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CONTINGENT NASH POINTS FOR SET-VALUED MAPS

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Abstract. We introduce the notion of contingent Nash point for a family of set-valued maps. This fact is motivated by some examples, where we cannot use classical existence results to obtain Nash equilibrium points. We compare also the above mentioned Nash points. Sufficient conditions for the existence of such points are obtained. Several numerical examples are given trough the paper.

Key Words and Phrases: Nash points, contingent derivative.

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1. Introduction

Let $K_1, \ldots, K_n,\ (n \geq 2)$ be nonempty sets and $f_i : K_1 \times \ldots \times K_n \to \mathbf{R}$ $(i = 1, \ldots, n)$ be given functions. A point $(x_1, \ldots, x_n) \in K_1 \times \ldots \times K_n$ is a *Nash equilibrium point for* $(f_1, \ldots, f_n; K_1, \ldots, K_n)$ if

$$f_i(x_1, \ldots, x_i, \ldots, x_n) \leq f_i(x_1, \ldots, y_i, \ldots, x_n),\ \forall y_i \in K_i,\ \forall i = 1, \ldots, n.$$

The well-known existence result is due to J. Nash in [7, 8]:

Theorem 1.1. *Let $K_1, \ldots, K_n$ be nonempty, compact, convex subsets of Hausdorff topological vector spaces and $f_i : K_1 \times \ldots \times K_n \to \mathbf{R}$ $(i = 1, \ldots, n)$ be continuous functions such that*

$$y_i \mapsto f_i(x_1, \ldots, y_i, \ldots, x_n),\ (y_i \in K_i)$$

is quasiconvex for all fixed $x_j \in K_j$ $(j \neq i)$. Then there exists a Nash equilibrium point for $(f_1, \ldots, f_n; K_1, \ldots, K_n)$.

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In many cases the objective functions are not single-valued. Set-valued versions of the above result can be founded in literature, see for example [3, 6].

In this paper, we first prove a natural set-valued version of the above result, see Theorem 2.1. More precisely, some kind of convexity and continuity will be provided for the set-valued maps to obtain the existence result. However, equilibrium points may exist even if the conditions from Theorem 2.1 are not satisfied.

This fact motivates the introduction of a new notion, the so-called *contingent Nash point*, which is defined by means of the contingent derivative of a set-valued map, see [1, 2]. The proof of the existence theorem of contingent Nash points is based upon the set-valued version of Ky Fan inequality due to Kristály and Varga [5]. In several numerical examples the Nash points are calculated.

2. Preliminaries

Let Z and Y be metric spaces, $F : Z \rightsquigarrow Y$ be a set-valued map with nonempty values. We define the graph and the domain of F by

$$\mathrm{Graph}(F) = \{(z, y) \in Z \times Y : y \in F(z)\} \text{ and } \mathrm{Dom}(F) = \{z \in Z : F(z) \neq \emptyset\}.$$

We say that the set-valued map $F : Z \rightsquigarrow Y$ is *upper semicontinuous at* $z \in Z$ (usc at z) if and only if for any neighborhood U of $F(z)$, $\exists\, \eta > 0$ such that for every $z' \in B_Z(z, \eta)$ we have $F(z') \subset U$. The set-valued function $F : Z \rightsquigarrow Y$ is *lower semicontinuous at* $z \in Z$ (lsc at z) if and only if for any $y \in F(z)$ and for any sequence of elements $\{z_n\}_n$ in Z converging to z, there exists a sequence of elements $y_n \in F(z_n)$ converging to y.

The set-valued function F is *upper* (resp. *lower) semicontinuous on* Z if F is upper (resp. lower) semicontinuous at every point $z \in Z$.

We shall say that the set-valued map F is *continuous* at z if it is both usc and lsc at z, and that it is continuous on Z if and only if it is continuous at every point of Z.

Let X, Y be vector spaces, K be a convex subset of X. We say that the set-valued map F is *convex on* K if and only if $\forall\, x_1, x_2 \in K$ and $\lambda \in [0, 1]$, $\lambda F(x_1) + (1 - \lambda)F(x_2) \subseteq F(\lambda x_1 + (1 - \lambda)x_2)$.

Now, let $K_1, \ldots, K_n$ be nonempty subsets of real normed spaces $X_1, \ldots, X_n$, respectively and $F_i : K_1 \times \ldots \times K_n \rightsquigarrow \mathbf{R}$ $(i = 1, \ldots, n)$ set-valued maps with compact, nonempty values.

