Fixed Point Theory, Volume 6, No. 1, 2005, 133-138
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.htm

ZHONG'S VARIATIONAL PRINCIPLE IS EQUIVALENT WITH EKELAND'S

MIHAI TURINICI

Seminarul Matematic "A. Myller", Universitatea "A. I. Cuza"
11, Copou Boulevard, 700506 Iaşi, Romania E-mail: mturi@uaic.ro

Abstract. The functional type extension of Ekeland's variational principle [7] due to Zhong [18] is logical equivalent with it.
Key Words and Phrases: Complete metric space, lsc function, variational principle, quasi-order, normal function, nonexpansiveness.
2000 Mathematics Subject Classification: Primary 49J53, Secondary 47J30.

1. Introduction

Let (M, d) be a complete metric space; and $f : M \to R \cup \{\infty\}$, some function with the properties

$$f \text{ is proper } (\mathrm{Dom}(f) \neq \emptyset), \text{ bounded below } (f_* = \inf[f(M)] > -\infty) \quad (101)$$

$$f \text{ is lsc over } M \quad (f(x) \leq \liminf_n f(x_n), \text{ whenever } x_n \to x). \quad (102)$$

A basic variational statement involving these data is the 1974 one in Ekeland [7] [referred to as Ekeland's variational principle (EVP); cf. Section 2]. It found some useful applications to control and optimization, generalized differential calculus and critical point theory; see the 1979 paper by Ekeland [8] for a survey of these. So, it cannot be surprising that, soon after its formulation, many extensions of it were obtained. Among these, the 1997 one obtained by Zhong [18] [and referred to as Zhong's variational principle (ZVP)] is of interest for us. Nevertheless, the argument used there is rather involved. It is our aim

This paper was presented at International Conference on Nonlinear Operators, Differential Equations and Applications held in Cluj-Napoca (Romania) from August 24 to August 27, 2004.

to show (in Section 3) that a simplification of it is possible; precisely, that ZVP is deductible from EVP (hence equivalent with EVP). Further aspects will be discussed elsewhere.

2. EVP (the relative form)

Let (M, d) be a complete metric space; and $\psi : M \to R_+ \cup \{\infty\}$, some function with

$$\psi \text{ is proper and lsc on } M \quad \text{(cf. Section 1).} \tag{201}$$

Denote by $(\leq)$ the relation (over M)

$$(x, y \in M) \;\; x \leq y \text{ iff } d(x, y) + \psi(y) \leq \psi(x). \tag{202}$$

It is not hard to see that $(\leq)$ is **(i)** a quasi-order (reflexive and transitive) on M, and **(ii)** an order (antisymmetric quasi-order) on $\mathrm{Dom}(\psi)$. Let also $(<)$ stand for the associated strict order

$$(x, y \in M) \;\; x < y \text{ iff } x \leq y \text{ and } x \neq y.$$

Call the point $z \in M$, $(\leq)$-*maximal* if

$$w \in M,\; z \leq w \Rightarrow z = w \;\; \text{(i.e.: } z < x \text{ is false, for each } x \in M). \tag{203}$$

Note that, by (201) above, any point with this property is necessarily in $\mathrm{Dom}(\psi)$. Concerning the maximal elements of the structure $(M, \leq)$, the following result is available.

Theorem 1. *Let the precised conditions be admitted. Then, for each $u \in M$, there exists $v = v(u)$ in $\mathrm{Dom}(\psi)$ with*

$$d(u, v) \leq \psi(u) - \psi(v) \quad \textit{(wherefrom } u \leq v, \;\; \psi(u) \geq \psi(v)\textit{)} \tag{204}$$

$$d(v, x) > \psi(v) - \psi(x), \; \forall x \in M \setminus \{v\} \; \textit{(hence } v \textit{ is } (\leq)\textit{-maximal).} \tag{205}$$

Some remarks are in order. A first proof of Theorem 1 was given in 1974 by Ekeland [7]. For this reason, it will be referred to as Ekeland's variational principle (EVP). On the other hand, this result is logically equivalent with the 1975 Caristi-Kirk fixed point theorem [5]. Hence, the transfinite induction reasoning used by the quoted authors is also working in our setting; see also Wong [17]. A sequential type argument (which, in fact, yields an ordering principle that extends Theorem 1) was given in the 1976 paper by Brezis and Browder [3]; for various extensions of it we refer to Altman [1], Turinici [15] or Kang and Park [9]. A proof of Theorem 1 involving the chains of the structure $(M, \leq)$ may be found in Turinici [14]; and its sequential translation has been developed in Dancs, Hegedus and Medvegyev [6]. Some pseudometric extensions of these facts were proposed in Tataru [13]; see also Valyi [16].

