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MEASURE OF WEAK COMPACTNESS AND FIXED POINT THEORY

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Abstract. In this paper, we study a class of Banach spaces, called Φ-spaces. In a natural way, we associate a measure of weak compactness in such spaces and prove an analogue of Sadovskii fixed point theorem for weakly sequentially continuous maps. A counter-example is given to justify our requirement. As an application, we establish an existence result for a Hammerstein equation in a Banach space.

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1. Introduction

The Kuratowski measure of noncompactness $\chi(C)$ of a bounded subset C of a metric space (X, d) is defined to be the infimum of the set of all $\varepsilon > 0$ with the following property:

C can be covered by finitely many sets, each of whose diameter is $\leq \varepsilon$.

Let X be a Banach space and let $\mathcal{B}(X)$ denote the collection of nonempty and bounded subsets of X. Measures of noncompactness appear in various

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contexts and are very useful tools in Nonlinear Analysis (see e.g., [2, 3, 4, 8] and the references therein). Measures of noncompactness allows us to obtain fixed point results. The first result in this direction was Sadovskii's fixed point theorem [11]. It states that if $M \in \mathcal{B}(X)$ is closed, convex and $T : M \to M$ is a condensing mapping (that is, T is continuous and $\chi(T(C)) < \chi(C)$ for all $C \in \mathcal{B}(X)$, then T has a fixed point.

Let $\Phi \in \mathscr{L}(X)$ be a linear continuous operator. The main purpose of this paper is to investigate what conditions will ensure that the function $\chi \circ \Phi \colon \mathcal{B}(X) \to [0, \infty)$ is a measure of weak compactness (in the sense that if $C \in \mathcal{B}(X)$ is weakly closed and $\chi(\Phi(C)) = 0$, then C is weakly compact) and then to derive from this fact an analogue of Sadovskii's theorem for weakly sequentially continuous maps. For the sake of convenience, X will be called a Φ-space if the existence of such a Φ is verified. Typical theorems for weakly sequentially continuous maps has been obtained in [1, 7, 9]. Nevertheless, the definition of our measure as well as its use in the study of fixed points for such maps seems to be new and original.

This paper is organized as follows: In Section 2, we first formulate the notion of Φ-spaces and give a sufficient condition for a Banach space to be a Φ-space. We will see that automatically reflexive spaces are Φ-spaces. However, there are nonreflexive Φ-spaces (see Example 2). In Section 3, we extend Sadovskii's fixed point theorem to the weak topology setting. More precisely, we prove that every map weakly sequentially continuous and Φ-condensing (see Definition 3.1) has a fixed point in nonempty closed, convex and bounded subsets of a Φ-space (see Theorem 3.1). Also, we will see that Φ-condensing is essential for the conclusion of our main result. Finally, in Section 4, we use our fixed point theory to establish an existence result for Hammerstein integral equation in a Banach space.

2. Preliminaries

Throughout this paper, X stands for a real Banach space with norm $\|\cdot\|$ and $\mathscr{L}(X)$ the space of all linear continuous operators on X. In what follows, we proceed to define the notion of a Φ-space. Let $\mathcal{F}$ be the family of all the bounded and weakly closed subsets of X. If I is the identity map on X, we shall denote by span{I} the vector space generated by I.

Definition 2.1. *Let $\Phi \in \mathscr{L}(X)$ be such that $\Phi \notin \operatorname{span}\{I\}$. We will say that X is a Φ-space if the following condition is satisfied*

$$\text{if } C \in \mathcal{F} \text{ and } \Phi(C) \text{ is relatively compact, then } C \text{ is weakly compact.} \qquad (\mathscr{C})$$

Example 2.1. *Every reflexive space is a Φ-space for any $\Phi \in \mathscr{L}(X)$ with $\phi \notin \operatorname{span}\{I\}$.*

In our next result, we give a sufficient condition for a Banach space to be a Φ-space.

Proposition 2.1. *Let X be a Banach space. Suppose that there is a $\Phi \in \mathscr{L}(X)$ and $c > 0$ such that*

$$\Phi \notin \operatorname{span}\{I\},\ \operatorname{Ker}(\Phi) = \{0\} \text{ and } \|\Phi^{-1}y\| \le c\|y\|, \qquad (\mathscr{H})$$

for all $y \in \Phi(X)$. Then X is a Φ-space.

