,x_0)} + \int_0^t v_2(s)\,ds,\ t \in J.$$

Now for any $\gamma \in [0,1]$,

$$\begin{aligned}
\lambda u_1(t) + (1-\lambda)u_2(t) &= \lambda\left(\frac{x_0}{f(0,x_0,x_0} + \int_0^t v_1(s)\,ds\right) \\
&\quad +(1-\lambda)\left(\frac{x_0}{f(0,x_0,x_0)} + \int_0^t v_2(s)\,ds\right) \\
&= \frac{x_0}{f(0,x_0,x_0)} + \int_0^t [\lambda v_1(s) + (1-\lambda)v_2(s)]\,ds \\
&= \frac{x_0}{f(0,x_0,x_0)} + \int_0^t v(s)\,ds
\end{aligned}$$

where $v(t) = \lambda v_1(t) + (1-\lambda)v_2(s) \in G\Big(t, x(\mu(t)), \displaystyle\int_0^{\sigma(t)} k(t,s,x(\eta(s))\,ds\Big)$ for all $t \in J$. Hence $\lambda u_1 + (1-\lambda)u_2 \in Bx$ and consequently Bx is convex for each $x \in X$. As a result A defines a multi-valued operator $B : X \to \mathcal{P}_{bd,cl,cv}(X)$. Again let $t, \tau \in J$. Then for any $u \in Bx$ we have

$$\begin{aligned}
|u(t) - u(\tau)| &\le \left|\int_0^t v(s)\,ds - \int_0^\tau v(s)\,ds\right| \\
&\le \left|\int_\tau^t |v(s)|\,ds\right| \\
&\le |p(t) - p(\tau)|
\end{aligned}$$

where $p(t) = \displaystyle\int_0^t h(s)\,ds$.

Since p is continuous on compact interval J, it is uniformly continuous. Hence Bx is compact by Arzela-Ascoli theorem. Thus we have $B : X \to \mathcal{P}_{cp,cv}(X)$. Hence $A, B : X \to \mathcal{P}_{cp,cv}(X)$.

Step II : To show A a contraction on X, let $x, y \in X$. Then

$$\begin{aligned}\|Ax - Ay\| &= \sup_{t\in J} |Ax(t) - Ay(t)| \\ &= \sup_{t\in J} |f(t, x(t), x(\theta(t))) - f(t, y(t), y(\theta(t)))| \\ &\leq \sup_{t\in J} \ell(t) \max\{|x(t) - y(t)|, |x(\theta(t)) - x(\theta(t))|\} \\ &\leq \|\ell\| \|x - y\|,\end{aligned}$$

showing that A is a Lipschitz on X with a Lipschitz constant $\|\ell\|$.

Step III : Next we show that B is compact and upper semi-continuous on X. First we prove that $B(X)$ is totally bounded on X. To do this, it is enough to prove that $B(X)$ is a uniformly bounded and equi-continuous set in X. To see this, let $u \in B(X)$ be arbitrary. Then there is a $v \in S_G^1(x)$ such that

$$u(t) = \frac{x_0}{f(0, x_0, x_0)} + \int_0^t v(s)\, ds.$$

for some $x \in X$. Hence

$$\begin{aligned}|u(t)| &\leq \left|\frac{x_0}{f(0, x_0, x_0)}\right| + \int_0^t |v(s)|\, ds \\ &\leq \left|\frac{x_0}{f(0, x_0, x_0)}\right| + \int_0^t \left\|G\Big(s, x(\mu(s)), \int_0^{\sigma(s)} k(s, \tau, x(\eta(\tau))\, d\tau\Big)\right\|_{\mathcal{P}} ds \\ &\leq \left|\frac{x_0}{f(0, x_0, x_0)}\right| + \int_0^t h(s) ds \\ &= \left|\frac{x_0}{f(0, x_0, x_0)}\right| + \|h\|_{L^1}\end{aligned}$$

for all $x \in S$ and so $B(X)$ is a uniformly bounded set in X. Again as in Step I , it is proved that

$$|u(t) - u(\tau)| \leq |p(t) - p(\tau)|$$

where $p(t) = \displaystyle\int_0^t h(s) ds$.