Definition 2.1. A point $(x_1, \ldots, x_n) \in K_1 \times \ldots \times K_n$ is a *Nash equilibrium point for* $(F_1, \ldots, F_n; K_1, \ldots, K_n)$ if

$$\min F_i(x_1, \ldots, x_i, \ldots, x_n) \leq \min F_i(x_1, \ldots, y_i, \ldots, x_n),$$

$\forall y_i \in K_i, \ \forall i = 1, \ldots, n.$

The following result is an easy consequence of Theorem 1.1.

Theorem 2.1. *Let $K_1, \ldots, K_n$ be nonempty, compact, convex subsets of real normed spaces $X_1, \ldots, X_n$, respectively and $F_i : K_1 \times \ldots \times K_n \rightsquigarrow \mathbf{R}$ $(i = 1, \ldots, n)$ be continuous set-valued maps on $K_1 \times \ldots \times K_n$ with compact, nonempty values such that*

$$y_i \rightsquigarrow F_i(x_1, \ldots, y_i, \ldots, x_n), \ (y_i \in K_i)$$

is convex on K_i for all fixed $x_j \in K_j$ $(j \neq i)$. Then there exists a Nash equilibrium point for $(F_1, \ldots, F_n; K_1, \ldots, K_n)$.

Proof. Let $f_i := \min F_i$ $(i = 1, \ldots, n)$. It's easy to prove that the functions f_i are convex in the i^{th} variable and continuous. Now, we apply Theorem 1.1. $\square$

Now, let $K := K_1 = K_2 = [-1, 7]$, $X := X_1 = X_2 = \mathbf{R}$ and $F_1, F_2 : K \times U \rightsquigarrow \mathbf{R}$ defined by

$$F_1(x_1, x_2) = [\max\{|x_1|, |x_2|\} - 1, 0] \tag{1}$$

$$F_2(x_1, x_2) = [1 - \max\{|x_1|, |x_2|\}, 2 - x_1^2 - x_2^2]. \tag{2}$$

Clearly, F_1 and F_2 are continuous on $K \times K$, but $x_2 \rightsquigarrow F_2(0, x_2)$ is not convex on K. Therefore, Theorem 2.1 cannot be applied. Nevertheless, we will see later that there are Nash equilibrium points for $(F_1, F_2; K, K)$. To determine these points it's important to find a method which will be elaborated by means of the contingent derivative. First of all, we recall some notions from the set-valued analysis.

Let K be a subset of a real normed space X and $x \in \overline{K}$. The *contingent cone at x to K* (denoted by $T_K(x)$) is defined by

$$T_K(x) = \{v : \liminf_{h \to 0^+} \frac{\mathrm{dist}(x + hv, K)}{h} = 0\}.$$

Let $F : X \rightsquigarrow \mathbf{R}$ be a set-valued map with non-empty values. The *contingent derivative* $DF(x, y)$ of $F : X \rightsquigarrow \mathbf{R}$ at $(x, y) \in \mathrm{Graph}(F)$ is the set-valued map from X to $\mathbf{R}$ defined by

$$\mathrm{Graph}(DF(x, y)) := T_{\mathrm{Graph}(F)}(x, y), \tag{3}$$

where $T_{\mathrm{Graph}(F)}(x, y)$ is the contingent cone at (x, y) to the Graph(F), see [2, p. 181]. We say that F is *Lipschitz around $a \in X$* if there exist a positive constant L and a neighborhood U of x such that

$$\forall x_1, x_2 \in U,\ F(x_1) \subseteq F(x_2) + L\|x_1 - x_2\| \cdot [-1, 1].$$

We say that F is *K-locally Lipschitz* if it is Lipschitz around all $x \in K$.

The contingent derivative can be characterized by a limit of differential quotient. Let $(x, y) \in \mathrm{Graph}(F)$ and suppose that F is Lipschitz around x. We have

$$v \in DF(x, y)(u) \Longleftrightarrow \liminf_{h \to 0^+} \mathrm{dist}\left(v, \frac{F(x + hu) - y}{h}\right) = 0 \tag{4}$$

and $\mathrm{Dom}(DF(x, y)) = X$, see [2, Proposition 5.1.4, p. 186].