Proof. Take an arbitrary set $C \in \mathcal{F}$ and suppose that $\Phi(C)$ is relatively compact. Let (C_i) be a decreasing sequence of nonempty closed, convex subsets of C. Since each C_i is closed and $\|\Phi^{-1}y\| \le c\|y\|$ for all $y \in \Phi(X)$, we have that each $\Phi(C_i)$ is closed too. Thus, $(\Phi(C_i))$ is a decreasing sequence of nonempty closed and convex subsets of $\overline{\Phi(C)}$. By the finite intersection property of compact sets, we conclude that $\cap_{i=1}^{\infty}\Phi(C_i) \neq \emptyset$. Now since $\Phi^{-1}(\Phi(C_i)) = C_i$, we get $\cap_{i=1}^{\infty} C_i \neq \emptyset$. Applying now the *Šmulian* nested interval principle as in [6, pp. 433], one has that C is weakly compact. □

Remark 2.1. From the open mapping theorem we know that if $\Phi \in \mathscr{L}(X)$ is one-to-one and $\Phi(X)$ is a Banach space then there exists $c > 0$ such that $\|\Phi^{-1}y\| \le c\|y\|$, for all $y \in \Phi(X)$.

Example 2.2. *This is an example of a Φ-space that is not reflexive. Let $C[0,1]$ be the space of the real-valued continuous functions and fix $m \in C[0,1]$ a increasing continuous function such that $m(t) \ge a$, for all $t \in [0,1]$ whith $0 < a < 1$. Let us define $\Phi \in \mathscr{L}(C[0,1])$ by $(\Phi x)(t) = m(t) \cdot x(t)$. Then $\Phi \notin \operatorname{span}\{I\}$ and $\operatorname{Ker}(\Phi) = \{0\}$. Moreover, if $(Sy)(t) = m^{-1}(t) \cdot y(t)$, for all $y \in C[0,1]$, then it follows that $\|Sy\|_\infty \le \|1/m\|_\infty \cdot \|y\|_\infty$, for all $y \in C[0,1]$ and $S_{|\Phi(C[0,1])} \equiv \Phi^{-1}$. By Proposition 2.1, $C[0,1]$ is a Φ-space.*

In fact, there are many other Φ-spaces. For instance, if X is a Banach space with a Schauder basis, can show that X is a Φ-space for some $\Phi \in \mathscr{L}(X)$.

We are now going to define a natural measure of weak compactness in a Φ-space. For convenience, we first summarize the main properties of χ which will be used here.

The function $\chi\colon \mathcal{B}(X) \to [0,\infty)$ enjoys the following properties:

(1) For any $C \in \mathcal{B}(X)$, $0 \leq \chi(C) \leq \delta(C)$ =diameter of C,
(2) $\chi(C) = 0$ iff C is relatively compact,
(3) if $C \subset D$, then $\chi(C) \leq \chi(D)$ for all $C, D \in \mathcal{B}(X)$,
(4) $\chi(C \cup D) = \max\{\chi(C), \chi(D)\}$, $\forall C, D \in \mathcal{B}(X)$,
(5) $\chi(\overline{C}) = \chi(C)$,
(6) $\chi(\overline{\text{co}}(C)) = \chi(\text{co(C)}) = \chi(\text{C})$.

The details of χ and their properties may be found in [12].

Definition 2.2. *Let X be a Φ-space. Define the measure of weak compactness on X as being the function $\chi_\Phi\colon \mathcal{B}(X) \to [0,\infty)$ given by*

$$\chi_\Phi(C) = \chi(\Phi(C)).$$

Remark 2.2. Clearly χ_Φ is well defined. Also, it follows from the definition of Φ-spaces and by (2) above that a set $C \in \mathcal{F}$ is weakly compact if $\chi_\Phi(C) = 0$.

3. Fixed Point Theory

Before going to the main result of this paper, we give a useful definition. In the proof of Sadovskii theorem the assumption $\chi(T(C)) < \chi(C)$ plays a crucial role.

This property suggests the following definition.

Definition 3.1. *Let M be a bounded subset of a Φ-space X. We will say that $T\colon M \to M$ is ws Φ-condensing (resp. w Φ-condensing) if T is weakly sequentially continuous (resp. weakly continuous) and Φ-condensing, i.e,*

$$\chi_\Phi(T(C)) < \chi_\Phi(C),$$

for all $C \in \mathcal{B}(M)$ such that $\chi_\Phi(C) > 0$, where $\mathcal{B}(M)$ denote the collection of all nonempty and bounded subsets of M.

Our main result is as follows.

Theorem 3.1. *Let X be a Φ-space and M a closed, convex and bounded subset of X. Suppose that $T\colon M \to M$ is a ws Φ-condensing mapping. Then T has a fixed point.*

Since every weakly continuous mapping is weakly sequentially continuous we immediately obtain the following corollary.

Corollary 1. *Assume M is as in Theorem 3.1. If T is a w Φ-condensing mapping from M into itself, then T has a fixed point.*

The proof of Theorem 3.1 is based in the following generalization of the Schauder-Tychonov fixed point principle which was obtained by Arino, Gautier and Penot [1].