Notice that p is a continuous function on J, so it is uniformly continuous on J. As a result we have that

$$|u(t) - u(\tau)| \to 0 \text{ as } t \to \tau.$$

This shows that $B(X)$ is a equi-continuous set in X. Next we show that B is a upper semi-continuous multi-valued mapping on X. Let $\{x_n\}$ be a sequence in

X such that $x_n \to x_*$. Let $\{y_n\}$ be a sequence such that $y_n \in Bx_n$ and $y_n \to y_*$. We shall show that $y_* \in Bx_*$. Since $y_n \in Bx_n$, there exists a $v_n \in S^1_G(x_n)$ such that

$$y_n(t) = \frac{x_0}{f(0,x_0,x_0)} + \int_0^t v_n(s)\,ds, \;\; t \in J.$$

We must prove that there is a $v_* \in S^1_G(x_*)$ such that

$$y_*(t) = \frac{x_0}{f(0,x_0,x_0)} + \int_0^t v_*(s)\,ds, \;\; t \in J.$$

Consider the continuous linear operator $\mathcal{K} : L^1(J, \mathbb{R}) \to C(J,E)$ defined by

$$\mathcal{K}y(t) = \int_0^t v(s)\,ds, \;\; t \in J.$$

Now we have

$$\left\| \left(y_n - \frac{x_0}{f(0,x_0,x_0)} \right) - \left(y_* - \frac{x_0}{f(0,x_0,x_0)} \right) \right\| \to 0 \text{ as } n \to 0.$$

From lemma 3.2, it follows that $\mathcal{K} \circ S^1_G$ is a closed graph operator. Also from the definition of K we have

$$y_n(t) - \frac{x_0}{f(0,x_0,x_0)} \in \mathcal{K} \circ S^1_G(x_n).$$

Since $y_n \to y_*$, there is a point $v_* \in S^1_G(x_*)$ such that

$$y_*(t) = \frac{x_0}{f(0,x_0,x_0)} + \int_0^t v_*(s)ds, \;\; t \in J.$$

This shows that B is a upper semi-continuous operator on X. Thus B is an upper semi-continuous and compact operator on X.

Step IV : Here we show that $AxBx$ is a convex subset of X for each $x \in X$. Let $x \in X$ be arbitrary and let $w, y \in X$. Then there are $u, v \in S^1_G(x)$ such that

$$w(t) = [f(t, x(t), x(\theta(t)))] \left(\frac{x_0}{f(0,x_0,x_0} + \int_0^t u(s)\,ds \right)$$

and

$$y(t) = [f(t, x(t), x(\theta(t)))] \left(\frac{x_0}{f(0,x_0,x_0} + \int_0^t v(s)\,ds \right).$$

Now for any $\lambda \in [0,1]$,

$$\lambda y(t) + (1-\lambda)w(t) = \lambda[f(t, x(t), x(\theta(t)))] \left(\frac{x_0}{f(0,x_0,x_0)} + \int_0^t v(s)\,ds \right)$$

$$+(1-\lambda)[f(t,x(t),x(\theta(t)))]\left(\frac{x_0}{f(0,x_0,x_0)}+\int_0^t v(s)\,ds\right)$$

$$=[f(t,x(t),x(\theta(t)))]\left(\lambda\frac{x_0}{f(0,x_0,x_0)}+\int_0^t \lambda v(s)\,ds\right)$$

$$+[f(t,x(t),x(\theta(t)))]\left((1-\lambda)\frac{x_0}{f(0,x_0,x_0)}+\int_0^t (1-\lambda)v(s)\,ds\right)$$

$$=[f(t,x(t),x(\theta(t)))]\left(\frac{x_0}{f(0,x_0,x_0)}+\int_0^t [\lambda u(s)+(1-\lambda)v(s)]\,ds\right).$$