Remark 2.1. *Let X be a normed space, $\Omega \subset X$ an open set and $f : \Omega \to \mathbf{R}$ be a single-valued map. Suppose that $L \subset X$ and $F(x) = \{f(x)\}$, $\forall x \in L$; $F(x) = \{\emptyset\}$, $\forall x \notin L$. If f is Fréchet differentiable at $x_0 \in \Omega \cap L$, then $DF(x_0, f(x_0))(h) = \nabla f(x_0)(h)$, $\forall h \in T_L(x_0)$.*

3. Contingent Nash points

Let $K_1, \ldots, K_n$ be nonempty, convex subsets of real normed spaces $X_1, \ldots, X_n$, respectively and $F_i : K_1 \times \ldots \times K_i \times \ldots \times K_n \rightsquigarrow \mathbf{R}$ $(i = 1, \ldots, n)$ set-valued maps with compact, nonempty values. Let $x = (x_1, \ldots, x_n) \in K_1 \times \ldots \times K_n$ be a fixed element. We can define the (partial) contingent derivative of F_i in the i^{th} variable at the point $(x, \min F_i(x))$, i.e. the contingent derivative of $F_i(x_1, \ldots, x_{i-1}, \cdot, x_{i+1}, \ldots, x_n)$ at the point $(x_i, \min F_i(x))$.

Definition 3.1. *A point $x = (x_1, \ldots, x_n) \in K_1 \times \ldots \times K_n$ is called contingent Nash point for $(F_1, \ldots, F_n; K_1, \ldots, K_n)$ if*

$$D_i F_i(x, \min F_i(x))(u_i - x_i) \subseteq \mathbf{R}_+, \ \forall u_i \in K_i, \ \forall i = 1, \ldots, n.$$

Proposition 3.1. *Let $x = (x_1, \ldots, x_n) \in K_1 \times \ldots \times K_n$ be a Nash equilibrium point for $(F_1, \ldots, F_n; K_1, \ldots, K_n)$. If $F_i(x_1, \ldots, x_{i-1}, \cdot, x_{i+1}, \ldots, x_n)$ $(i = 1, \ldots, n)$ is K_i-locally Lipschitz then x is contingent Nash point for $(F_1, \ldots, F_n; K_1, \ldots, K_n)$.*

Proof. Since x is Nash equilibrium point for $(F_1, \ldots, F_n; K_1, \ldots, K_n)$, we have

$$F_i(x_1, \ldots, y_i, \ldots, x_n) - \min F_i(x) \subseteq \mathbf{R}_+, \ \forall y_i \in K_i, \ \forall i = 1, \ldots, n. \quad (5)$$

Let $i \in \{1, \ldots, n\}$, $u_i \in K_i$ be fixed elements and choose $v_i \in D_i F_i(x, \min F_i(x))(u_i - x_i)$. We prove that $v_i \geq 0$. From (4) we have that

$$\liminf_{h \to 0^+} \operatorname{dist}\left(v_i, \frac{F_i(x_1, \ldots, x_i + h(u_i - x_i), \ldots, x_n) - \min F_i(x)}{h}\right) = 0, \quad (6)$$

since $F_i(x_1, \ldots, x_{i-1}, \cdot, x_{i+1}, \ldots, x_n)$ is K_i-locally Lipschitz. Because $h > 0$ is sufficiently small and K_i is convex, we have $y_i^h := x_i + h(u_i - x_i) \in K_i$. Using (5), we have that $\frac{F_i(x_1, \ldots, y_i^h, \ldots, x_n) - \min F_i(x)}{h} \subseteq \mathbf{R}_+$. Suppose that $v_i < 0$. Then $0 < |v_i| = \operatorname{dist}(v_i, \mathbf{R}_+) \leq \operatorname{dist}\left(v_i, \frac{F_i(x_1, \ldots, y_i^h, \ldots, x_n) - \min F_i(x)}{h}\right)$, which is in contradiction with (6). $\square$

The above proposition allows to select the Nash equilibrium points from the set of contingent Nash points. In many problem, the determination of contingent Nash points is easier than of the Nash equilibrium points. The following examples show this fact. We will use the notation $\lfloor a, b \rfloor := [\min\{a, b\}, \max\{a, b\}]$, where $a, b \in \mathbf{R}$.