Theorem 3.2. *Let M be a nonempty, convex and weakly compact subset of a Banach space X and let $T\colon M \to M$ be a weakly sequentially continuous operator. Then T has at least one fixed point in the set M.*

Proof of Theorem 3.1. We proceed as in the proof of the Sadovskii's theorem. Fix a point $x \in M$ and let Σ denote the system of all closed, convex subsets K of M for which $x \in K$ and $T(K) \subset K$. Now set

$$B = \cap_{K\in\Sigma} K, \quad \text{and} \quad C = \overline{\text{co}}\{T(B) \cup \{x\}\}.$$

Then $B = C$, $T(C) \subset C$ and $\Phi(C) \subset \overline{\text{co}}\{\Phi(T(B)) \cup \{\Phi(x)\}\}$. Using the properties (3),(4) and (6) of χ, one has

$$\chi(\Phi(C)) \leq \chi(\Phi(T(B))) = \chi(\Phi(T(C)).$$

Since T is Φ-condensing, we conclude that $\chi_\Phi(C) = 0$. Then definition 2.1 implies that C is weakly compact since $C \in \mathcal{F}$. Now Theorem 3.2 gives a fixed point $y \in C$ for T. This ends the proof. $\square$

Evidently, if X is a reflexive space the assumption $\chi_\Phi(T(C)) < \chi_\Phi(C)$ in the Theorem 3.1 is unnecessary. But, in a nonreflexive Φ-space this requirement may be essential as shown the next example.

Example 3.1. *Consider the Banach space* $X = \ell^1$*, of all scalar sequences* $x = (\alpha_n)_{n=1}^{\infty}$ *for which* $\Sigma_{n=1}^{\infty}|\alpha_n| < \infty$*, endowed with its usual norm. Let us define* $\Phi\colon \ell^1 \to \ell^1$ *by*

$$\Phi\big(\sum_{n\in\mathbb{N}} \alpha_n \mathrm{e}_n\big) = \sum_{n\in\mathbb{N}} \alpha_n \mathrm{e}_{n+1}.$$

Then $\Phi \in \mathscr{L}(\ell^1)$, $\mathrm{Ker}(\Phi) = \{0\}$ *and* $\Phi \notin \mathrm{span}\{I\}$*. Now, defining* $\Psi\colon \ell^1 \to \ell^1$ *by*

$$\Psi\big(\sum_{n\in\mathbb{N}} \alpha_n \mathrm{e}_n\big) = \sum_{n\geq 2} \alpha_n \mathrm{e}_{n-1},$$

we have $\Psi_{|\Phi(\ell^1)} \equiv \Phi^{-1}$ *and* $\|\Psi(\eta)\| \leq \|\eta\|$*, for all* $\eta \in \ell^1$*. Consequently, we get* $\|\Phi^{-1}y\| \leq \|y\|$ *for all* $y \in \Phi(\ell^1)$*. By Proposition 2.1,* ℓ^1 *is a* Φ*-space. Let now* M *be the set*

$$M = \{\sum_{n\in\mathbb{N}} \alpha_n \mathrm{e}_n\colon \quad \alpha_n \geq 0, \quad \sum_{n\in\mathbb{N}} \alpha_n = 1\}.$$

Next define the map $T\colon M \to M$ *by* $T \equiv \Phi_{|M}$*. Then,* T *is fixed point free on* M*. Moreover, one can check that* T *is an isometry on* M*. Also, it is easy to see that* T *is weakly sequentially continuous. However, from the fact that* Φ *is an isometry, we obtain*

$$\|\Phi(T(\eta)) - \Phi(T(\mu))\| = \|\Phi(\eta) - \Phi(\mu)\|,$$

for all $\eta, \mu \in M$*, which implies that* $\chi_\Phi(T(C)) \geq \chi_\Phi(C)$ *for all bounded subset* C *of* M*.*

Remark 3.3. The counterexample above was inspired by the paper [5, pp. 444].

4. Hammerstein integral equations in Banach spaces

Let E be a Banach space and consider the Hammerstein integral equation

$$y(t) = h(t) + \int_0^1 \kappa(t,s) f(s, y(s)) ds, \quad t \in [0,1], \tag{4.1}$$

where

$$\begin{cases} f\colon [0,1]\times E \to E \text{ is such that, } f_t = f(t,\cdot)\colon E \to E \text{ is weakly} \\ \text{sequentially continuous, for each } t \in [0,1], \end{cases} \tag{4.2}$$

$$h \in C([0,1], E) \text{ is arbitrary,} \tag{4.3}$$

and

$$\begin{cases} \kappa(t,\cdot) \in L^1([0,1],\mathbb{R}) \text{ for each } t \in [0,1] \text{ and} \\ \text{the map } t \mapsto \kappa(t,s) \text{ is continuous from } [0,1] \text{ to } L^1([0,1],\mathbb{R}) \end{cases} \tag{4.4}$$

hold.