Since

$$G\Big(t,x(\mu(t)),\int_0^{\sigma(t)} k(t,s,x(\eta(s))\,ds\Big)$$

is convex, we have

$$z(t)=\lambda y(t)+(1-\lambda)w(t)\in G\Big(t,x(\mu(t)),\int_0^{\sigma(t)} k(t,s,x(\eta(s))\,ds\Big)$$

for all $t\in J$ and so $z\in S^1_G(x)$. As a result $\lambda y+(1-\lambda)w\in AxBx$. Hence $AxBx$ is a convex subset of X.

Step V : Finally from condition (1.1) it follows that

$$2Mk=2\|\ell\|\left(\left|\frac{x_0}{f(0,x_0,x_0)}\right|+\|h\|_{L^1}\right)<1.$$

Thus all the conditions of Theorem 2.1 are satisfied and hence a direct application of it yields that either the conclusion (i) or the conclusion (ii) holds. We show that the conclusion (ii) is not possible. Let $u\in\mathcal{E}$ be arbitrary. Then we have, for any $\lambda>1$,

$$\lambda u(t)\in Au(t)Bu(t)$$

$$=[f(t,u(t),u(\theta(t))]\left(\frac{x_0}{f(0,x_0,x_0)}+\int_0^t G\Big(s,u(\mu(s)),\int_0^{\sigma(s)} k(s,\tau,x(\eta(\tau)))\,d\tau\Big)\,ds\right),$$

for all $t\in J$ and for some real number $\lambda>1$. Therefore

$$\lambda u(t)\in\Big[f(t,u(t),u(\theta(t)))\Big]\Big(\frac{x_0}{f(0,x_0,x_0)}+\int_0^t G\Big(s,u(\mu(s)),\int_0^{\sigma(s)} k(s,\tau,x(\eta(\tau))\,d\tau\Big)\,ds\Big)$$

or

$$u(t)=\lambda^{-1}\Big[f(t,u(t),u(\theta(t)))\Big]\Big(\frac{x_0}{f(0,x_0,x_0)}+\int_0^t v(s)\,ds\Big)$$

for some $v\in S^1_G(u)$.

Now

$$|u(t)| = \Big|\lambda^{-1}\Big[f(t,u(t),u(\theta(t)))\Big]\Big|\; \Big|\Big(\frac{x_0}{f(0,x_0,x_0)} + \int_0^t v(s)\,ds\Big)\Big|$$

$$\leq \Big|\Big[f(t,u(t),u(\theta(t)))\Big]\Big|\; \Big(\Big|\frac{x_0}{f(0,x_0,x_0)}\Big| + \int_0^t |v(s)|\,ds\Big)$$

$$\leq \Big[|f(t,u(t),u(\theta(t))) - f(t,0,0)| + |f(t,0,0)|\Big]$$

$$\times\Big(\Big|\frac{x_0}{f(0,x_0,x_0)}\Big| + \int_0^t \Big\|G\Big(s,u(\mu(s)),\int_0^{\sigma(s)} k(s,\tau,u(\eta(\tau))\,d\tau\Big)\Big\|_{\mathcal{P}}\,ds\Big)$$

$$\leq |\ell(t)|\max\{|u(t)|,u(\theta(t))\}\Bigg(\Big|\frac{x_0}{f(0,x_0,x_0)}\Big|$$

$$+\int_0^t \Big\|G\Big(s,u(\mu(s)),\int_0^{\sigma(s)} k(s,\tau,u(\eta(\tau))\,d\tau\Big)\Big\|_{\mathcal{P}}\,ds\Bigg)$$