Note that Theorem 1 is just the original Ekeland's principle, via

$$\psi(x) = (\lambda/\varepsilon)(f(x) - f_*), \quad x \in M \quad \text{(where } f_* \text{ is as in (101))}. \tag{206}$$

Moreover the very formulation of (204)+(205) in terms of $(\leq)$ makes our statement be viewed as a version of Zorn's maximality principle (cf. Moore [10, ch 4, Sect 4]) for the class of quasi-orders introduced via (202). Further aspects may be found in Brunner [4].

3. Main result

Let $b : R_+ \to R_+$ be some *normal* function; i.e.

$$b \text{ is decreasing and strictly positive on } R_+ \tag{301}$$

$$B(\infty) = \infty, \text{ where } B(t) = \int_0^t b(\tau)d\tau, \quad t \geq 0. \tag{302}$$

Some basic facts involving the couple (b, B) are being collected in

Lemma 1. *The following are true*

$$sb(t+s) \leq B(t+s) - B(t) \leq sb(t), \ \forall t, s \in R_+ \tag{303}$$

$$B \text{ and } B^{-1} \text{ are order homeomorphisms of } R_+ \tag{304}$$

$$\tau \vdash [B(\tau + t) - B(\tau)] \text{ is decreasing on } R_+, \ \forall t \geq 0 \tag{305}$$

$$B \text{ is sub-additive and } B^{-1} \text{ is super-additive.} \tag{306}$$

Now, let (M, d) be a complete metric space; and $f : M \to R \cup \{\infty\}$, some function taken as in (101)+(102). Further, take a couple (ε, λ) of strictly positive numbers; and a function $\Gamma : M \to R_+$ with

$$|\Gamma(x) - \Gamma(y)| \leq d(x, y), \text{ for all } x, y \in M \quad \text{(nonexpansiveness)}. \tag{307}$$

Let $\psi = \psi(f; \varepsilon, \lambda; \Gamma)$ be the function from M to $R_+ \cup \{\infty\}$ given as

$$\psi(x) = B^{-1}(B(\Gamma(x)) + (\lambda/\varepsilon)(f(x) - f_*)) - \Gamma(x), \quad x \in M; \tag{308}$$

or equivalently (in the implicit way)

$$f(x) = f_* + (\varepsilon/\lambda)[B(\Gamma(x) + \psi(x)) - B(\Gamma(x))], \quad x \in M. \tag{309}$$

By the assumptions upon f, it is clear that ψ fulfils (201); and, moreover,

$$f(x) = \infty \text{ iff } \psi(x) = \infty \quad (\text{hence } \mathrm{Dom}(f) = \mathrm{Dom}(\psi)).$$

The following property will be useful for us:

Lemma 2. *Under these conventions,*

$$(\varepsilon/\lambda)b(\Gamma(x))d(x,y) + f(y) \le f(x) \Longrightarrow d(x,y) + \psi(y) \le \psi(x). \tag{310}$$

In particular, when

$$\Gamma(x) = d(a,x), \quad x \in M, \quad \text{for some } a \in M, \tag{311}$$

this result covers the one due to Park and Bae [11].

We are now in position to state the main result of this exposition.

Theorem 2. *Let* $\varepsilon, \lambda > 0$ *be as before and* $u \in \mathrm{Dom}(f)$ *be arbitrary fixed. There exists then a point* $v = v(\varepsilon, \lambda; u) \in \mathrm{Dom}(f)$ *with*

$$f(u) \ge f(v), \quad d(u,v) \le \psi(u) - \psi(v) \ (\textit{where } \psi \textit{ is that of (308)}) \tag{312}$$

$$(\varepsilon/\lambda)b(\Gamma(v))d(v,x) > f(v) - f(x), \quad \textit{for all} \quad x \in M \setminus \{v\}. \tag{313}$$

In particular, when $(\varepsilon, \lambda; u)$ *fulfil (for some* $\rho > 0$*)*

$$f(u) \le f_* + \varepsilon, \quad \lambda \le B(\Gamma(u) + \rho) - B(\Gamma(u)), \tag{314}$$

the above evaluation (312) gives

$$f(u) \ge f(v), \quad d(u,v) \le \rho \quad (\textit{hence } |\Gamma(u) - \Gamma(v)| \le \rho). \tag{315}$$

Proof. Denote for simplicity

$$M_u = \{x \in M; f(x) \le f(u)\} \quad (\text{where } u \in M \text{ is the above one}).$$

By the lsc property (102), M_u is closed (hence complete) and nonempty (since $u \in M_u$). Let again ψ stand for the restriction to M_u of the function $\psi : M \to R_+ \cup \{\infty\}$ (introduced via (308)). By the remarks above, Theorem 1 is applicable to (M_u, d) and ψ. So, for the starting point $u \in M_u$, there exists