When E is a reflexive space and f is weakly-weakly continuous, O'Regan [7] proved that (4.1) has a solution if the additional conditions

$$\begin{cases} \text{there exists a nondecreasing continuous function } \Omega\colon [0,\infty) \to (0,\infty) \\ \text{such that } \|f(s,x)\| \leq \Omega(\|x\|), \text{ for all } s \in [0,1] \text{ and } x \in E, \end{cases} \tag{4.5}$$

and

$$\big(K = \sup_{t\in[0,1]} \int_0^1 |\kappa(t,s)|ds\big) \cdot \limsup_{t\to\infty} \frac{\Omega(t)}{t} \leq 1, \tag{4.6}$$

are satisfied. The proof in [7] depends strongly on the reflexivity of E since a weak version of the Arzela-Ascoli theorem is used to guarantee that the right-side in (4.1) defines a weakly compact operator on $C([0,1],E)$. However, in an arbitrary Banach space E this technique may fails due to lack of weak compactness of balls. Thus it is natural to ask what conditions are needed if the word reflexive is removed. Here as an application of our fixed point theory, we offer the following result as a partial answer to this question.

Theorem 4.1. *Let E be a Banach space. Assume (4.2)-(4.6) and, in addition, suppose that for some $\Phi \in \mathscr{L}(E)$ satisfying ($\mathscr{H}$), we have*

$$\|\Phi \circ \Big(\int_0^1 \kappa(\cdot,s)\big(f(s,x)-f(s,y)\big)ds\Big)\|_{C([0,1],E)} \leq a\|\Phi\circ(x-y)\|_{C([0,1],E)}, \tag{4.7}$$

for all $t \in [0,1]$, $x, y \in C([0,1],E)$ with $0 < a < 1$. Then, (4.1) has a solution $y \in C([0,1],E)$.

Proof. Let us consider the space $X = C([0,1],E)$ endowed with its usual norm $\|\cdot\|$. We define $\tilde{\Phi} \in \mathscr{L}(X)$ by $(\tilde{\Phi}x)(t) = \Phi(x(t))$. Then, $\tilde{\Phi}$ verifies the assumption ($\mathscr{H}$). Hence Proposition 2.1 implies that X is a $\tilde{\Phi}$-space. From (4.5) and (4.6), we can find $R > 0$ such that

$$\|h\| + K\Omega(R) \leq R. \tag{4.8}$$

Let us now consider the set $M = \{x \in X : \|x\| \leq R\}$. From estimate (4.8), we can define the map $T\colon M \to M$ by

$$(Tx)(t) = h(t) + \int_0^1 \kappa(t,s) f(s, x(s)) ds, \quad t \in [0,1].$$

Also, from (4.3) and (4.4) one follows that T is well defined, i.e, Tx is norm continuous if $x \in M$. Thus, any fixed point of T is a solution of (4.1). We claim now that T is weakly sequentially continuous. Indeed, let $(x_n) \subset M$ such that $x_n \rightharpoonup x$ in X. Then, $x_n(s) \rightharpoonup x(s)$ in E for each $s \in [0,1]$. By (4.2), one has that $f(s, x_n(s)) \rightharpoonup f(s, x(s))$ in E for each $s \in [0,1]$, as $n \to \infty$. Combining now the Hahn-Banach theorem with the Dominated convergence theorem we conclude that $(Tx_n)(t) \to (Tx)(t)$ in E for each $t \in [0,1]$, as $n \to \infty$. Now, from (4.4) it is easy to check that $\{(Tx_n)\colon n \in \mathbb{N}\}$ is an equicontinuous subset of X. Thus, by Ascoli-Arzela theorem, we may conclude that $Tx_{n_j} \to Tx$ in X, for some subsequence (x_{n_j}) of (x_n). Hence $Tx_n \rightharpoonup Tx$ in X, as $n \to \infty$. If not there exists a subsequence (Tx_{n_k}) of (Tx_n) and a weak neighborhood $V^w(Tx)$ of Tx such that $Tx_{n_k} \notin V^w(Tx)$, for all $k \in \mathbb{N}$. Now arguing as before, we find a subsequence $Tx_{n_{k_l}} \rightharpoonup Tx$, which is a contradiction. This proves the claim. On the other hand, one easily verifies from (4.7) that T is $\tilde{\Phi}$-condensing. Theorem 3.1 now yields a fixed point for T. The proof is complete. □

Remark 4.2. Notice (4.6) could be replaced by condition: there exists $R > 0$ such that $\|h\| + K\Omega(R) \leq R$.

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