$$+F\Big(\Big|\frac{x_0}{f(0,x_0,x_0)}\Big| + \int_0^t \Big\|G\Big(s,u(\mu(s)),\int_0^{\sigma(s)} k(s,\tau,u(\eta(\tau))\,d\tau\Big)\Big\|_{\mathcal{P}}\,ds\Big)$$

$$\leq \|\ell\|\max\{|u(t)|,u(\theta(t))\}\left(\Big|\frac{x_0}{f(0,x_0,x_0)}\Big| + \|h\|_{L^1}\right)$$

$$+F\left(\Big|\frac{x_0}{f(0,x_0,x_0)}\Big| + \int_0^t \gamma(s)\psi\Big(|u(\mu(s))| + \alpha(s)|u(\eta(s))|\,ds\Big)\,ds\right) \tag{3.5}$$

Put $m(t) = \max_{s\in[0,t]}|u(s)|$, $t \in J$. Then, $\max\{|u(t)|, |u(\theta(t))|\} \leq m(t)$ for all $t \in J$. Since u is continuous, there is a $t^* \in [0,t]$ such that $m(t) = |u(t^*)|$.

Then from above inequality (3.5), we obtain

$$
\begin{aligned}
m(t) &= |u(t^*)| \\
&\leq \|\ell\| \max\{|u(t^*)|, |u(\theta(t^*)|)\} \left(\Big| \frac{x_0}{f(0,x_0,x_0)} \Big| + \|h\|_{L^1} \right) \\
&\quad + F \left(\Big| \frac{x_0}{f(0,x_0,x_0)} \Big| + \int_0^{t^*} \gamma(s) \psi \Big(|u(\mu(s))| + \alpha(s) |u(\eta(s))| \, ds \Big) \, ds \right) \\
&\leq \|\ell\| m(t) \left(\Big| \frac{x_0}{f(0,x_0,x_0)} \Big| + \|h\|_{L^1} \right) \\
&\quad + F \left(\Big| \frac{x_0}{f(0,x_0,x_0)} \Big| + \int_0^{t} \gamma(s) \psi \Big(m(\mu(s)) + \alpha(s)\, m(\eta(s)) \, ds \Big) \, ds \right) \\
&\leq C_1 + C_2 \int_0^t \gamma(s) \psi \Big(m(s) + \alpha(s) m(s) \, ds \Big) \, ds \\
&\leq C_1 + C_2 \int_0^t \gamma(s) \big[1 + \alpha(s) \big] \psi(m(s)) \, ds \qquad (3.6)
\end{aligned}
$$

where

$$
C_1 = \frac{F \Big| \frac{x_0}{f(0,x_0,x_0)} \Big|}{1 - \|\ell\| \left(\Big| \frac{x_0}{f(0,x_0,x_0)} \Big| + \|h\|_{L^1} \right)},
$$

and

$$
C_2 = \frac{F}{1 - \|\ell\| \left(\Big| \frac{x_0}{f(0,x_0,x_0)} \Big| + \|h\|_{L^1} \right)}.
$$

Let

$$
w(t) = C_1 + C_2 \int_0^t \gamma(s) \big[1 + \alpha(s) \big] \psi(m(s)) \, ds.
$$

Then $m(t) \leq w(t)$ and a direct differentiation of $w(t)$ yields

$$
\left.
\begin{array}{rcl}
w'(t) & \leq & C_2 \gamma(s) \big[1 + \alpha(s) \big] \psi(w(s)) \\
w(0) & = & C_1,
\end{array}
\right\} \qquad (3.7)
$$

that is,

$$
\int_0^t \frac{w'(s)}{\psi(w(s))} \, ds \leq C_2 \int_0^t \gamma(s) \big[1 + \alpha(s) \big] \, ds.
$$

A change of variables in the above integral gives that

$$
\int_{C_1}^{w(t)} \frac{ds}{\psi(s)} \leq C_2 \int_0^1 \gamma(s) \big[1 + \alpha(s) \big] \, ds < \int_{C_1}^{\infty} \frac{ds}{\psi(s)}.
$$