$v = v(\varepsilon, \lambda; u) \in M_u$ fulfiling (204)+(205). The former of these is just (312). Moreover, if our data are like in (314) then (cf. (308))

$$d(u,v) \le \psi(u) = B^{-1}[B(\Gamma(u)) + (\lambda/\varepsilon)(f(u) - f_*)] - \Gamma(u) \le$$
$$B^{-1}(B(\Gamma(u)) + \lambda) - \Gamma(u) \le \rho;$$

and so, (315) follows. On the other hand, the latter of these gives at once (313) if we take Lemma 2 into account. This ends the argument. ■

Now, Theorem 2 includes Theorem 1, to which it reduces when $b = 1$ (hence ψ is that of (206)). The reciprocal inclusion also holds, by the argument above. Summing up,

$$\text{Theorem 1} \Longleftrightarrow \text{Theorem 2} \quad \text{(from a logical viewpoint).} \tag{316}$$

In particular, this is valid for Γ taken as in (311) and $b(.) = 1/(1+h(.))$ (when Theorem 2 becomes ZVP). Hence, the "functional" extension of EVP assured by ZVP has a technical significance only (related to (313)). This, however, may be sometimes useful for concrete applications; a result of this type may be found in Zhong [18]. Note that Theorem 2 cannot be deduced in the way described by Bao and Khanh [2]; because the ψ-localizing evaluation (312) is not accessible there. Further, by the remarks in Section 2, Theorem 2 is also reductible to the Brezis-Browder ordering principle [3]. A direct proof of this may be deduced under the lines in Ray and Walker [12]. Note finally that Theorem 2 is the best result we can have under the normality setting (301)+(302); we shall discuss this fact elsewhere.

References

[1] M. Altman, *A generalization of the Brezis-Browder principle on ordered sets*, Nonlinear Analysis, **6** (1981), 157-165.

[2] T. Q. Bao, P. Q. Khanh, *Are several recent generalizations of Ekeland's variational principle more general than the original principle ?*, Acta Math. Vietnamica, **28** (2003), 345-350.

[3] H. Brezis, F. E. Browder, *A general principle on ordered sets in nonlinear functional analysis*, Advances in Math., **21** (1976), 355-364.

[4] N. Brunner, *Topologische Maximalprinzipien*, Zeitschr. Math. Logik Grundl. Math., **33** (1987), 135-139.

[5] J. Caristi, W. A. Kirk, *Geometric fixed point theory and inwardness conditions*, in "The Geometry of Metric and Linear Spaces (Michigan State Univ., 1974)" , pp. 74-83, Lecture Notes Math. vol. 490, Springer, Berlin, 1975.

[6] S. Dancs, M. Hegedus, P. Medvegyev, *A general ordering and fixed-point principle in complete metric space*, Acta Sci. Math. (Szeged), **46** (1983), 381-388.

[7] I. Ekeland, *On the variational principle*, J. Math. Anal. Appl., **47** (1974), 324-353.

[8] I. Ekeland, *Nonconvex minimization problems*, Bull. Amer. Math. Soc. (New Series), **1** (1979), 443-474.

[9] B. G. Kang, S. Park, *On generalized ordering principles in nonlinear analysis*, Nonlinear Analysis, **14** (1990), 159-165.

[10] G. H. Moore, *Zermelo's Axiom of Choice: its Origin, Development and Influence*, Springer, New York, 1982.

[11] S. Park, J. S. Bae, *On the Ray-Walker extension of the Caristi-Kirk fixed point theorem*, Nonlinear Analysis, **9** (1985), 1135-1136.

[12] W. O. Ray, A. Walker, *Mapping theorems for Gateaux differentiable and accretive operators*, Nonlinear Analysis, **6** (1982), 423-433.

[13] D. Tataru, *Viscosity solutions of Hamilton-Jacobi equations with unbounded nonlinear terms*, J. Math. Anal. Appl., **163** (1992), 345-392.

[14] M. Turinici, *Maximal elements in a class of order complete metric spaces*, Math. Japonica, **5** (1980), 511-517.

[15] M Turinici, *A generalization of Altman's ordering principle*, Proc. Amer. Math. Soc., **90** (1984), 128-132.

[16] I. Valyi, *A general maximality principle and a fixed point theorem in uniform space*, Periodica Math. Hungarica, **16** (1985), 127-134.

[17] C. S. Wong, *On a fixed point theorem of contractive type*, Proc. Amer. Math. Soc., **57** (1976), 283-284.

[18] C. K. Zhong, *A generalization of Ekeland's variational principle and application to the study of the relation between the weak P. S. condition and coercivity*, Nonlinear Analysis, **29** (1997) 1421-1431.

Received October 12, 2004.