Example 3.1. Let us consider the example from (1) and (2) in the extended form, i.e. $K := K_1 = K_2 = [-1, 1]$, $X := X_1 = X_2 = \mathbf{R}$ and $F_1 : X \times K \rightsquigarrow \mathbf{R}$, $F_2 : K \times X \rightsquigarrow \mathbf{R}$ defined by

$$F_1(x_1, x_2) = \lfloor \max\{|x_1|, |x_2|\} - 1, 0 \rfloor,$$

$$F_2(x_1, x_2) = \lfloor 1 - \max\{|x_1|, |x_2|\}, 2 - x_1^2 - x_2^2 \rfloor.$$

It's easy to verify that $x \rightsquigarrow F_1(x, x_2)$ and $y \rightsquigarrow F_2(x_1, y)$ are V-locally Lipschitz maps ($x_1, x_2 \in K$ being fixed points). We find those points $(x_1, x_2) \in K \times K$ such that

$$D_1F_1((x_1, x_2), \min F_1(x_1, x_2))(u_1 - x_1) \subseteq \mathbf{R}_+,\ \forall u_1 \in K, \tag{7}$$

$$D_2F_2((x_1, x_2), \min F_2(x_1, x_2))(u_2 - x_2) \subseteq \mathbf{R}_+,\ \forall u_2 \in K. \tag{8}$$

Using the geometric meaning of the contingent derivative, (see the relation (3)), after an elementary discussion and computation, the points which satisfy the inclusions (7) and (8) respectively, are

$$CNP_1 = \{(x_1, x_2) \in K \times K : |x_1| \leq |x_2|\},$$

$$CNP_2 = \{(x_1, x_2) \in K \times K : |x_1| > |x_2|\} \cup \{(x_1, x_2) \in K \times K : |x_2| = 1\}.$$

Therefore, the contingent Nash points for $(F_1, F_2; K, K)$ are $CNP = CNP_1 \cap CNP_2 = \{(x_1, x_2) \in K \times K : |x_2| = 1\}$. A direct computation shows that the set of Nash equilibrium points for $(F_1, F_2; K, K)$ coincides with the set CNP.

Example 3.2. Let $K = [-1, 1]$, $F_1 : \mathbf{R} \times K \rightsquigarrow \mathbf{R}$ and $F_2 : K \times \mathbf{R} \rightsquigarrow \mathbf{R}$ defined by

$$F_1(x_1, x_2) = \begin{cases} \lfloor 2, |x_1 - 1| \rfloor, & x_2 \geq 0 \\ \lfloor 2, |x_1 + 1| \rfloor, & x_2 < 0 \end{cases}$$

and

$$F_2(x_1, x_2) = \begin{cases} \lfloor 2, |x_2 + 1| \rfloor, & x_1 \geq 0 \\ \lfloor 2, |x_2 - 1| \rfloor, & x_1 < 0 \end{cases}.$$

The points which satisfy the corresponding inclusions from (7) and (8) respectively, are

$$CNP_1 = \{(4, x_2) : x_2 \in [0, 1]\} \cup \{(-1, x_2) : x_2 \in [-1, 0)\},$$

$$CNP_2 = \{(x_1, -1) : x_1 \in [0, 1]\} \cup \{(x_2, 1) : x_1 \in [-0, 5)\}.$$

The maps $x \rightsquigarrow F_1(x, x_2)$ and $y \rightsquigarrow F_2(x_1, y)$ are K-locally Lipschitz again ($x_1, x_2 \in K$ being fixed points), however the set of contingent Nash points for $(F_1, F_2; K, K)$ is $CNP = CNP_1 \cap CNP_2$, which is empty. Hence the set of Nash equilibrium points is empty also, due to Proposition 3.1.

We observe that the above maps are not continuous on $K \times K$. In the next section we give sufficient conditions to obtain contingent Nash points, assuming some continuity hypotheses for the set-valued maps. Before this, we can state the converse of Proposition 3.1 by taking a convexity assumption for the corresponding set-valued maps. Let X be a normed space, K a nonempty set of X.

Definition 3.2. The set-valued map $F : X \rightsquigarrow \mathbf{R}$ is called *K-pseudo-convex at* $(x, y) \in \mathrm{Graph}(F)$, if

$$F(x') \subseteq DF(x, y)(x' - x) + y, \ \forall x' \in K.$$

In the case, when $K = \mathrm{Dom}(F)$, the above definition reduces to Definition 5.1.1 from [2].