Now an application of mean value theorem yields that there is a constant $M > 0$ such that $w(t) \leq M$ for all $t \in J$. This further implies that

$$|u(t)| \leq w(t) \leq M.$$

for all $t \in J$. Thus the conclusion (ii) of Corollary 2.2 does not hold. Therefore the operator inclusion $x \in AxBx$ and consequently the FDI (1.1) has a solution on J. This completes the proof. □

4. Existence of Extremal Solutions

A non-empty closed set K in a Banach algebra X is called a **cone** if (i) $K + K \subseteq K$, (ii) $\lambda K \subseteq K$ for $\lambda \in \mathbb{R}, \lambda \geq 0$ and (iii) $\{-K\} \cap K = 0$, where 0 is the zero element of X. A cone K is called to be **positive** if (iv) $K \circ K \subseteq K$, where "$\circ$" is a multiplication composition in X. We introduce an order relation $\leq$ in K as follows. Let $x, y \in X$. Then $x \leq y$ if and only if $y - x \in K$. A cone K is called to be **normal** if the norm $\|\cdot\|$ is monotone increasing on K. It is known that if the cone K is normal in X, then every order-bounded set in X is norm-bounded. The details of cones and their properties appear in Guo and Lakshmikantham [13].

We equip the space $C(J, \mathbb{R})$ with the order relation $\leq$ with the help of the cone defined by

$$K = \{x \in C(J, \mathbb{R}) : x(t) \geq 0 \text{ for all } t \in J\}. \tag{4.1}$$

It is well known that the cone K is positive and normal in $C(J, \mathbb{R})$. As a result of positivity of the cone K in $C(J, \mathbb{R})$ we have:

Lemma 4.1. (Dhage [10]). Let $u_1, u_2, v_1, v_2 \in K$ be such that $u_1 \leq v_1$ and $u_2 \leq v_2$. Then $u_1 u_2 \leq v_1 v_2$.

For any $a, b \in X = C(J, \mathbb{R})$ with $a \leq b$, the order interval $[a, b]$ is a set in X defined by

$$[a, b] = \{x \in X : a \leq x \leq b\}.$$

We use the following fixed point theorem of Dhage [10] for proving the existence of extremal solutions of the FDI(1.1) under certain monotonicity conditions.

Definition 4.1. *A multi-valued operator $T : X \to \mathcal{P}_p(X)$ is called strictly monotone increasing if for any $x, y \in X$ with $x < y$ implies $Tx \leq Ty$.*

Theorem 4.1. (Dhage [10]). *Let* $[a,b]$ *be an order interval in a Banach algebra* X*. Suppose that* $A : [a,b] \to K$ *and* $B : [a,b] \to \mathcal{P}_{cl}(K)$ *are two operators such that*

(a) A *is Lipschitz with a Lipschitz constant* α,
(b) B *is completely continuous,*
(c) $AxBx \subset [a,b]$ *for each* $x \in [a,b]$, *and*
(d) A *and* B *are strictly monotone increasing on* $[a,b]$.

Further if the cone K *is positive and normal, then the operator equation* $x \in Ax\,Bx$ *has a greatest and a least positive solution in* $[a,b]$, *whenever* $2\alpha M < 1$, *where* $M = \| \cup B([a,b])\| := \sup\{\|Bx\| : x \in [a,b]\}$.

We need the following definitions in the sequel.

