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EXISTENCE IN THE FUTURE BY FIXED POINT METHOD

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Abstract. We deal with the problem of finding the conditions which assure that the solutions of some differential equations or systems of differential equations exist in the future. We consider a perturbed differential equation and a system of two differential equations. In the proofs we use the Schauder's fixed point theorem.
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1. Introduction

In this paper we give conditions under which all solutions of a system of differential equations are continuable in the future.

Bernfeld [1], Hara, Yoneyama, Okazaki have studied the continuability of solutions of perturbed scalar differential equations. Burton [2], Hara, Yoneyama, Sugie [5], [6], Haddock [4], Hatvani [7] have studied the continuability of solutions of a system of two differential equations using Liapunov functions.

Stokes has studied the continuability in the future of solutions of scalar differential equations using the Schauder's fixed point theorem.

We give here two results which also use the Schauder's fixed point theorem.

2. Perturbation theorem

Consider the following differential equations:

$$x' = f(t, x) \tag{1}$$

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$$x' = f(t,x) + g(t,x) \tag{2}$$

where f, g are continuous on $[0,\infty[\times\mathbb{R}$.

For all $T > 0$ we denote

$$F_T^+(x) = \max\{\max\{f(t,x)|\ t \in [0,T]\}, 0\}$$

$$F_T^-(x) = \max\{\max\{-f(t,x)|\ t \in [0,T]\}, 0\}.$$

Lemma 1. *If for every $T > 0$ we have*

$$\int_0^\infty \frac{ds}{1+F_T^+(s)} = \infty, \quad \int_{-\infty}^0 \frac{ds}{1+F_T^-(s)} = \infty,$$

then all the solutions of (1) exist in the future.

Proof. Suppose that there exists a solution $x(t)$ unbounded from above on a finite interval $[t_0, t_1[$ and $x(t_0) \geq 0$. Let $a > 0$ be such that

$$\int_{x(t_0)}^a \frac{ds}{1+F_{t_1}^+(s)} > t_1 - t_0. \tag{3}$$

There exists $t_2 \in]0, t_1[$ such that $x(t_2) \geq a$. Then from

$$x'(t) = f(t, x(t)) \leq F_{t_1}^+(x(t)) < 1 + F_{t_1}^+(x(t))$$

follows that

$$\int_{x(t_0)}^a \frac{ds}{1+F_{t_1}^+(s)} \leq \int_{x(t_0)}^{x(t_2)} \frac{ds}{1+F_{t_1}^+(s)} \leq t_2 - t_0 < t_1 - t_0$$

which contradicts (3). Hence $x(t)$ is bounded from above.

In the same way we can prove that all solutions of (1) are bounded from below.

We state now a result for the perturbed equation (2).

Theorem 2. *Let $f : [0,\infty[\times\mathbb{R} \to \mathbb{R}$ be regular enough to ensure the uniqueness of the solution of the Cauchy's problem for (1).*

If:

i) f is increasing with respect to x for each $t \in [0,\infty[$,

ii) all the solutions of (1) exist in the future,

iii) there exists a continuous function $h : [0,\infty[\to \mathbb{R}$ such that

$$|g(t,x)| \leq h(t),\ \forall\ x \in \mathbb{R},$$

then all the solutions of (2) exist in the future.

Proof. Let $T > 0$ be fixed. We prove that all solutions of (2) exist on an interval of length T.

Let $(t_0, x_0) \in [0, \infty[\times\mathbb{R}$ be arbitrary. Denoting by

$$a = \int_{t_0}^{t_0+T} h(t)dt$$

We consider $x_1(t), x_2(t)$ be the solutions of (1) verifying

$$x_1(t_0) = x_0 - a, \quad x_2(t_0) = x_0 + a,$$

respectively.

Denote $D = \{x \in C[t_0, t_0 + T]|\ x_1(t) \leq x(t) \leq x_2(t),\ t \in [t_0, t_0 + T]\}$.

This set is a convex subset of Banach space $C[t_0, t_0 + T]$.

Define the complete continuous operator $L : C[t_0, t_0 + T] \to C[t_0, t_0 + T]$

$$L(x)(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds + \int_{t_0}^{t} g(s, x(s))ds.$$

We have

$$L(x)(t) \leq x_0 + \int_{t_0}^{t} f(s, x_2(s))ds + a = x_2(t)$$

and

$$L(x)(t) \geq x_0 + \int_{t_0}^{t} f(s, x_1(s))ds - a = x_1(t).$$

Hence $L[D] \subset D$.

Applying the Schauder's fixed point theorem we conclude that the fixed point of L is the solution of (2) verifying the condition

$$x(t_0) = x_0.$$

Theorem 3. *Let $f : [0, \infty[\times\mathbb{R} \to \mathbb{R}$ be regular enough to ensure the uniqueness of the solution of the Cauchy's problem for (1).*

If:

i) f is decreasing with respect to x for each $t \in [0, \infty[$,

ii) all solutions of (1) exist in the future

iii) there exists a continuous function $h : [0, \infty[\to \mathbb{R}$ such that

$$|g(t, x)| \leq h(t),\ \forall\ x \in \mathbb{R},$$

then all solutions of (2) exist in the future.