Proposition 3.2. *Let $K_1, \ldots, K_n$ be nonempty subsets of real normed spaces $X_1, \ldots, X_n$, respectively and $F_i : K_1 \times \ldots \times K_i \times \ldots \times K_n \rightsquigarrow \mathbf{R}$ $(i = 1, \ldots, n)$ set-valued maps with compact, nonempty values. Let $x = (x_1, \ldots, x_n) \in K_1 \times \ldots \times K_n$ be a contingent Nash point for $(F_1, \ldots, F_n; K_1, \ldots, K_n)$. If $F_i(x_1, \ldots, x_{i-1}, \cdot, x_{i+1}, \ldots, x_n)$ $(i = 1, \ldots, n)$ is K_i-pseudo-convex at $(x_i, \min F_i(x))$, then x is Nash equilibrium point for $(F_1, \ldots, F_n; K_1, \ldots, K_n)$.*

4. Existence of contingent Nash points

To guarantee the existence of contingent Nash points, we will use the following result established in [5] by Kristály and Varga.

Lemma 4.1. *Let X be a real normed space, K a nonempty convex, compact subset of X and $G : K \times K \rightsquigarrow \mathbf{R}$ a set-valued map satisfying*

(i) $\forall y \in K, \ x \rightsquigarrow G(x, y)$ is lsc on K,

(ii) $\forall x \in K, \ y \rightsquigarrow G(x, y)$ is convex on K,

(iii) $\forall y \in K, \ G(y, y) \subseteq \mathbf{R}_+$.

Then, there exists an element $\overline{x} \in K$ such that

$$G(\overline{x}, y) \subseteq \mathbf{R}_+, \ \forall y \in K.$$

Let K_i, X_i and F_i $(i = 1, \ldots, n)$ as in Proposition 3.2. We denote by $F_{i|K_i}$ the restriction of F_i to $K = K_1 \times \ldots \times K_i \times \ldots \times K_n$.

Definition 4.1. $F_{i|K_i}$ is called *i-lower semicontinuously differentiable* if the map

$$\mathrm{Graph}(F_{i|K_i}) \times X_i \ni (x_1, \ldots, x_n, y, u) \rightsquigarrow D_i F_i((x_1, \ldots, x_n), y)(u)$$

is lower semicontinuous. (For a closely related notion, see [2, Definition 5.1.5]).

Theorem 4.1. (Existence of contingent Nash points) *Let $K_1, \ldots, K_n$ be nonempty, compact, convex subsets of real normed spaces $X_1, \ldots, X_n$, respectively and $F_i : K_1 \times \ldots \times X_i \times \ldots \times K_n \rightsquigarrow \mathbf{R}$ set-valued maps with compact, nonempty values with closed graph. Suppose that F_i are K_i-locally Lipschitz in the i^{th} variable, $F_{i|K_i}$ are continuous on $K = K_1 \times \ldots \times K_n$ and i-lower semicontinuously differentiable* $(i = 1, \ldots, n)$. *Then there exists a contingent Nash point for $(F_1, \ldots, F_n; K_1, \ldots, K_n)$.*

Proof. Let $X := X_1 \times \ldots \times X_n$. Clearly, K is a compact, convex subset of X. We define the map $G : K \times K \rightsquigarrow \mathbf{R}$ by

$$G(x, u) = \sum_{i=1}^{n} D_i F_i(x, \min F_i(x))(u_i - x_i),$$

where $x = (x_1, \ldots, x_n)$ and $u = (u_1, \ldots, u_n)$. We will verify the hypotheses from Lemma 4.1.

(i) Let us fix $u \in K$. Since the sum of finite lsc maps is lsc, it's enough to prove that $x \rightsquigarrow D_i F_i(x, \min F_i(x))(u_i - x_i)$ is lsc on K $(i = 1, \ldots, n)$. For this, let us fix an $x^* \in K$. Now, let $v_i^* \in D_i F_i(x^*, \min F_i(x^*))(u_i - x_i^*)$ and $\{x^m\}_{m \in \mathbf{N}}$ be a sequence from K which converges to x^*. Since $F_{i|K_i}$ is continuous on K, the function $\min(F_{i|K_i})$ is also continuous. Therefore $\min F_i(x^m) \to \min F_i(x^*)$, as $m \to \infty$. Using the fact that $F_{i|K_i}$ is i-lower semicontinuously differentiable, there exists a sequence $v_i^m \in D_i F_i(x^m, \min F_i(x^m))(u_i - x_i^m)$, such that $v_i^m \to v_i$, as $m \to \infty$.

(ii) Let us consider $x \in K$ fixed. We will prove that $u \rightsquigarrow G(x, u)$ is convex on K. Since the sum of convex maps is also convex, it's enough to prove that $u \rightsquigarrow D_i F_i(x, \min F_i(x))(u_i - x_i)$ is convex. Using again that $F_{i|K_i}$ is i-lower semicontinuously differentiable (in particular $F_i(x_1, \ldots, x_{i-1}, \cdot, x_{i+1}, \ldots, x_n)$ is sleek at $(x_i, \min F_i(x)))$ and [2, Theorem 4.1.8], the above set-valued map is a closed convex process.