Definition 4.2. *A function* $a \in C(J, \mathbb{R})$ *is called a lower solution of the FDI (1.1) on* J *if for each* $v \in S^1_G(a)$, *we have*

$$\frac{d}{dt}\left[\frac{a(t)}{f(t, a(t), a(\theta(t)))}\right] \leq v(t), \quad a.e\ t \in J, \quad and \quad a(0) \leq x_0.$$

Again a function $b \in C(J, \mathbb{R})$ *is called an upper solution of the BVP (1.1) on* J *if for each* $v \in S^1_G(b)$, *we have*

$$\frac{d}{dt}\left[\frac{b(t)}{f(t, b(t), b(\theta(t)))}\right] \geq v(t), \quad a.e\ t \in J, \quad and \quad b(0) \geq x_0.$$

Definition 4.3. *A solution* x_M *of the FDI (1.1) is said to be maximal if for any other solution* x *to FDI(1.1) one has* $x(t) \leq x_M(t)$, *for all* $t \in J$. *Again a solution* x_m *of the FDI (1.1) is said to be minimal if* $x_m(t) \leq x(t)$, *for all* $t \in J$, *where* x *is any solution of the FDI (1.1) on* J.

Definition 4.4. *A Carathéodory multi-valued function* $\beta : I \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}_p(\mathbb{R})$ *is called* L^1*-Carathéodory if for each* $r > 0$ *there is a function* $h_r \in L^1(I, \mathbb{R})$ *such that* $\|\beta(t, x, y)\|_{\mathcal{P}} \leq h_r(t)$ *a.e.* $t \in I$ *for all* $x, y \in \mathbb{R}$ *with* $|x| \leq r, |y| \leq r$.

Definition 4.5. *A multi-valued function* $\beta(t, x, y)$ *is said to be nondecreasing in* x *if for all* $t \in J$ *and* $y \in \mathbb{R}$, *we have* $G(t, x_1, y) \leq G(t, x_2, y)$ *for all* $x_1, x_2 \in \mathbb{R}$ *for which* $x_1 < x_2$. *Similarly the monotonicity of* $\beta(t, x, y)$ *in the argument* y *is defined.*

We consider the following set of assumptions:

(B_0) $f : J \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+ - \{0\}$, $G : J \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}_{cl}(\mathbb{R}^+)$ and $x(0) \geq 0$.

(B_1) G is L^1-Carathéodory.

(B_2) The functions $f(t,x,y)$ and $G(t,x,y)$ are nondecreasing in x and y almost everywhere for $t \in I$.

(B_3) The FDI (1.1) has a lower solution a and an upper solution b on J with $a \leq b$.

Remark 4.1. *Assume that (B_1)-(B_3) hold. Define a function $c : J \to \mathbb{R}^+$ by*

$$c(t) = \Big\|G\Big(t, a(\mu(t)), \int_0^{\sigma(t)} k(t,s,a(\eta(s)))\,ds\Big)\Big\|_{\mathcal{P}} + \Big\|G\Big(t, b(\mu(t)), \int_0^{\sigma(t)} k(t,s,b(\eta(s)))\,ds\Big)\Big\|_{\mathcal{P}}, \tag{4.2}$$

for all $t \in I$. Then c is Lebesgue integrable and

$$\Big\|G\Big(t, x(\mu(t))), \int_0^{\sigma(t)} k(t,s,x(\eta(s)))\,ds\Big)\Big\|_{\mathcal{P}} \leq c(t), \quad a.e.\ t \in I,$$

for all $x \in [a,b]$.

Theorem 4.2. *Suppose that the assumptions (H_1)-(H_3) and (B_0)-(B_3) hold. Further if $\|\ell\|\Big(\Big|\dfrac{x_0}{f(0,x_0,x_0)}\Big| + \|c\|_{L^1}\Big) < 1/2$, and c is given in Remark 4.1, then FDI (1.1) has a minimal and a maximal positive solution on J.*