Proof. Let $(t_0, x_0) \in [0, \infty[\times\mathbb{R}$ and $T > 0$ be arbitrary. Denote by $x(t)$ the solution of (1) verifying $x(t_0) = x_0$ and by $y(t)$ the solution of (2) verifying

$y(t_0) = x_0$. If we denote

$$a = \int_{t_0}^{t_0+T} h(t)dt$$

then we will prove that $x(t) - a \leq y(t) \leq x(t) + a$ for $t \in [t_0, t_0 + T]$.

Let

$$E_1 = \{t \in [t_0, t_0 + T[\mid y(t) > x(t)\}$$

and

$$E_2 = \{t \in [t_0, t_0 + T[\mid y(t) < x(t)\}.$$

E_1 and E_2 are open sets. Hence

$$E_1 = \bigcup_{n \in I}]a_n, b_n[\text{ and } x(a_n) = y(a_n).$$

Then for $t \in]a_n, b_n[$ we have:

$$y(t) = y(a_n) + \int_{a_n}^{t} f(s, y(s))ds + \int_{a_n}^{t} h(s)ds$$

$$\leq x(a_n) + \int_{a_n}^{t} f(s, x(s))ds + a = x(t) + a.$$

Analogously we can show that: if $y(t) < x(t)$ then $y(t) \geq x(t) - a$.

Hence $y(t)$ exists on $[t_0, t_0 + T]$ with arbitrary $T > 0$.

3. Existence in the future for a system of differential equations

Consider the system:

$$\begin{aligned} x' &= f(t, x, y) \\ y' &= g(t, x, y) \end{aligned} \tag{4}$$

where $f, g \in C([0, \infty[\times \mathbb{R})$.

Theorem 4. *Assume that:*

i) f is local Lipschitz with respect to x and y

ii) for each $y \in C[0, \infty[$, all solutions of differential equation $x' = f(t, x, y(t))$ exist on $[0, \infty[$.

Moreover suppose that there exist the functions $G_1, G_2 : [0, \infty[\times \mathbb{R} \to \mathbb{R}$, regular enough to ensure the uniqueness of the solutions of the Cauchy's problem for the equations:

$$y' = G_i(t, y), \quad i = 1, 2 \tag{5}$$

and such that

iii) all solutions of (5) exist in the future
iv) $G_1(t,y) \leq g(t,x,y) \leq G_2(t,y),\ \forall\ (t,x,y) \in [0,\infty[\times\mathbb{R}^2$
v) G_1 *and* G_2 *are increasing with respect to* y.
Then all solutions of (4) exist in the future.

Proof. Let $(t_0,x_0,y_0) \in [0,\infty[\times\mathbb{R}^2$ and let $T > 0$ be arbitrary. Denote by $y_1(t), y_2(t)$ the solutions of (5) satisfying the inequalities:

$$y_1(t_0) < y_0 < y_2(t_0).$$

Then $y_1(t)$ and $y_2(t)$ exist on $[t_0, t_0+T]$.

Define the operator $S : C[t_0,t_0+T] \to C[t_0,t_0+T]$

$$S(y)(t) = x(t),$$

where $x(t)$ is the solution of Cauchy's problem

$$x' = f(t,x,y(t)), \quad x(t_0) = x_0.$$

From Kurzweil's theorem follows that S is continuous.

Define the operator $L : C[t_0,t_0+T] \to C[t_0,t_0+T]$

$$L(y)(t) = y_0 + \int_{t_0}^{t} g(s,x(s),y(s))ds$$

where $x(s) = S(x)(s)$.

Denote $D = \{y \in C[t_0,t_0+T] |\ y(t_0) = y_0,\ y_1(t) \leq y(t) \leq y_2(t),\ t \in [t_0,t_0+T]\}$. There exists $M > 0$ such that

$$|G_1(t,y(t))| \leq M_1 |G_2(t,y(t))| \leq M \text{ if } t \in [t_0,t_0+T],\ y \in D.$$

If $y \in D$ then

$$L(y)(t) \leq y_0 + \int_{t_0}^{t} G_2(s,y_2(s))ds \leq y_2(t)$$
$$L(y)(t) \geq y_0 + \int_{t_0}^{t} G_1(s,y_1(s))ds \geq y_1(s)$$

and

$$|L(y)(t_2) - L(y)(t_1)| \leq M|t_2 - t_1|.$$

Hence $L(D) \subset D$, D is a convex subset of the Banach space $C[t_0,t_0+T]$ and L is a completely continuous operator. From Schauder's fixed point theorem it follows that there exists a fixed point $y(t)$ of L. If $x(t) = S(y)(t)$ then $(x(t),y(t))$ is the solution on $[t_0,t_0+T]$ of (4) which verifies $x(t_0) = x_0$, $y(t_0) = y_0$.

Example. Consider the system corresponding to Liénard equation

$$x' = y - F(x)$$
$$y' = -g(x) + e(t)$$

Denote

$$G(x) = \int_0^x g(s)ds.$$

Theorems related to the existence in the future, given by Hara, Yoneyama, Sugie, Burton, Graef contain one of the following conditions:

i) $G(x) \geq -P$

ii) $g(x)F(x) \geq -Q$

iii) $g(x)F(x) \geq -Q \cdot G(x)$

iv) $\exists\ a > 0$ such that $xg(x) > 0$ if $|x| \geq 0$.

Consider

$$g(x) = \frac{-x^2 + x - 1}{x^2 + 1}, \quad F(x) = x^3, \quad G(x) = \frac{1}{2}\ln(x^2 h) - x.$$

The conditions i)-iv) are not satisfied, but the hypotheses of the above Theorem 4 are satisfied.

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