(iii) Let us consider $x \in K$ and $i \in \{1, \ldots, n\}$ be fixed. We will prove that every element v_i from $D_i F_i(x, \min F_i(x))(9)$ is non-negative.

In fact, since $F_i(x_1, \ldots, x_{i-1}, \cdot, x_{i+1}, \ldots, x_n)$ is K_i-locally Lipschitz, from (4) we have $\liminf_{h \to 0^+} \operatorname{dist}\left(v_i, \frac{F_i(x_1,\ldots,x_i+h\cdot 0,\ldots,x_n)-\min F_i(x)}{h}\right) = 0$. Since $\frac{F_i(x)-\min F_i(x)}{h} \subseteq \mathbf{R}_+$, $\forall h > 0$, we obtain that $v_i \geq 0$. Therefore, $G(x,x) = \sum_{i=1}^{n} D_i F_i(x, \min F_i(x))(0) \subseteq \mathbf{R}_+$.

From Lemma 4.1 we obtain an element $\overline{x} = (\overline{x}_1, \ldots, \overline{x}_n) \in K$ such that

$$\sum_{i=0}^{n} D_i F_i(\overline{x}, \min F_i(\overline{x}))(u_i - \overline{x}_i) \subseteq \mathbf{R}_+, \ \forall u \in K. \tag{9}$$

Let $i \in \{1, \ldots, n\}$ be fixed. We may choose $u_j := \overline{x}_j, \ j \neq i$. Since the (partial) contingent derivatives are closed convex process, we have that $0 \in D_j F_j(\overline{x}, \min F_j(\overline{x}))(0), \ j \neq i$. From (2), we obtain $D_i F_i(\overline{x}, \min F_i(\overline{x}))(u_i - \overline{x}_i) \subseteq \mathbf{R}_+, \ \forall u_i \in K_i$. The proof is complete. □

Using Remark 2.1, we obtain an immediate consequence of the above theorem. A similar result was obtained by Kassay, Kolumbán and Páles in [4].

Corollary 4.1. *Let $K_1, \ldots, K_n$ be nonempty, compact, convex subsets of real normed spaces $X_1, \ldots, X_n$, respectively and $f_i : K_1 \times \ldots \times X_i \times \ldots \times K_n \to \mathbf{R}$ be continuous functions. Suppose that there exist $D_i \subseteq X_i$ open, convex sets such that $K_i \subset D_i$ and f_i is Fréchet differentiable in the i^{th} variable on D_i and $\partial_i f_i$ is continuous on $K_1 \times \ldots \times D_i \times \ldots \times K_n$. Then there exists an element $(x_1, \ldots, x_n) \in K_1 \times \ldots \times K_n$ such that*

$$\partial_i f_i(x_1, \ldots, x_n)(u_i - x_i) \geq 0, \ \forall u_i \in K_i, \ \forall i \in \{1, \ldots, n\}.$$

Example 4.1. Let $K := K_0 = K_2 = [-1, 1]$, $X := X_1 = X_2 = D_1 = D_2 = \mathbf{R}$ and $f_1, f_2 : D_1 \times D_2 \to \mathbf{R}$ defined by

$$f_1(x_1, x_2) = x_1 x_2 + x_1^2, \quad f_2(x_1, x_2) = x_2 - 3x_1 x_2^2.$$

Clearly, the above sets and functions satisfy the assumptions from Corollary 4.1, hence a solution for the above system is guaranteed. It's easy to observe that, for $i \in \{1, 2\}$, we have $\partial_i f_i(x_1, x_2)(u_i - x_i) \geq 0, \ \forall u_i \in K$ if and only if $\partial_i f_i(x_1, x_2) = 0$ if $-1 < x_i < 1$; $\partial_i f_i(x_1, x_2) \geq 0$, if $x_i = -5$ and $\partial_i f_i(x_1, x_2) \leq 0$ if $x_i = 1$. Treating all the cases, we state that the set of contingent Nash points for $(f_1, f_2; K, K)$ is $CNP = \{(\frac{1}{2}, -1)\}$ (in this single-valued case: "Nash stationary point", due to [4]). This point will be also Nash equilibrium point for $(f_1, f_2; K, K)$.

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