Proof. Now FDI (1.1) is equivalent to FII (3.1)-(3.2) on J. Let $X = C(J, \mathbb{R})$. Define two operators A and B on $[a,b]$ by (3.3) and (3.4) respectively. Then FIE (1.1) is transformed into an operator inclusion $x(t) \in Ax(t)Bx(t)$ in a Banach algebra X. Notice that (B_1) implies $A : [a,b] \to K$ and $B : [a,b] \to \mathcal{P}_{cl}(K)$. Since the cone K in X is normal, $[a,b]$ is a norm bounded set in X. Now it is shown, as in the proof of Theorem 3.1, that A is a Lipschitz with a Lipschitz constant $\|\ell\|$ and B is completely continuous operator on $[a,b]$. Again the hypothesis (B_2) implies that A and B are nondecreasing on $[a,b]$. To see this, let $x, y \in [a,b]$ be such that $x < y$. Then by (B_2),

$$Ax(t) = f(t, x(t), x(\theta(t))) \leq f(t, y(t), y(\theta(t))) = Ay(t)$$

for all $t \in J$. Similarly,

$$\begin{aligned} Bx(t) &= \frac{x_0}{f(0,x_0,x_0)} + \int_0^t G\Big(t, x(\mu(t)), \int_0^{\sigma(t)} k(t,s,x(\eta(s))\,ds\Big)\,ds \\ &\le \frac{x_0}{f(0,x_0,x_0)} + \int_0^t G\Big(t, x(\mu(t)), \int_0^{\sigma(t)} k(t,s,x(\eta(s))\,ds\Big)\,ds \\ &= By(t) \end{aligned}$$

for all $t \in I$.

So A and B are nondecreasing operators on $[a,b]$. Again Lemma 4.1 and hypothesis (B_3) implies that

$$a(t) \le$$

$$\le [f(t,a(t),a(\theta(t)))]\Big(\frac{x_0}{f(0,x_0,x_0)} + \int_0^t G\Big(t, a(\mu(t)), \int_0^{\sigma(t)} k(t,s,a(\eta(s))ds\Big)ds\Big)$$

$$\le [f(t,x(t),x(\theta(t)))]\Big(\frac{x_0}{f(0,x_0,x_0)} + \int_0^t G\Big(t, x(\mu(t)), \int_0^{\sigma(t)} k(t,s,x(\eta(s))ds\Big)ds\Big)$$

$$\le [f(t,b(t),b(\theta(t)))]\Big(\frac{x_0}{f(0,x_0,x_0)} + \int_0^t G\Big(t, b(\mu(t)), \int_0^{\sigma(t)} k(t,s,b(\eta(s))\,ds\Big)\,ds\Big)$$

$$\le b(t),$$

for all $t \in J$ and $x \in [a,b]$. As a result $a(t) \le Ax(t)\,Bx(t) \le b(t), \forall\, t \in J$ and $x \in [a,b]$. Hence $Ax\,Bx \in [a,b], \forall x \in [a,b]$.

Again

$$M = \|B([a,b])\| = \sup\{\|Bx\| : x \in [a,b]\}$$

$$\le \sup_{x\in[a,b]} \Bigg\{\Big|\frac{x_0}{f(0,x_0,x_0)}\Big| + \sup_{t\in J} \int_0^t \Big\|G\Big(t, x(\mu(t)), \int_0^{\sigma(t)} k(t,s,x(\eta(s))\,ds\Big)\Big\|_{\mathcal{P}}\,ds\Bigg\}$$

$$\le \Big|\frac{x_0}{f(0,x_0,x_0)}\Big| + \int_0^1 c(s)\,ds = \Big|\frac{x_0}{f(0,x_0,x_0)}\Big| + \|c\|_{L^1}.$$

Since $\alpha M \le \|\ell\|\Big(\Big|\frac{x_0}{f(0,x_0,x_0)}\Big| + \|c\|_{L^1}\Big) < 1/2$, we apply Theorem 4.1 to the operator inclusion $x \in AxBx$ to yield that the FDI (1.1) has a minimal and a maximal positive solution on J. This completes the proof. □

Remark 4.2. *Finally while concluding this paper, we mention that our existence results of this paper include the existence results of Dhage* [7, 8] *and improve the existence result proved in Dhage* [9] *for the FDI (1.2) in Banach algebras.*

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